Some explicit identities for the modified higher-order degenerate $q$-Euler polynomials and their zeroes

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Abstract


Furthermore, we demonstrate the shapes and zeroes of the modified higher-order $q$-Euler polynomials and the modified higher-order degenerate $q$-Euler polynomials by using a computer. ©2017 All rights reserved.

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1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of $p$-adic integers, the field of $p$-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. We normalized the $p$-adic norm as $|p|_p = \frac{1}{p}$. Let $q$ be an indeterminate in $\mathbb{C}_p$ such that $|1 - q|_p < p^{-\frac{1}{p-1}}$ and the $q$-analogue of the number $x$ is defined as $[x]_q = \frac{1 - q^x}{1 - q}$. Note that $\lim_{q \to 1}[x]_q = x$.

Let $f(x)$ be a continuous function on $\mathbb{Z}_p$. Then, the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim et al.
(see [11–13, 18, 20]) to be

\[ I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)(-q)^x = \frac{2}{2} \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)q^x(-1)^x, \quad (1.1) \]

where \([x]_{-q} = \frac{1-(-q)^x}{1+q}\). Note that

\[ \lim_{q \to 1} \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) \quad (1.2) \]

is the ordinary fermionic \(p\)-adic integral on \(\mathbb{Z}_p\) (see [2, 4, 5, 9, 14, 17, 19, 22, 25, 26]). From (1.1), we have

\[ qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \quad \text{where } f_1(x) = f(x+1). \quad (1.3) \]

From (1.2), we have

\[ I_{-1}(f_1) + I_{-1}(f) = 2f(0), \quad \text{where } f_1(x) = f(x+1). \quad (1.4) \]

Recall that the Carlitz’s \(q\)-Euler numbers are defined by the \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\) as follows:

\[ \int_{\mathbb{Z}_p} [x]^m_q d\mu_{-q}(x) = \mathcal{E}_{m,q} \quad (\text{see [10, 12]}). \]

From (1.3) with \(f(x) = [x]^m_q\), we can derive

\[ q \int_{\mathbb{Z}_p} [x+1]^m_q d\mu_{-q}(x) + \int_{\mathbb{Z}_p} [x]^m_q d\mu_{-q}(x) = \begin{cases} [2]_q, & \text{if } m = 0, \\ 0, & \text{if } m > 0. \end{cases} \]

We note that

\[ [x+1]_q^m = \left( \frac{1 - q^{x+1}}{1 - q} \right)^m = (1 + qx)_q^m = \sum_{l=0}^{m} \binom{m}{l}_q q^l [x]_q^l \quad (1.5) \]

and hence

\[ \int_{\mathbb{Z}_p} [x+1]^m_q d\mu_{-q}(x) = \sum_{l=0}^{m} \binom{m}{l}_q q^l \int_{\mathbb{Z}_p} [x]^l_q d\mu_{-q}(x) = \sum_{l=0}^{m} \binom{m}{l}_q q^l \mathcal{E}_l.q = (q \mathcal{E}_q + 1)^m. \quad (1.6) \]

Combining (1.6) and (1.3), the Carlitz’s \(q\)-Euler numbers \(\mathcal{E}_{m,q}\) satisfy as follows:

\[ q(q\mathcal{E}_q + 1)^m + \mathcal{E}_{m,q} = \begin{cases} [2]_q, & \text{if } m = 0, \\ 0, & \text{if } m > 0, \end{cases} \]

with the usual convention about replacing \(\mathcal{E}_q^m\) by \(\mathcal{E}_{m,q}\) (see [1, 3–5, 8]).

Then, the modified \(q\)-Euler numbers \(E_{m,q}\) are defined by Kim et al. (see [8, 12, 23]) as follows:

\[ \int_{\mathbb{Z}_p} [x]^m_q d\mu_{-1}(x) = E_{m,q}. \]

From (1.5), we have

\[ \int_{\mathbb{Z}_p} [x+1]^m_q d\mu_{-1}(x) = \sum_{l=0}^{m} \binom{m}{l}_q q^l \int_{\mathbb{Z}_p} [x]^l_q d\mu_{-1}(x) = \sum_{l=0}^{m} \binom{m}{l}_q q^l E_{l,q} = (qE_q + 1)^m. \quad (1.7) \]
Combining (1.7) and (1.4), the modified q-Euler numbers $E_{m,q}$ satisfy the followings:

$$(qE_q + 1)^m + E_{m,q} = \begin{cases} 2, & \text{if } m = 0, \\ 0, & \text{if } m > 0. \end{cases} \quad (1.8)$$

It is well-known that the Euler numbers are defined by the generating function

$$\frac{2}{e^t + 1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad (1.9)$$

with the usual convention about replacing $E^n$ by $E_n$. From (1.9), we have

$$2 = e^{Et}(e^t + 1) = e^{(E+1)t} + e^{Et} = \sum_{n=0}^{\infty} ((E + 1)^n + E_n) \frac{t^n}{n!}.$$

Thus, we have

$$(E + 1)^n + E_n = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases} \quad (1.10)$$

We note that $\lim_{q \to 1} E_{n,q} = E_n$ and that if $q$ approaches to 1, then the equation (1.8) is equal to the equation (1.10).

The purpose of this paper is to define the modified higher-order degenerate q-Euler polynomials which are defined from fermionic $p$-adic integral on $Z_p$, and to give some explicit identities for those polynomials. Furthermore, we demonstrate the shapes of the modified higher-order q-Euler polynomials $E_{n,q}(x)$ and the modified higher-order degenerate q-Euler polynomials $E_{n,\lambda, q}(x)$ (see Figure 1) and investigated the zeroes of $E_{n,q}(x)$ and $E_{n,\lambda, q}(x)$ by using a computer.

### 2. The modified higher-order degenerate q-Euler polynomials

Let $r \in \mathbb{N}$ and $\lambda, t \in \mathbb{C}$ be such that $|\lambda t| < p^{-\frac{1}{r+1}}$. We note that if we take $f(x) = e^{xt}$, then, by (1.4), we have

$$\int_{Z_p} e^{xt}d\mu_{-1}(x) = \frac{2}{e^t + 1}, \quad (2.1)$$

By (2.1), we have

$$\int_{Z_p} \cdots \int_{Z_p} e^{(x_1 + x_2 + \cdots + x_r)t}d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \int_{Z_p} e^{x_1t}d\mu_{-1}(x_1) \cdots \int_{Z_p} e^{x_r t}d\mu_{-1}(x_r)$$

$$= \left(\frac{2}{e^t + 1}\right)^r = \sum_{n=0}^{\infty} E_n^{(r)} \frac{t^n}{n!}, \quad (2.2)$$

where $E_n^{(r)}$ are called the higher-order Euler numbers (see [15, 19, 26]). We also note that

$$\int_{Z_p} \cdots \int_{Z_p} e^{(x_1 + x_2 + \cdots + x_r)t}d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)$$

$$= \sum_{n=0}^{\infty} \int_{Z_p} \cdots \int_{Z_p} (x_1 + \cdots + x_r)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \frac{t^n}{n!}. \quad (2.3)$$

From (2.2) and (2.3), we obtain the following theorem.
\textbf{Theorem 2.1.} Let \( n \in \mathbb{N} \cup \{0\} \). Then we have

\[
E_{n}^{(r)} = \int_{Z_{p}} \cdots \int_{Z_{p}} (x_{1} + \cdots + x_{r})^{n} d\mu_{-1} \cdots d\mu_{-1}(x_{r}).
\]

In [11], the modified higher-order q-Euler numbers are defined by Kim to be

\[
E_{n,q}^{(r)} = \int_{Z_{p}} \cdots \int_{Z_{p}} [x_{1} + x_{2} + \cdots + x_{r}]^{n} d\mu_{-1} \cdots d\mu_{-1}(x_{r}).
\]

The next diagram illustrates the variations of several types of degenerate q-Euler polynomials and numbers. Those polynomials in the first row and the third row of the diagram are introduced by Carliz et al. [1, 3–5, 8] and Kim et al. [12, 18, 20], respectively. A research of these has yielded fruitful results in number theory and combinatorics (see [6, 7, 21, 24]). The motivation of this paper is to investigate some explicit identities for those polynomials in the second row of the diagram.

\[
\begin{align*}
\int_{Z_{p}} (1 + \lambda t)^{[x+y]+d} d\mu_{-1}(y) &= \sum_{n=0}^{\infty} E_{n,\lambda,q}(x)^{\frac{t_{n}}{n!}} (\text{degenerate q-Euler polynomials}) \\
\int_{Z_{p}} (1 + \lambda)^{[x+y]+d} d\mu_{-1}(y) &= \sum_{n=0}^{\infty} E_{n,\lambda,q}(x)^{\frac{t_{n}}{n!}} (\text{modified degenerate q-Euler polynomials}) \\
\int_{Z_{p}} e^{[x+y]+d} d\mu_{-1}(y) &= \sum_{n=0}^{\infty} E_{n,q}(x)^{\frac{t_{n}}{n!}} (\text{modified q-Euler polynomials}) \tag{2.4}
\end{align*}
\]

Recently, Kim defined the higher-order degenerate q-Euler polynomials given by the generating function (see [18, 20]) as follows:

\[
\int_{Z_{p}} \cdots \int_{Z_{p}} (1 + \lambda t)^{[x+y]+d} d\mu_{-1} \cdots d\mu_{-1}(x_{r}) = \sum_{n=0}^{\infty} E_{n,\lambda,q}(x) \frac{t_{n}}{n!}.
\]

Accordingly, we define the modified higher-order degenerate q-Euler polynomials given by the generating function as follows:

\[
\int_{Z_{p}} \cdots \int_{Z_{p}} (1 + \lambda t)^{[x+y]+d} d\mu_{-1} \cdots d\mu_{-1}(x_{r}) = \sum_{n=0}^{\infty} E_{n,\lambda,q}(x) \frac{t_{n}}{n!} \tag{2.4}
\]

Note that \( \lim_{\lambda \to 0} E_{n,\lambda,q}(x) = E_{n,q}(x) \), where \( E_{n,q}(x) \) are the higher-order q-Euler polynomials.
We observe that
\[
(1 + \lambda t)^\frac{1}{n} \left[ x_1 + \cdots + x_r + x \right]_q = \sum_{n=0}^{\infty} \left( \frac{1}{n} [x_1 + \cdots + x_r + x]_q \right) \lambda^n t^n n!
\]
\[
= \sum_{n=0}^{\infty} \left( \frac{1}{n} [x_1 + \cdots + x_r + x]_q \right) \frac{\lambda^n t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left( \frac{1}{n} [x_1 + \cdots + x_r + x]_q \right) \left( \frac{1}{n} [x_1 + \cdots + x_r + x]_q - 1 \right) \frac{\lambda^n t^n}{n!}
\]
\[
\cdots \left( \frac{1}{n} [x_1 + \cdots + x_r + x]_q - n + 1 \right) \frac{\lambda^n t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left( [x_1 + x_2 + \cdots + x_r + x]_q \right) \left( [x_1 + x_2 + \cdots + x_r + x]_q - \lambda \right) \frac{\lambda^n t^n}{n!}
\]
\[
\cdots \left( [x_1 + x_2 + \cdots + x_r + x]_q - (n-1)\lambda \right) \frac{\lambda^n t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left( [x_1 + x_2 + \cdots + x_r + x]_q \right) \frac{\lambda^n t^n}{n!},
\]
where \([x]_q = [x]_q [x]_q - \lambda ([x]_q - 2\lambda) \cdots ([x]_q - (n-1)\lambda]). By (2.4), we have
\[
\int_{Z_p} \cdots \int_{Z_p} (1 + \lambda t)^\frac{1}{n} [x_1 + \cdots + x_r + x]_q d\mu_1 \cdots d\mu_r
\]
\[
= \sum_{n=0}^{\infty} \int_{Z_p} \cdots \int_{Z_p} \left( [x_1 + x_2 + \cdots + x_r + x]_q \right)_{n,\lambda} d\mu_1 \cdots d\mu_r \frac{t^n}{n!}.
\]
Using (2.4) and (2.6), we obtain the following Witt’s formula.

**Theorem 2.2** (Witt’s formula). For \( n \in \mathbb{N} \cup \{0\} \), we have
\[
\int_{Z_p} \cdots \int_{Z_p} \left( [x_1 + x_2 + \cdots + x_r + x]_q \right)_{n,\lambda} d\mu_1 \cdots d\mu_r = E^{(r)}_{n,\lambda,q}(x).
\]

We observe that
\[
\left( [x_1 + x_2 + \cdots + x_r + x]_q \right)_{n,\lambda} = \sum_{l=0}^{n} S_1(n,1) \lambda^{n-l} [x_1 + x_2 + \cdots + x_r + x]_q^l,
\]
where \( S_1(n,1) \) is the Stirling numbers of the first kind. By (2.7) and (2.8), we have
\[
E^{(r)}_{n,\lambda,q}(x) = \sum_{l=0}^{n} S_1(n,1) \lambda^{n-l} E_{n,\lambda,q}(x).
\]

Thus, we obtain the following theorem.
Theorem 2.3. For $n \in \mathbb{N} \cup \{0\}$, we have

$$E_{n,\lambda,q}^{(r)}(x) = \sum_{l=0}^{n} S_{1}(n, l)\lambda^{n-l}E_{l,q}^{(r)}(x).$$

Remark that $\lim_{\lambda \to 0} E_{n,\lambda,q}^{(r)}(x) = E_{n,q}^{(r)}(x)$ are the modified higher-order q-Euler polynomials and that $\lim_{q \to 1} E_{n,q}^{(r)}(x) = E_{n}^{(r)}(x)$ are the higher-order Euler polynomials. We note that

$$E_{n,q} = \int_{Z_p} [x]_{q}^{n} d\mu_{-1}(x)$$

$$= \frac{1}{(1-q)^{n}} \sum_{l=0}^{n} \binom{n}{l} (-1)^{l} \int_{Z_p} q^{lx} d\mu_{-1}(x)$$

$$= \frac{1}{(1-q)^{n}} \sum_{l=0}^{n} \binom{n}{l} (-1)^{l} \lim_{N \to \infty} \sum_{x=0}^{p^{N-1}} (-1)^{x} q^{lx}$$

$$= \frac{1}{(1-q)^{n}} \sum_{l=0}^{n} \binom{n}{l} (-1)^{l} \lim_{N \to \infty} \frac{1 + q^{lp^{N}}}{1 + q^{l}}$$

$$= \frac{2}{(1-q)^{n}} \sum_{l=0}^{n} \binom{n}{l} (-1)^{l} \sum_{m=0}^{\infty} (-1)^{m} q^{ml}$$

$$= 2 \sum_{m=0}^{\infty} (-1)^{m} \binom{[m]}{q}^{n}.$$

Summarizing this, we have the following equation.

Theorem 2.4. For $n \in \mathbb{N} \cup \{0\}$, we have

$$E_{n,q} = \int_{Z_p} [x]_{q}^{n} d\mu_{-1}(x) = 2 \sum_{m=0}^{\infty} (-1)^{m} [m]_{q}^{n}.$$

For $r \in \mathbb{N}$, we derive

$$E_{n,q}^{(r)}(x) = \int_{Z_p} \cdots \int_{Z_p} [x_1 + x_2 + \cdots + x_r + x]_{q}^{n} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)$$

$$= \left( \frac{1}{1-q} \right)^{n} \int_{Z_p} \cdots \int_{Z_p} (1 - q^{x_1 + \cdots + x_r + x})^{n} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)$$

$$= \left( \frac{1}{1-q} \right)^{n} \sum_{l=0}^{n} \binom{n}{l} (-1)^{l} \int_{Z_p} \cdots \int_{Z_p} q^{x_1 + \cdots + x_r + x} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)$$

$$= \frac{1}{(1-q)^{n}} \sum_{l=0}^{n} \binom{n}{l} (-1)^{l} \left( \lim_{N \to \infty} \sum_{x_1, \ldots, x_r=0}^{p^{N-1}} (-1)^{x_1 + \cdots + x_r} q^{lx_1 + \cdots + lx_r} \right) q^{lx} \quad (2.9)
expressions

For Theorem 2.6.

\[ \sigma \text{ generating function of the higher-order q}\]

\[ q^3. \text{ The modified higher-order degenerate} \]


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By (2.9), we obtain the following theorem.

**Theorem 2.5.** For \( n \in \mathbb{N} \cup \{0\} \), we have

\[ E_{n,q}^{(r)}(x) = 2^r \sum_{m_1, \ldots, m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r}[m_1 + \cdots + m_r + x]_q^n, \quad \text{(see [8, 12, 23]).} \]

**Theorem 2.6.** For \( w_1, w_2, \ldots, w_n \in \mathbb{N} \) with \( w_i \equiv 1 \pmod{2} \), \( (i = 1, 2, \ldots, n) \), and \( m \geq 0 \), the following expressions

\[ \sum_{p=0}^{m} \sum_{i=1}^{\infty} \binom{p}{i} \lambda^{m-p} S_1(m, p) \left( \frac{[w_{\sigma(i)}]_q}{[\prod_{j=1}^{n-1} w_{\sigma(j)}]_q} \right)^{p-i} \]

\[ \times E_{\lambda q^{w_{\sigma(1)}}w_{\sigma(2)}}^{w_{\sigma(n-1)}}(w_{\sigma(n)}x) \binom{p}{n} \lambda^{w_{\sigma(n)}}(w_{\sigma(1)}, \ldots, w_{\sigma(n-1)}|i + 1) \]

are the same for any permutation \( \sigma \) in the symmetry group of degree \( n \).

3. The modified higher-order degenerate q-Euler polynomials and the higher-order q-zeta functions

In [11, 12], Kim introduced the generating function of the higher-order q-Euler polynomials. From the generating function of the higher-order q-Euler polynomials, we have

\[ F_{q}^{(r)}(x, t) = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{n!} \]

\[ = 2^r \sum_{m_1, \ldots, m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \sum_{n=0}^{\infty} [m_1 + \cdots + m_r + x]_q^n \frac{t^n}{n!}, \quad (3.1) \]

From (3.1), Kim [11] defined the higher-order q-zeta functions as follows:

\[ \zeta_{E,q}^{(r)}(s, x) = \frac{1}{\Gamma(s)} \int_0^{\infty} F_{q}^{(r)}(x, -t)t^{s-1}dt, \quad (3.2) \]

where \( \Gamma(s) = \int_0^{\infty} y^{s-1}e^{-y}dy \). By (3.1) and (3.2) we derive

\[ \zeta_{E,q}^{(r)}(s, x) = \frac{1}{\Gamma(s)} 2^r \sum_{m_1, \ldots, m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \int_0^{\infty} e^{-[m_1+\cdots+m_r+x]_q t} t^{s-1}dt \]

\[ = \frac{2^r}{\Gamma(s)} \sum_{m_1, \ldots, m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \frac{1}{[m_1 + \cdots + m_r + x]_q^s} \int_0^{\infty} y^{s-1}e^{-y}dy \]

\[ = 2^r \sum_{m_1, \ldots, m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \frac{1}{[m_1 + \cdots + m_r + x]_q^s}. \]

(3.3)
By (3.3), we obtain the following theorem.

**Theorem 3.1.** For \( r \in \mathbb{N}, s \in \mathbb{C} \) with \( \text{Re}(s) > 0 \), we have

\[
\zeta_{E,q}^{(r)}(s,x) = 2^r \sum_{m_1, \ldots, m_r = 0}^{\infty} \frac{(-1)^{m_1 + \cdots + m_r}}{[m_1 + \cdots + m_r + x]^s_q}, \quad \text{(see [11, 12]).}
\]

For \( s, x \in \mathbb{C} \) with \( \text{Re}(x) > 0 \), \( a_1, \ldots, a_r \in \mathbb{C} \), the Barnes-type multiple \( q \)-zeta functions are defined by Kim [12] as follows:

\[
\zeta_{E,q}^{(r)}(s,x|w_1, \cdots, w_r; a_1, \cdots, a_r) = 2^r \sum_{m_1, \cdots, m_r = 0}^{\infty} \frac{(-1)^{m_1 + \cdots + m_r} q^{m_1 a_1 + \cdots + m_r a_r}}{[x + w_1 m_1 + \cdots + w_r m_r]^s_q},
\]

where the parameters \( w_1, \ldots, w_r \) are positive. Note that \( \zeta_{E,q}^{(r)}(s,x|1, \cdots, 1,0, \cdots, 0) = \zeta_{E,q}^{(r)}(s,x) \).

By (3.1), we have

\[
\zeta_{E,q}^{(r)}(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{r^{(r)}(x,t) t^{s-1} dt}{\Gamma(r)} = \frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} E_{m,q}^{(r)}(x) \frac{(-1)^m}{m!} \int_0^\infty t^{s-1+m} dt.
\]  \( (3.4) \)

Let \( s = -n \) (\( n \in \mathbb{N} \)). Then, by (3.4), we have

\[
\zeta_{E,q}^{(r)}(-n,t) = \lim_{s \to -n} \frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} E_{m,q}^{(r)}(x) \frac{(-1)^m}{m!} \int_0^\infty t^{-n-1+m} dt
\]

\[
= \left( \lim_{s \to -n} \frac{1}{\Gamma(s)} \right) \left( E_{n,q}^{(r)}(x) \frac{(-1)^n}{n!} \right) 2\pi i
\]

\[
= \frac{n!}{2\pi i} (-1)^n E_{n,q}^{(r)}(x) \frac{(-1)^n}{n!} 2\pi i = E_{n,q}^{(r)}(x).
\]  \( (3.5) \)

where

\[
\Gamma(-n) = \int_0^\infty e^{-t} t^{-n-1} dt = \lim_{t \to 0} 2\pi i \frac{1}{n!} \left( \frac{d}{dt} \right)^n \left( t^{n+1} e^{-t} t^{-n-1} \right) = 2\pi i \frac{1}{n!} (-1)^n \lim_{t \to 0} e^{-t} = 2\pi i \frac{1}{n!} (-1)^n.
\]

By (3.5), we obtain the following theorem.

**Theorem 3.2.** For \( n \in \mathbb{N} \), we have

\[
\zeta_{E,q}^{(r)}(-n,x) = E_{n,q}^{(r)}(x), \quad \text{(see [12]).}
\]

From Theorem 2.3 and Theorem 2.5, and (2.8), we have

\[
E_{n,\lambda,q}^{(r)}(x) = \sum_{l=0}^{n} S_1(n,1) \lambda^{n-l} E_{l,q}^{(r)}(x)
\]

\[
= 2^r \sum_{l=0}^{n} \sum_{m_1, \cdots, m_r = 0}^{\infty} (-1)^{m_1 + \cdots + m_r} [m_1 + \cdots + m_r + x]_q S_1(n,1) \lambda^{n-l}
\]

\[
= 2^r \sum_{m_1, \cdots, m_r = 0}^{\infty} (-1)^{m_1 + \cdots + m_r} \sum_{l=0}^{n} [m_1 + \cdots + m_r + x]_q S_1(n,1) \lambda^{n-l}
\]

\[
= 2^r \sum_{m_1, \cdots, m_r = 0}^{\infty} (-1)^{m_1 + \cdots + m_r} \left( [m_1 + \cdots + m_r + x] \right)_{n,\lambda}.
\]  \( (3.6) \)

By (3.6), we obtain the following theorem.
By (3.8), we obtain the following theorem.

**Theorem 3.5.** For \( n \in \mathbb{N} \cup \{0\} \), we have

\[
E_{n,\lambda,q}^{(r)}(x) = 2^r \sum_{m_1, \ldots, m_r = 0}^{\infty} (-1)^{m_1 + \cdots + m_r} \left( [m_1 + \cdots + m_r + x]_q \right) n, \lambda. \tag{3.7}
\]

Applying (3.7) and using (2.5), we have

\[
\sum_{n=0}^{\infty} E_{n,\lambda,q}^{(r)}(x) \frac{t^n}{n!} = 2^r \sum_{m_1, \ldots, m_r = 0}^{\infty} (-1)^{m_1 + \cdots + m_r} \sum_{n=0}^{\infty} \left( [m_1 + \cdots + m_r + x]_q \right) n, \lambda \frac{t^n}{n!} \tag{3.8}
\]

By (3.8), we obtain the following theorem.

**Theorem 3.4.** For \( r \in \mathbb{N} \), we have

\[
\sum_{n=0}^{\infty} E_{n,\lambda,q}^{(r)}(x) \frac{t^n}{n!} = 2^r \sum_{m_1, \ldots, m_r = 0}^{\infty} (-1)^{m_1 + \cdots + m_r} (1 + \lambda t)^{[m_1 + \cdots + m_r + x]_q} \lambda t^n. \tag{3.9}
\]

Replacing \( r \) by \( \frac{1}{\lambda} (e^{\lambda t} - 1) \) in (3.9), and by using (3.1), we have

\[
\sum_{m=0}^{\infty} E_{m,\lambda,q}^{(r)}(x) \frac{t^n}{n!} = 2^r \sum_{m_1, \ldots, m_r = 0}^{\infty} (-1)^{m_1 + \cdots + m_r} e^{[m_1 + \cdots + m_r + x]_q} = \sum_{n=0}^{\infty} E_{n,\lambda,q}^{(r)}(x) \frac{t^n}{n!},
\]

and

\[
\sum_{m=0}^{\infty} E_{m,\lambda,q}^{(r)}(x) \frac{t^n}{n!} = \sum_{m=0}^{\infty} E_{m,\lambda,q}^{(r)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} S_2(n, m) \lambda^n \frac{t^n}{n!} \tag{3.10}
\]

By (3.10), we obtain the following theorem.

**Theorem 3.5.** For \( n \in \mathbb{N} \cup \{0\} \), we have

\[
E_{n,\lambda,q}^{(r)}(x) = \sum_{m=0}^{n} \lambda^{n-m} E_{m,\lambda,q}^{(r)}(x) S_2(n, m). \tag{3.11}
\]

By replacing \( r \) by \( \frac{1}{\lambda} \log(1 + \lambda t) \) by in (3.9), and using (3.8), we have

\[
\sum_{m=0}^{\infty} E_{m,\lambda,q}^{(r)}(x) \frac{t^n}{n!} = 2^r \sum_{m_1, \ldots, m_r = 0}^{\infty} (-1)^{m_1 + \cdots + m_r} e^{[m_1 + \cdots + m_r + x]_q} \frac{1}{\lambda} \log(1 + \lambda t) \tag{3.11}
\]

By replacing \( m_1, \ldots, m_r \) by \( \frac{1}{\lambda} r \) in (3.11), and using (3.8), we have

\[
\sum_{n=0}^{\infty} E_{n,\lambda,q}^{(r)}(x) \frac{t^n}{n!}.
\]
and
\[
\sum_{m=0}^{\infty} E_{m,q}^{(r)}(x) \lambda^{-m} \left( \log(1 + \lambda t) \right)^m = \sum_{m=0}^{\infty} E_{m,q}^{(r)}(x) \lambda^{-m} \sum_{n=m}^{\infty} S_1(n, m) \frac{\lambda^n t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \lambda^{n-m} E_{m,q}^{(r)}(x) S_1(n, m) \right) t^n. \tag{3.12}
\]

By comparing the coefficients of (3.11) and (3.12), we obtain the following theorem.

**Theorem 3.6.** For \( n \in \mathbb{N} \cup \{0\} \), we have
\[
E_{n,\lambda,q}^{(r)}(x) = \sum_{m=0}^{n} \lambda^{n-m} E_{m,q}^{(r)}(x) S_1(n, m).
\]

### 4. Zeroes of the modified higher-order q-Euler polynomials and the modified higher-order degenerate q-Euler polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting patterns of the zeroes of the modified higher-order q-Euler polynomials \( E_{n,q}^{(r)}(x) \) and the modified higher-order degenerate q-Euler polynomials \( E_{n,\lambda,q}^{(r)}(x) \). We display the shapes of the modified higher-order q-Euler polynomials \( E_{n,q}^{(r)}(x) \) and the modified higher-order degenerate q-Euler polynomials \( E_{n,\lambda,q}^{(r)}(x) \). Next we investigate the zeroes of the modified higher-order q-Euler polynomials \( E_{n,q}^{(r)}(x) \) and the modified higher-order degenerate q-Euler polynomials \( E_{n,\lambda,q}^{(r)}(x) \). Let \( q \in \mathbb{C}, |q| < 1 \). For \( n = 1, \ldots, 10 \), we can draw a plot of the modified higher-order q-Euler polynomials \( E_{n,q}^{(r)}(x) \) and the modified higher-order degenerate q-Euler polynomials \( E_{n,\lambda,q}^{(r)}(x) \), respectively. This shows the ten plots combined into one. We display the shape of \( E_{n,\lambda,q}^{(r)}(x) \) and \( E_{n,q}^{(r)}(x) \), \(-5 \leq x \leq 5\) (Figure 1).

![Figure 1: Curve of the \( E_{n,\lambda,q}^{(r)}(x) \) and \( E_{n,q}^{(r)}(x) \).](image)

In Figure 1 (left), we choose \( r = 5, \lambda = 1/2 \) and \( q = 1/2 \). In Figure 1 (middle), we choose \( r = 5, \lambda = 1/10000 \) and \( q = 1/2 \). In Figure 1 (right), we choose \( r = 5 \) and \( q = 1/2 \). It is obvious that, by letting \( \lambda \) tend to 1 from the curve of \( E_{n,\lambda,q}^{(r)}(x) \) of left side, we lead to the curve of the \( E_{n,q}^{(r)}(x) \). By using computer, the modified higher-order q-Euler numbers \( E_{n,q}^{(r)} \) and the modified higher-order degenerate q-Euler numbers \( E_{n,\lambda,q}^{(r)} \) are listed in Table 1.

We investigate the beautiful zeroes of the modified higher-order q-Euler polynomials \( E_{n,q}^{(r)}(x) \) and the modified higher-order degenerate q-Euler polynomials \( E_{n,\lambda,q}^{(r)}(x) \) by using a computer. We plot the zeroes...
of the modified higher-order $q$-Euler polynomials $E_{n,q}^{(r)}(x)$ and the modified higher-order degenerate $q$-Euler polynomials $E_{n,\lambda,q}^{(r)}(x)$ for $n = 50$, $q = 1/2$ and $x \in \mathbb{C}$ (Figure 2).

Table 1: The first few $E_{n,q}^{(r)}$ and $E_{n,\lambda,q}^{(r)}$.

<table>
<thead>
<tr>
<th>degree $n$</th>
<th>$E_{n,q}^{(r)}$ $q = 1/2, r = 5$</th>
<th>$E_{n,\lambda,q}^{(r)}$ $q = 1/2, r = 5, \lambda = 1/10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$-1562/243$</td>
<td>$-1562/243$</td>
</tr>
<tr>
<td>2</td>
<td>$9287996/759375$</td>
<td>$10264246/759375$</td>
</tr>
<tr>
<td>3</td>
<td>$3037448168/184528125$</td>
<td>$4674974089/922640625$</td>
</tr>
<tr>
<td>4</td>
<td>$-1425517528162096/262003549978125$</td>
<td>$-240516181113919276/6550088749453125$</td>
</tr>
</tbody>
</table>

Figure 2: Zeroes of $E_{n,\lambda,q}^{(r)}(x)$ and $E_{n,q}^{(r)}(x)$.

In Figure 2 (top-left), we choose $n = 50$, $q = 1/2$, and $\lambda = 1/100$. In Figure 2 (top-right), we choose $n = 50$, $q = 1/2$ and $\lambda = 1/1000$. In Figure 2 (bottom-left), we choose $n = 50$, $q = 1/2$ and $\lambda = 1/10000$. In Figure 2 (bottom-right), we choose $n = 50$, $q = 1/2$ and $\lambda \to 0$.

Stacks of zeroes of the modified higher-order $q$-Euler polynomials $E_{n,q}^{(r)}(x)$ and the modified higher-order degenerate $q$-Euler polynomials $E_{n,\lambda,q}^{(r)}(x)$ for $1 \leq n \leq 40$ from a 3-D structure are presented in Figure 3.
In Figure 3 (left), we choose $1 \leq n \leq 40, q = 1/2$ and $\lambda = 1/10$. In Figure 3 (right), we choose $1 \leq n \leq 40, q = 1/2$, and $\lambda \to 0$.

It was known that $E_n^{(r)}(x), x \in \mathbb{C}$, has $\text{Im}(x) = 0$ reflection symmetry analytic complex functions, (see [12]). However, we observe that $E_n^{(r)}(x), x \in \mathbb{C}$, has not $\text{Im}(x) = 0$ reflection symmetry analytic complex functions (Figures 2 and 3).

Our numerical results for approximate solutions of real zeroes of the modified higher-order $q$-Euler polynomials $E_n^{(r)}(x)$ and the modified higher-order degenerate $q$-Euler polynomials $E_n^{(r)}_{n,\lambda}(x)$ are displayed in Tables 2, 3, and 4. We observe a remarkably regular structure of the complex roots of the modified higher-order $q$-Euler polynomials $E_n^{(r)}_{n,\lambda}(x)$ and the modified higher-order degenerate $q$-Euler polynomials $E_n^{(r)}_{n,\lambda}(x)$ are displayed in Table 2. We hope to verify a remarkably regular structure of the complex roots of the modified higher-order $q$-Euler polynomials $E_n^{(r)}_{n,\lambda}(x)$ and the modified higher-order degenerate $q$-Euler polynomials $E_n^{(r)}_{n,\lambda}(x)$ (Table 2).

![Image of stacks of zeroes of $E_n^{(r)}_{n,\lambda}(x)$ and $E_n^{(r)}_{n,q}(x)$ for $1 \leq n \leq 40$.]

**Table 2: Numbers of real and complex zeroes of $E_n^{(r)}_{n,\lambda}(x)$ and $E_n^{(r)}_{n,q}(x)$.**

<table>
<thead>
<tr>
<th>degree $n$</th>
<th>$E_{n,1/10,1/2}^{(5)}(x)$</th>
<th>$E_{n,1/2}^{(5)}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>real zeroes</td>
<td>complex zeroes</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>1</td>
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<tr>
<td>5</td>
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</tr>
<tr>
<td>6</td>
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<td>4</td>
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<td>9</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

Plot of real zeroes of $E_n^{(r)}_{n,\lambda}(x)$ and $E_n^{(r)}_{n,q}(x)$ for $1 \leq n \leq 40$ structure are presented in Figure 4.

In Figure 4 (left), we choose $r = 5, \lambda = 1/10$ and $q = 1/2$. In Figure 4 (middle), we choose $r = 5, \lambda = 1/1000$ and $q = 1/2$. In Figure 4 (right), we choose $r = 5$ and $q = 1/2$. It is obvious that, by letting $\lambda$ tend to 1 from the real zeroes of $E_n^{(r)}_{n,\lambda}(x)$ of left side, we lead to the real zeroes of the $E_n^{(r)}_{n,q}(x)$. 


Next, we calculated an approximate solution satisfying $E^{(r)}_{n,\lambda,q}(x) = 0$, $E^{(r)}_{n,q}(x) = 0$, and $x \in \mathbb{R}$. The results are given in Tables 3 and 4.

Table 3: Approximate solutions of $E^{(5)}_{n,\lambda,q}(x) = 0$, $q = 1/2$, $\lambda = 1/10$, $x \in \mathbb{R}$.

<table>
<thead>
<tr>
<th>degree $n$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.07519</td>
</tr>
<tr>
<td>2</td>
<td>0.674416, 2.86795</td>
</tr>
<tr>
<td>3</td>
<td>-0.0853565, 1.46616, 3.59324</td>
</tr>
<tr>
<td>4</td>
<td>-0.507236, 0.0495496, 1.56211, 3.60538</td>
</tr>
<tr>
<td>5</td>
<td>0.064954, 1.2468, 2.5000, 3.7532, 4.9350</td>
</tr>
<tr>
<td>6</td>
<td>0.642491, 2.23658, 4.2821</td>
</tr>
<tr>
<td>7</td>
<td>-0.00924544, 1.3286, 3.12019, 8.46041</td>
</tr>
<tr>
<td>8</td>
<td>-0.422343, 0.65015, 2.17641, 4.17786</td>
</tr>
</tbody>
</table>

Table 4: Approximate solutions of $E^{(5)}_{n,\lambda,q}(x) = 0$, $q = 1/2$, $x \in \mathbb{R}$.

<table>
<thead>
<tr>
<th>degree $n$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.07519</td>
</tr>
<tr>
<td>2</td>
<td>0.601538, 2.78882</td>
</tr>
<tr>
<td>3</td>
<td>-0.29846, 1.19492, 3.25392</td>
</tr>
<tr>
<td>4</td>
<td>-0.707974, 0.0540258, 1.61452, 3.60212</td>
</tr>
<tr>
<td>5</td>
<td>0.380137, 1.93959, 3.88122</td>
</tr>
<tr>
<td>6</td>
<td>-0.577628, 0.651902, 2.20503, 4.1144</td>
</tr>
<tr>
<td>7</td>
<td>-0.820786, -0.396232, 0.88272, 2.42936, 4.31478</td>
</tr>
<tr>
<td>8</td>
<td>-0.201325, 1.08272, 2.6236, 4.49051</td>
</tr>
</tbody>
</table>

Finally, we shall consider the more general problems. How many zeroes does $E^{(r)}_{n,q}(x)$ have? Prove or disprove: $E^{(r)}_{n,q}(x) = 0$ has $n$ distinct solutions. Find the numbers of complex zeroes $C_{E^{(r)}_{n,q}(x)}$ of $E^{(r)}_{n,q}(x)$, $\text{Im}(x) \neq 0$. Since $n$ is the degree of the polynomial $E^{(r)}_{n,q}(x)$, the number of real zeroes $R_{E^{(r)}_{n,q}(x)}$ lying on the real plane $\text{Im}(x) = 0$ is then $R_{E^{(r)}_{n,q}(x)} = n - C_{E^{(r)}_{n,q}(x)}$, where $C_{E^{(r)}_{n,q}(x)}$ denotes complex zeroes. See Table 2 for tabulated values of $R_{E^{(r)}_{n,q}(x)}$ and $C_{E^{(r)}_{n,q}(x)}$. 
5. Conclusions

Kim et al., [17–20] studied some identities of symmetry on the higher-order degenerate q-Euler polynomials. The motivation of this paper is to investigate some explicit identities for the modified higher-order degenerate q-Euler polynomials in the second row of the diagram at page 4. So we defined the modified higher degenerate q-Euler polynomials in the equation (2.4) and obtained the formulas (see Theorems 2.2-2.5). We also obtained the explicit identities related with the modified higher-order degenerate q-Euler polynomials and the higher-order q-zeta functions (see Theorems 3.1-3.6).

Finally, we demonstrated the comparing three facts between modified higher-order q-Euler polynomials $E_{n,q}^{(r)}(x)$ and modified higher-order degenerate q-Euler polynomials $E_{n,\lambda,q}^{(r)}(x)$ as follows:

1. We displayed the shape of $E_{n,q}^{(r)}(x)$ and $E_{n,\lambda,q}^{(r)}(x)$ (see Figure 1) and investigated the zeroes of $E_{n,q}^{(r)}(x)$ and $E_{n,\lambda,q}^{(r)}(x)$ by using a computer (see Figure 2 and Table 1).

2. We presented stacks of zeroes of $E_{n,q}^{(r)}(x)$ and $E_{n,\lambda,q}^{(r)}(x)$ for $1 \leq n \leq 40$ from a 3-D structure (see Figure 3) and verified a regular structure of the complex roots of $E_{n,q}^{(r)}(x)$ and $E_{n,\lambda,q}^{(r)}(x)$ (see Figure 4 and Table 2).

3. We calculated an approximate solution satisfying $E_{n,q}^{(r)}(x) = 0$, $E_{n,\lambda,q}^{(r)}(x) = 0$, and $x \in \mathbb{R}$ (see Tables 3-4).

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References