Approximation of the mixed additive and cubic functional equation in paranormed spaces

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Abstract

In this paper, we prove some theorems about the Hyers-Ulam stability of the functional equation
\[
f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 2|f(2x) - 2f(x)|
\]
in paranormed spaces. From these theorems, as corollaries, we obtain the stability of the above functional equation with weaker conditions controlled by product of powers of norms and mixed-type product-sum of powers of norms. ©2017 All rights reserved.

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1. Introduction

In 1940, Ulam [18] posed the first stability problem concerning group homomorphisms: let \((G_1, \ast)\) be a group and \((G_2, \odot, d)\) be a metric group with the metric \(d(\cdot, \cdot)\). Given \(\varepsilon > 0\), does there exist a \(\delta(\varepsilon) > 0\) such that if a mapping \(f : G_1 \to G_2\) satisfies the inequality
\[
d(f(x \ast y), f(x) \odot f(y)) < \delta
\]
for all \(x, y \in G_1\), then there is a homomorphism \(g : G_1 \to G_2\) with \(d(f(x), g(x)) < \varepsilon\) for all \(x \in G_1\)?

In 1941, Hyers [4] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ result was generalized by Aoki [1] for additive mappings and by Rassias [14] for linear mappings. In 1982, Rassias [15] considered a further generalization of the results of Hyers and proved a theorem using weaker conditions controlled by a product of different powers of norms. Forti [2] and Găvruţa [3] obtained generalized Rassias’ result which allows the Cauchy difference to be controlled by a general unbounded function in the spirit of Rassias’ approach. Since then, the stability of several functional equations has been extensively investigated by several mathematicians (see [5–7, 16, 17] and references

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In this section, using the direct method, we prove the Hyers-Ulam stability of the functional equation \(D_f(x, y) = 0\) in paranormed spaces. It is easy to check that the mapping \(x \mapsto f(x) - f(0)\) also satisfies the equation \(D_f(x, y) = 0\). Thus, throughout this section, without loss of generality, we assume that \(f(0) = 0\), otherwise we work on the function \(x \mapsto f(x) - f(0)\) instead of \(f\).

Before proceeding to the proof of the main results, we present the following two lemmas.
Lemma 3.1 ([9]). Let $V$ and $W$ be real vector spaces. If a mapping $f : V \to W$ satisfies \eqref{1.1}, then the mapping $g : V \to W$ defined by $g(x) = f(2x) - 2f(x)$ is cubic.

Lemma 3.2 ([9]). Let $V$ and $W$ be real vector spaces. If a mapping $f : V \to W$ satisfies \eqref{1.1}, then the mapping $h : V \to W$ defined by $h(x) = f(2x) - 8f(x)$ is additive.

Theorem 3.3. Let $r$ be positive real number with $r < 3$. Suppose that $f : X \to Y$ is a mapping satisfying

$$
\|D_f(x,y)\| \leq P(x)^r + P(y)^r
$$

\text{(3.1)}

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying

$$
\|f(2x) - 2f(x) - C(x)\| \leq \frac{7 + 2r}{8 - 2r} P(x)^r
$$

\text{(3.2)}

for all $x \in X$.

Proof. Setting $x = 0$ in \eqref{3.1}, we have

$$
\|f(y) + f(-y)\| \leq P(y)^r
$$

\text{(3.3)}

for all $y \in X$. Replacing $y$ by $x$ first and then $y$ by $2x$ in \eqref{3.1}, we obtain the inequalities

$$
\|f(3x) - 4f(2x) + 5f(x)\| \leq 2P(x)^r,
$$

\text{(3.4)}

$$
\|f(4x) - 2f(3x) - 2f(2x) - 2f(x) + 4f(x)\| \leq (2^r + 1)P(x)^r
$$

\text{(3.5)}

for all $x \in X$. It follows from \eqref{3.3}, \eqref{3.4}, and \eqref{3.5} that

$$
\|f(4x) - 10f(2x) + 16f(x)\| \leq \|2(f(x) + f(-x))\| + \|2(f(3x) - 4f(2x) + 5f(x))\|
$$

$$
+ \|f(4x) - 2f(3x) - 2f(2x) - 2f(x) + 4f(x)\|
$$

$$
\leq 2P(x)^r + 2 : 2P(x)^r + (2^r + 1)P(x)^r = (7 + 2^r)P(x)^r
$$

\text{(3.6)}

for all $x \in X$. Defining $g : X \to Y$ by $g(x) := f(2x) - 2f(x)$ and using \eqref{3.6} we have

$$
\|g(x) - \frac{1}{8^n} g(2^n x)\| \leq \frac{7 + 2^r}{8} P(x)^r
$$

\text{(3.7)}

for all $x \in X$. Hence

$$
\frac{1}{8^n} g(2^n x) - \frac{1}{8^n} g(2^n x) \mid \leq \sum_{j=m}^{n-1} \frac{(7 + 2^r)2^j}{8^{j+1}} P(x)^r
$$

\text{(3.8)}

for all nonnegative integers $n$ and $m$ with $n \geq m$ and all $x \in X$. Therefore we conclude from \eqref{3.8} that the sequence $\left\{ \frac{1}{8^n} g(2^n x) \right\}$ is a Cauchy sequence in $Y$ for all $x \in X$. Since $Y$ is complete, the sequence $\left\{ \frac{1}{8^n} g(2^n x) \right\}$ converges for all $x \in X$. So one can define the mapping $C : X \to Y$ by

$$
C(x) := \lim_{n \to \infty} \frac{1}{8^n} g(2^n x)
$$

\text{(3.9)}

for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in \eqref{3.8}, we get \eqref{3.2}. Now, we show that the mapping $C$ is cubic. By \eqref{3.7} and \eqref{3.9},

\[
\|C(2x) - 8C(x)\| = \lim_{n \to \infty} \| \frac{1}{8^n} g(2^{n+1} x) - \frac{1}{8^n} g(2^n x) \|
\]

\[
= \lim_{n \to \infty} \frac{1}{8^n} \| g(2^{n+1} x) - g(2^n x) \|
\]

\[
\leq \lim_{n \to \infty} \frac{(7 + 2^r)2^{nr}}{8^n} P(x)^r = 0
\]
for all $x \in X$. Therefore, we obtain
\[
C(2x) = 8C(x)
\] (3.10)
for all $x \in X$. On the other hand it follows from (3.1) and (3.9) that
\[
\|D_C(x, y)\| = \lim_{n \to \infty} \left\| \frac{1}{8^n} D_f(2^n x, 2^n y) \right\|
\]
\[
= \lim_{n \to \infty} \left( \frac{1}{8^n} \|D_f(2^{n+1} x, 2^{n+1} y) - 2D_f(2^n x, 2^n y)\| \right)
\]
\[
\leq \lim_{n \to \infty} \left( \frac{1}{8^n} (2^{nr+r}[P(x)^r + P(y)^r] + 2^{n+1}[P(x)^r + P(y)^r]) \right)
\]
\[
= \lim_{n \to \infty} \left( \frac{2^n r}{8^n} (2^r + 2)[P(x)^r + P(y)^r] \right) = 0
\]
for all $x, y \in X$. Hence the mapping $C$ satisfies (1.1). So by Lemma 3.1, the mapping $x \mapsto C(2x) - 2C(x)$ is cubic. Hence (3.10) implies that the mapping $C$ is cubic. To prove the uniqueness of $C$, let $C' : X \to Y$ be another cubic mapping satisfying (3.2). Then we have
\[
\|C(x) - C'(x)\| = \left\| \frac{1}{8^n} C(2^n x) - \frac{1}{8^n} C'(2^n x) \right\|
\]
\[
\leq \left\| \frac{1}{8^n} (C(2^n x) - g(2^n x)) \right\| + \left\| \frac{1}{8^n} (C'(2^n x) - g(2^n x)) \right\|
\]
\[
\leq \frac{2(7 + 2^r)2^{nr}}{(8 - 2^r)8^n} P(x)^r,
\]
which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $C(x) = C'(x)$ for all $x \in X$. This completes the proof of the theorem.

**Corollary 3.4.** Let $\tau, s$ be positive real numbers such that $\rho := \tau + s < 3$. Suppose that $f : X \to Y$ is a mapping satisfying
\[
\|D_f(x, y)\| \leq \begin{cases} P(x)^r P(y)^s, \\ P(x)^r P(y)^s + P(x)^r + P(y)^r + s \end{cases}
\] (3.11)
for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying
\[
\|f(2x) - 2f(x) - C(x)\| \leq \begin{cases} \frac{2 + 2^s}{8 - 2^r} P(x)^\rho, \\ \frac{(9 + 2^s) + 2^s}{8 - 2^r} P(x)^\rho \end{cases}
\]
for all $x \in X$.

**Proof.** The proof is similar to the proof of Theorem 3.3 and thus it is omitted.

**Theorem 3.5.** Let $\tau, \theta$ be positive real numbers such that $\tau > 3$. Suppose that $f : Y \to X$ is a mapping satisfying
\[
P(D_f(x, y)) \leq \theta (\|x\|^\tau + \|y\|\tau)
\] (3.12)
for all $x, y \in Y$. Then there exists a unique cubic mapping $C : Y \to X$ satisfying
\[
P(f(2x) - 2f(x) - C(x)) \leq \frac{2^\tau + 7}{2^\tau - 8} \|x\|^\tau
\] (3.13)
for all $x \in Y$. 
Proof. Setting \( x = 0 \) in (3.12), we obtain

\[
P(f(y) + f(-y)) \leq \theta \|y\|^r
\]

for all \( y \in Y \). Replacing first \( y \) by \( x \) and then \( y \) by \( 2x \) in (3.12), we get

\[
\begin{align*}
P(f(3x) - 4f(2x) + 5f(x)) & \leq 20\|x\|^r, \\
P(f(4x) - 2f(3x) - 2f(2x) - 2f(-x) + 4f(x)) & \leq (2^r + 1)\|x\|^r
\end{align*}
\]

for all \( x \in Y \). It follows from (3.14), (3.15), and (3.16) that

\[
P(f(4x) - 10f(2x) + 16f(x)) \leq P(2(f(x) + f(-x))) + P(2(f(3x) - 4f(2x) + 5f(x)))
\]

\[
\quad + P(f(4x) - 2f(3x) - 2f(2x) - 2f(-x) + 4f(x))
\]

\[
\quad \leq 20\|x\|^r + 2 \cdot 20\|x\|^r + (2^r + 1)\|x\|^r
\]

\[
= (2^r + 7)\|x\|^r
\]

for all \( x \in Y \). Defining \( g : Y \to X \) by \( g(x) := f(2x) - 2f(x) \) and using (3.17) we have

\[
P(g(x) - 8g(x)) \leq \frac{2^r + 7}{2^r} \theta \|x\|^r
\]

for all \( x \in Y \). Hence

\[
P(8^ng(\frac{x}{2^m}) - 8^ng(\frac{x}{2^n})) \leq \sum_{j=m}^{n-1} \frac{(2^r + 7)8^j}{2^r + r} \theta \|x\|^r
\]

for all nonnegative integers \( n \) and \( m \) with \( n \geq m \) and all \( x \in Y \). It follows from (3.19) that the sequence \( \{8^ng(\frac{x}{2^n})\} \) is a Cauchy sequence in \( X \) for all \( x \in Y \). Since \( X \) is complete, the sequence \( \{8^ng(\frac{x}{2^n})\} \) converges for all \( x \in Y \). So one can define the mapping \( C : Y \to X \) by

\[
C(x) := \lim_{n \to \infty} 8^ng(\frac{x}{2^n})
\]

for all \( x \in Y \). Moreover, letting \( m = 0 \) and passing the limit \( n \to \infty \) in (3.19), we get (3.13). Now, we show that the mapping \( C \) is cubic. By (3.18) and (3.20),

\[
P(C(x) - 8C(x)) = \lim_{n \to \infty} P(8^ng(\frac{x}{2^n}) - 8^{n+1}g(\frac{x}{2^{n+1}})) = \lim_{n \to \infty} 8^{n+1}P(g(\frac{x}{2^n}) - 8g(\frac{x}{2^{n+1}}))
\]

\[
\quad \leq \lim_{n \to \infty} \frac{(2^r + 7)8^n}{2^{n+r}} \theta \|x\|^r = 0
\]

for all \( x \in Y \). Therefore, we obtain

\[
C(x) = 8C(x)
\]

for all \( x \in Y \). On the other hand it follows from (3.12) and (3.20) that

\[
P(D_C(x, y)) = \lim_{n \to \infty} P(8^ng(\frac{x}{2^n}, \frac{y}{2^n}))
\]

\[
= \lim_{n \to \infty} 8^nP(\frac{x}{2^n} - 2\frac{y}{2^n}) - 2D(\frac{x}{2^n}, \frac{y}{2^n})
\]

\[
\quad \leq \lim_{n \to \infty} \frac{8^n}{2^{nr}} (2^r + 2) \theta (\|x\|^r + \|y\|^r) = 0
\]
for all $x, y \in Y$. Hence the mapping $C$ satisfies (1.1). So by Lemma 3.1, the mapping $x \mapsto C(2x) - 2C(x)$ is cubic. Hence (3.21) implies that the mapping $C$ is cubic. To prove the uniqueness of $C$, let $C' : Y \to X$ be another cubic mapping satisfying (3.13). Then we have
\[
P(C(x) - C''(x)) = P(8^nC(x) - 8^nC''(x)) \\
\leq 8^n(P(C(x) - g(x)) + P(C''(x) - g(x))) \\
\leq 2(2^r + 7)8^n(2r - 8)\theta \|x\|^r,
\]
which tends to zero as $n \to \infty$ for all $x \in Y$. So we can conclude that $C(x) = C''(x)$ for all $x \in Y$. This completes the proof of the theorem.

**Corollary 3.6.** Let $r, s, \theta$ be positive real numbers such that $\theta := r + s > 3$. Suppose that $f : Y \to X$ is a mapping satisfying
\[
P(D_f(x, y)) \leq \begin{cases} 
\theta \|x\|^r \|y\|^s, \\
\theta \|x\|^r \|y\|^s + (\|x\|^{r+s} + \|y\|^{r+s})
\end{cases}
\]
for all $x, y \in Y$. Then there exists a unique cubic mapping $C : Y \to X$ satisfying
\[
P(f(2x) - 2f(x) - C(x)) \leq \begin{cases} 
\frac{2r + 7}{2^r - 8}\theta \|x\|^r, \\
\frac{2^{2r} + (2^r + 9)}{2^r - 8}\theta \|x\|^r
\end{cases}
\]
for all $x \in Y$.

**Proof.** The proof is similar to the proof of Theorem 3.5 and thus it is omitted.

**Theorem 3.7.** Let $r$ be positive real number with $r < 1$. Suppose that $f : X \to Y$ is a mapping satisfying
\[
\|D_f(x, y)\| \leq P(x)^r + P(y)^r
\]
for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying
\[
\|f(2x) - 8f(x) - A(x)\| \leq \frac{7 + 2r}{2 - 2r}P(x)^r
\]
for all $x \in X$.

**Proof.** The proof is similar to the proof of Theorem 3.3 and thus it is omitted.

**Corollary 3.8.** Let $r, s$ be positive real numbers such that $\rho := r + s < 1$. Suppose that $f : X \to Y$ is a mapping satisfying (3.11) for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying
\[
\|f(2x) - 8f(x) - A(x)\| \leq \begin{cases} 
\frac{2^{2r} + 7}{2^r - 2}\rho P(x)^\rho, \\
\frac{(2^r + 9) + 2\rho}{2^r - 2}\rho P(x)^\rho
\end{cases}
\]
for all $x \in X$.

**Proof.** The proof is similar to the proof of Theorem 3.7 and thus it is omitted.

**Theorem 3.9.** Let $r, \theta$ be positive real numbers such that $r > 1$. Suppose that $f : Y \to X$ is a mapping satisfying
\[
P(D_f(x, y)) \leq \theta (\|x\|^r + \|y\|^r)
\]
for all $x, y \in Y$. Then there exists a unique additive mapping $A : Y \to X$ satisfying
\[
P(f(2x) - 8f(x) - A(x)) \leq \begin{cases} 
\frac{2^r + 7}{2^r - 2}\theta \|x\|^r, \\
\frac{2^{2r} + (2^r + 9) + 2\theta}{2^r - 2}\theta \|x\|^r
\end{cases}
\]
for all $x \in Y$. 

\[\]
Proof. The proof is similar to the proof of Theorem 3.5 and thus it is omitted.

**Corollary 3.10.** Let $r, s, \theta$ be positive real numbers such that $\rho := r + s > 1$. Suppose that $f : Y \to X$ is a mapping satisfying (3.22) for all $x, y \in Y$. Then there exists a unique additive mapping $A : Y \to X$ satisfying

$$P(f(2x) - 8f(x) - A(x)) \leq \left\{ \begin{array}{ll} \frac{2^s+\theta}{2^r-2} \|x\|^{\rho}, & \\
\frac{2^s+(2^r+9)}{2^r-2} \|x\|^{\rho} & \end{array} \right.$$  \(\text{for all } x \in Y.\)

Proof. The proof is similar to the proof of Theorem 3.9 and thus it is omitted.

Applying Theorems 3.3 and 3.7, we obtain the following result.

**Theorem 3.11.** Let $r$ be positive real number with $r < 1$. Suppose that $f : X \to Y$ is a mapping satisfying (3.1). Then there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{7+2^r}{2-2^r} P(x)^r, \quad \|f(2x) - 2f(x) - C(x)\| \leq \frac{7+2^r}{8-2^r} P(x)^r$$  \(\text{for all } x \in X.\)

Proof. Let $g(x) = f(2x) - 2f(x)$ and $h(x) = f(2x) - 8f(x)$. Then $6f(x) = g(x) - h(x)$. By Theorems 3.3 and 3.7, there exists an additive mapping $A_0 : X \to Y$ and a cubic mapping $C_0 : X \to Y$ such that

$$\|f(2x) - 8f(x) - A_0(x)\| \leq \frac{7+2^r}{2-2^r} P(x)^r, \quad \|f(2x) - 2f(x) - C_0(x)\| \leq \frac{7+2^r}{8-2^r} P(x)^r$$

for all $x \in X$. Therefore it follows from the last inequalities that

$$\|6f(x) + A_0(x) - C_0(x)\| = \|g(x) - h(x) + A_0(x) - C_0(x)\|$$

$$\leq \|h(x) - A_0(x)\| + \|g(x) - C_0(x)\|$$

$$\leq \frac{7+2^r}{2-2^r} + \frac{7+2^r}{8-2^r} P(x)^r$$

for all $x \in X$. So we obtain (3.23) by letting $A(x) = -A_0(x)$ and $C(x) = C_0(x)$ for all $x \in X$.

To prove the uniqueness of $A$ and $C$, let $A_1, C_1 : X \to Y$ be another additive and cubic mappings satisfying (3.23), and let $A' = A - A_1$ and $C' = C - C_1$. So

$$\|A'(x) + C'(x)\| \leq \|6f(x) - A(x) - C(x)\| + \|6f(x) - A_1(x) - C_1(x)\| \leq 2\left( \frac{7+2^r}{2-2^r} + \frac{7+2^r}{8-2^r} \right) P(x)^r$$  \(3.24\)

for all $x \in X$. Since

$$\lim_{n \to \infty} \frac{1}{8^n} P(2^n x)^r = \lim_{n \to \infty} \frac{1}{2^n} P(2^n x)^r = 0$$

for all $x \in X$, (3.24) implies

$$\lim_{n \to \infty} \frac{1}{8^n} \|A'(2^n x) + C'(2^n x)\| = 0$$

for all $x \in X$. Therefore $C'(x) = 0$ for all $x \in X$. By (3.24), we have $A'(x) = 0$ for all $x \in X$. This completes the proof of the theorem. \(\square\)
Corollary 3.12. Let \( r, s \) be positive real numbers such that \( \rho := r + s < 1 \). Suppose that \( f : X \to Y \) is a mapping satisfying (3.11) for all \( x, y \in X \). Then there exist a unique additive mapping \( A : X \to X \) and a unique cubic mapping \( C : X \to Y \) such that

\[
\|6f(x) - A(x) - C(x)\| \leq \left\{ \begin{array}{ll}
\left( \frac{2^r + 2^s}{2^{r+s}-8} + \frac{2^r+2^s}{2^{r+s}-2} \right) \rho \|x\|^\rho, \\
\left( \frac{9+2^r+2^s}{2^{r+s}-8} + \frac{9+2^r+2^s}{2^{r+s}-2} \right) \rho \|x\|^\rho
\end{array} \right.
\]

for all \( x \in X \).

Proof. The result follows from Theorem 3.11.

Similar to Theorems 3.5 and 3.9, we also obtain the following results.

Theorem 3.13. Let \( r, \theta \) be positive real numbers such that \( r > 3 \). Suppose that \( f : Y \to X \) is a mapping satisfying (3.12). Then there exist a unique additive mapping \( A : Y \to X \) and a unique cubic mapping \( C : Y \to X \) such that

\[
P(6f(x) - A(x) - C(x)) \leq \left( \frac{2^r + 7}{2^r-2} + \frac{2^r + 7}{2^r-8} \right) \theta \|x\|^\rho
\]

for all \( x \in Y \).

Proof. The proof is similar to the proof of Theorem 3.11 and the result follows from Theorems 3.5 and 3.9.

Corollary 3.14. Let \( r, s, \theta \) be positive real numbers such that \( \rho := r + s > 3 \). Suppose that \( f : Y \to X \) is a mapping satisfying (3.22) for all \( x, y \in Y \). Then there exist a unique additive mapping \( A : Y \to X \) and a unique cubic mapping \( C : Y \to X \) such that

\[
P(6f(x) - A(x) - C(x)) \leq \left\{ \begin{array}{ll}
\left( \frac{2^r + 2^s}{2^{r+s}-8} + \frac{2^r+2^s}{2^{r+s}-2} \right) \theta \|x\|^\rho, \\
\left( \frac{2^r+(2^r+9)}{2^{r+s}-8} + \frac{2^r+(2^r+9)}{2^{r+s}-2} \right) \theta \|x\|^\rho
\end{array} \right.
\]

for all \( x \in Y \).

Proof. The result follows from Theorem 3.13.

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