Continuous dependence of semilinear Petrovsky equation

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Abstract

In this study, we obtain the continuous dependence on the coefficients of solutions of semilinear Petrovsky equation. Such models are involved in various fields of mathematical physics likewise geophysical and oceanic applications. ©2017 All rights reserved.

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1. Introduction

In this paper we study the continuous dependence of semilinear Petrovsky equation.

\[ u_{tt} + \Delta^2 u + au_t |u_t|^{p-2} + bu |u|^{q-2} = 0, \tag{1.1} \]

\[ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \tag{1.2} \]

\[ u(x,t) = \Delta u = 0 \quad \text{on} \quad [0, T] \times \partial \Omega, \tag{1.3} \]

where \( a, b > 0 \), \( \Omega \subset \mathbb{R}^n \) is a bounded region with a smooth boundary \( \partial \Omega \), \( 2 \leq p < \infty \), \( 2 < q < \infty \) if \( n = 1, 2 \) and \( 2 \leq p \leq \frac{2(n-1)}{n-2} \), \( 2 < q \leq \frac{2(n-1)}{n-2} \) if \( n \geq 3 \).

The subject of continuous dependence of solutions of hyperbolic type problems in partial differential equations on the coefficients in the equations has been extensively studied in the last decades. The question of continuous dependence means, one wishes to know whether a small changes in the coefficients or the parameters will inference a small changes in the solutions of the problem.

In [4], the authors studied multidimensional marine riser equations:

\[ u_{tt} + k \Delta^2 u + a \Delta u + \vec{g} \cdot \nabla u_t + b |u_t|^p u_t = 0, \quad x \in \Omega, \quad t > 0. \]

They obtained continuous dependence of the parameters \( a \) (coriolis force), \( b \) (drag force), and \( g \) (effective tension).

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In the following problem, it is studied the continuous dependence of solutions for the damped nonlinear hyperbolic equation [15]:

\[ u_{tt} + \alpha \Delta^2 u + \beta \Delta^2 u_t + \Delta g(\Delta u) = 0, \quad x \in \Omega, \ t > 0. \]

Some relevant papers on continuous dependence problems on hyperbolic type are [5, 7, 12].

In 1990, You [16] considered energy decay rate and controllability for the Petrovsky equation in a higher dimensional bounded domain with homogeneous boundary conditions. Chen and Zhou [6] considered a semilinear Petrovsky equation with damping and source terms and they proved the solution blows up in finite time for suitable condition.

In [2] the authors considered the nonlinear damped semilinear Petrovsky equation

\[ u'' - \Delta^2 u + g(u') = bu|u|^{p-2} \]

and proved the global existence of its solutions with the Faedo Galerkin procedure. In addition, they studied the asymptotic behavior of solutions. In [8], Han and Wang investigated asymptotic behavior for the solution of the Petrovsky equation with locally distributed damping. In [11], the authors considered the following Petrovsky equations:

\[ u_{tt} + \Delta^2 u + |u_t|^{p-1} u_t = F_u(u,v), \]
\[ v_{tt} + \Delta^2 v + |v_t|^{p-1} v_t = F_v(u,v), \]

and investigated global existence, uniform decay and blow up solutions. In [14], Tahamtani and Shahrooz considered the following semilinear Petrovsky equation

\[ u_{tt} + \Delta^2 u - t \int_0^t g(t-s) \Delta^2 u(s) \, ds = |u|^p u \]

and proved the existence of weak solutions. We also refer to the papers [9, 10, 13, 17] on the semilinear Petrovsky equation. Some papers about continuous dependence results can be seen in [1, 3]. Since there is no study on the continuous dependence of Petrovsky equation, our goal is to investigate the continuous dependence of solutions to the problem (1.1)-(1.3) on the coefficients \( a \) and \( b \).

**Lemma 1.1** (Sobolev-Poincaré inequality). Let \( r \) be a number with \( 2 \leq r < \infty \) (\( n = 1, 2 \)) or \( 2 \leq r \leq \frac{2(n-1)}{n-2} \) \( (n \geq 3) \), then there is a constant \( d = d(\Omega, r) \) such that

\[ \|u\|_r \leq d \|\Delta u\|_2 \]

for \( u \in H^0_0(\Omega) \).

**2. Priori estimates**

**Theorem 2.1.** Let \( u_0 \in H^0_0(\Omega) \) and \( u_1 \in L^2(\Omega) \). Then the solution \( u \) of the problem (1.1)-(1.3) satisfies the following estimates:

\[ \|u_t(t)\|^2 \leq D_1, \quad \|\Delta u\|^2 \leq D_1, \quad \text{and} \quad \|u\|_q^q \leq D_2, \]

where \( D_1 \) and \( D_2 \) are positive constants depending on the initial data and the parameters of (1.1).

**Proof.** If the equation (1.1) is multiplied by \( u_t \) in \( L^2(\Omega) \), then

\[ \frac{d}{dt} \left[ \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\Delta u\|^2 + \frac{b}{q} \|u\|_q^q \right] + a \|u_t\|^p_p = 0 \]
is obtained. Let $$E_1(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\Delta u\|^2 + \frac{b}{q} \|u\|^q.$$ Thus, from the equation (2.2) we have

$$\frac{d}{dt} E_1(t) \leq 0$$

and

$$E_1(t) \leq E_1(0). \quad (2.3)$$

The inequality (2.3) gives us all estimates in (2.1).

**Theorem 2.2.** Let $$u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$$. Then the solution $$u$$ of the problem (1.1)-(1.3) satisfies the following estimates

$$\|u_{tt}\|^2 \leq D_4 e^{D_3 t}, \quad \|\Delta u_t\|^2 \leq D_4 e^{D_3 t}, \quad (2.4)$$

where $$D_3$$ and $$D_4$$ are positive constants depending on the initial data and the parameters of (1.1).

**Proof.** We differentiate the equation (1.1) with respect to $$t$$

$$u_{ttt} + \Delta^2 u_t + a(p-1)|u_t|^{p-2}u_t + bu^q u_t = 0. \quad (2.5)$$

We multiply the equation (2.5) by $$u_{tt}$$ in $$L^2(\Omega)$$, then we get

$$\frac{d}{dt} E_2(t) + a(p-1) \int_{\Omega} |u_t|^{p-2} u_t^2 u_{tt} dx + b(q-1) \int_{\Omega} |u|^{q-2} u_t u_{tt} dx = 0, \quad (2.6)$$

where $$E_2(t) = \frac{1}{2} \|u_{tt}\|^2 + \frac{1}{2} \|\Delta u_t\|^2$$. Then from the equation (2.6) we have the following inequality

$$\frac{d}{dt} E_2(t) \leq b(q-1) \int_{\Omega} |u|^{q-2} u_t u_{tt} dx. \quad (2.7)$$

Using Hölder and Sobolev-Poincaré inequalities and the estimate (2.1) we obtain the inequality

$$\left| b(q-1) \int_{\Omega} |u|^{q-2} u_t u_{tt} dx \right| \leq b(q-1) \|u\|^{q-2} \|u_{tt}\| \|u_t\| \leq D_3 \|u_{tt}\| \|\Delta u_t\|, \quad (2.8)$$

where $$D_3 = b d(q-1)(D_1)^{\frac{q-2}{2}}$$. Apply the Cauchy inequality on the right hand side of the inequality (2.8) and rewrite (2.7) to get

$$\frac{d}{dt} E_2(t) \leq D_3 E_2(t).$$

The last inequality gives the estimate

$$E_2(t) \leq e^{D_3 t} E_2(0).$$

Hence, proof is completed. \(\square\)

3. Continuous dependence on parameters

3.1. Continuous dependence on the coefficient $$a$$

Assume that $$u$$ is the solution of the problem

$$u_{tt} + \Delta^2 u + au_t |u_t|^{p-2} + bu^q u_t = 0,$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega,$$
Assume that \( v \) is the solution of the problem

\[
\nu_{tt} + \Delta^2 v + (a + \alpha)\nu_t|\nu_t|^{p-2} + b|\nu|^{q-2}v_t = 0,
\]

\( v(x, 0) = u_0(x), \quad \nu_t(x, 0) = u_1(x) \quad \text{in} \quad \Omega, \)

\( v(x, t) = \Delta v = 0 \quad \text{on} \quad [0, T] \times \partial \Omega, \)

and \( w = u - v \) is the solution of the problem

\[
w_{tt} + \Delta^2 w + a (|u_t|^{p-2}u_t - |\nu_t|^{p-2}v) - \alpha |\nu_t|^{p-2}v_t + b (|u|^{q-2}u - |\nu|^{q-2}v) = 0, \tag{3.1}
\]

\( w(x, 0) = 0, \quad w_t(x, 0) = 0 \quad \text{in} \quad \Omega, \)

\( w(x, t) = \Delta w = 0 \quad \text{on} \quad [0, T] \times \partial \Omega. \tag{3.3}
\]

**Theorem 3.1.** If \( w \) is the solution of the problem (3.1)-(3.3), then

\[
||\Delta w|| \to 0 \quad \text{as} \quad \alpha \to 0.
\]

**Proof.** We multiply the equation (3.1) by \( w_t \) in \( L^2(\Omega) \) to obtain

\[
\frac{d}{dt} E_3(t) + a \int_\Omega \left( |u_t|^{p-2}u_t - |\nu_t|^{p-2}v_t \right) w_t dx
\]

\[
- \alpha \int_\Omega |\nu_t|^{p-2}v_tw_t dx + b \int_\Omega \left( |u|^{q-2}u - |\nu|^{q-2}v \right) w_t dx = 0,
\]

where \( E_3(t) = \frac{1}{2} ||w_t||^2 + \frac{1}{2} ||\Delta w||^2. \) Since \( a \int_\Omega \left( |u_t|^{p-2}u_t - |\nu_t|^{p-2}v_t \right) w_t dx > 0, \) then we get from the relation (3.4)

\[
\frac{d}{dt} E_3(t) \leq \alpha \int_\Omega |\nu_t|^{p-2}v_tw_t dx + b \int_\Omega \left( |u|^{q-2}u - |\nu|^{q-2}v \right) w_t dx. \tag{3.5}
\]

Now, we evaluate first term on the right hand side of (3.5)

\[
\alpha \int_\Omega |\nu_t|^{p-2}v_tw_t dx \leq |\alpha| \int_\Omega |\nu_t|^{p-1} |w_t| dx \leq |\alpha||w_t|| ||\nu_t||^{p-1}_2 \leq \frac{1}{2} ||w_t||^2 + \frac{\alpha^2}{2} ||v||^{2(p-1)}_2. \tag{3.6}
\]

Using the Sobolev-Poincaré inequality we obtain from (3.6)

\[
\alpha \int_\Omega |\nu_t|^{p-2}v_tw_t dx \leq \frac{1}{2} ||w_t||^2 + \frac{\alpha^2}{2} d ||\Delta v||^{2(p-1)}_2, \tag{3.7}
\]

where \( d \) is the Sobolev constant. If we use the mean value theorem and Hölder’s inequality to the second term on the right hand side of (3.5), then we have

\[
b \int_\Omega \left( |u|^{q-2}u - |\nu|^{q-2}v \right) w_t dx \leq b (q - 1) \int_\Omega \left( |u|^{q-2} + |\nu|^{q-2} \right) |w| |w_t| dx
\]

\[
\leq b (q - 1) ||w||_\infty ||w_t|| \left( ||u||^{q-2}_{(q-2)n} + ||v||^{q-2}_{(q-2)n} \right). \tag{3.8}
\]
We use the Sobolev inequality in (3.8) to get
\[
\left| \int_{\Omega} (|u|^{q-2}u - |v|^{q-2}v) w_t \, dx \right| \leq b (q - 1) d^2 \|\Delta w\| \|w_t\| \left( \|\Delta u\|^{q-2} + \|\Delta v\|^{q-2} \right),
\] (3.9)
Hence using the estimate (2.1) in the inequality (3.9) we get
\[
\left| \int_{\Omega} (|u|^{q-2}u - |v|^{q-2}v) w_t \, dx \right| \leq D_5 \|\Delta w\| \|w_t\| \leq \frac{D_5}{2} \|\Delta w\|^2 + \frac{D_5}{2} \|w_t\|^2,
\] (3.10)
where \(D_5 = 2b (q - 1) d^2 (D_1)^{\frac{q-2}{2}}\). The estimates (2.4), (3.5), (3.7), and (3.10) give us the following differential inequality
\[
\frac{d}{dt} E_3 (t) \leq M_3 E_3 (t) + K_1 \alpha^2 e^{K_3 t},
\] (3.11)
where \(M_3 = 1 + D_5, K_1 = \frac{d}{dt} D_4^{p-1}, K_2 = D_3 (p - 1)\). Thus the inequality (3.11) concludes the estimate
\[
E_3 (t) \leq \frac{K_1}{K_2} e^{K_3 t} \alpha^2,
\]
where \(K_3 = K_2 + M_3\). Therefore we have \(\|\Delta w\| \to 0\) as \(\alpha \to 0\). \(\square\)

3.2. Continuous dependence on the coefficient \(b\)

Assume that \(u\) is the solution of the problem
\[
\begin{align*}
    &u_{tt} + \Delta^2 u + au_t |u_t|^{p-2} + bu |u|^{q-2} = 0, \\
    &u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad \Omega, \\
    &u(x, t) = \Delta u = 0 \quad \text{on} \quad [0, T] \times \partial \Omega.
    \end{align*}
\] (3.12)
Assume that \(v\) is the solution of the problem
\[
\begin{align*}
    &v_{tt} + \Delta^2 v + av_t |v_t|^{p-2} + (b + \beta) v |v|^{q-2} = 0, \\
    &v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x) \quad \text{in} \quad \Omega, \\
    &v(x, t) = \Delta v = 0 \quad \text{on} \quad [0, T] \times \partial \Omega,
    \end{align*}
\] (3.13)
and \(w = u - v\) is the solution of the problem
\[
\begin{align*}
    &w_{tt} + \Delta^2 w + a \left( |u_t|^{p-2} u_t - |v_t|^{p-2} v_t \right) + b \left( |u|^{q-2} u - |v|^{q-2} v \right) - \beta |v|^{q-2} v = 0, \\
    &w(x, 0) = 0, \quad w_t(x, 0) = 0 \quad \text{in} \quad \Omega, \\
    &w(x, t) = \Delta w = 0 \quad \text{on} \quad [0, T] \times \partial \Omega.
    \end{align*}
\] (3.14)

\textbf{Theorem 3.2.} If \(w\) is the solution of the problem (3.13)-(3.15), then
\[
\|\Delta w\| \to 0 \quad \text{as} \quad \beta \to 0.
\]

\textbf{Proof.} We multiply the equation (3.12) by \(w_t\) in \(L^2(\Omega)\) to get
\[
\frac{d}{dt} E_3 (t) + a \int_{\Omega} \left( |u_t|^{p-2} u_t - |v_t|^{p-2} v_t \right) w_t \, dx \\
- \beta \int_{\Omega} |v|^{q-2} v w_t \, dx + b \int_{\Omega} \left( |u|^{q-2} u - |v|^{q-2} v \right) w_t \, dx = 0.
\] (3.16)
By using the inequality (3.10) in (3.16) we obtain

\[ \frac{d}{dt}E_3(t) \leq \left| \int_{\Omega} v^{q-2}v w_t \, dx \right| + \frac{D_5}{2} \| \Delta w \|^2 + \frac{D_5}{2} \| w_t \|^2. \]  

(3.17)

Apply the Hölder, Cauchy, Sobolev-Poincaré inequalities, and estimate (2.1) to the first term on the right hand side of (3.17) to obtain the following estimate

\[ \beta \left| \int_{\Omega} v^{q-2}v w_t \, dx \right| \leq \beta \| w_t \| \| v \|^{\frac{q-1}{2}} \]  

(3.18)

\[ \leq \frac{1}{2} \| w_t \|^2 + \frac{\beta^2}{2} \| v \|^{2(q-1)} \]  

\[ \leq \frac{1}{2} \| w_t \|^2 + \frac{\beta^2}{2} d \| \Delta v \|^{2(q-1)} \leq \frac{1}{2} \| w_t \|^2 + \beta^2 D_7, \]

where \( D_7 = \frac{d}{2} (D_1)^{q-1} \). Therefore by substituting (3.18) in (3.17) we have

\[ \frac{d}{dt}E_3(t) \leq M_3 E_3(t) + \beta^2 D_7, \]  

(3.19)

where \( M_3 \) is positive constant. Solving the differential inequality (3.19) we obtain the following estimate

\[ E_3(t) \leq \frac{D_7}{M_3} e^{M_3 t} \beta^2. \]

Hence the proof is completed. \( \square \)

Conclusion

In this article, by using multiplier method, we conclude that the solution of the problem (1.1)-(1.3) describing semilinear Petrovsky equation is continuously dependent on the coefficients \( a \) and \( b \).

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References


