Optimal tracking performance of discrete-time systems with quantization

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Communicated by X. Liu

Abstract

This paper studies optimal tracking performance issues for linear time invariant system with two-channel constraints. The specific problem under consideration is quantization for up-link and down-link communication channel which satisfies some constraints. Logarithmic quantization law is employed in the quantizers. The tracking performance is defined in a square sense, and the reference signal under consideration is a step signal. The system’s reference signal is considered as a step signal. The tracking performance is measured by the minimum mean square error between the reference input and the system’s output. By using dynamic programming approach, discrete-time algebraic Riccati equation (ARE) is obtained. The optimal tracking performance is obtained by output feedback control, in terms of the space equation of the given system and the unique solution of the discrete-time algebraic Riccati equation. And, the impact of quantizer for optimal tracking performance is analyzed. Finally, simulation example is given to illustrate the theoretical results. ©2017 All rights reserved.

Keywords: Optimal tracking performance, quantization, two-channel constraints, discrete-time systems, algebraic Riccati equation (ARE).


1. Introduction

In recent years, more and more researchers are interested in networked control systems (NCSs), for example [2, 3, 10, 15, 20, 21, 24, 25, 27], and references therein. The most problems under consideration focus on how to model the networked control system and stabilization analysis with quantization effects [9, 16, 22], time delays [9, 13, 14, 28], bandwidth constraint [5, 6, 19], and signal-to-noise ratio (SNR) constraint [4, 7, 17, 18] over the communication channels. These studies investigate mostly the problem of stability analysis and stabilization for networked control systems. However, from the angle of application, only considering the stability of the networked control system is not enough, the performance of the networked control system should also be considered. A poor performance may deteriorate stability of the system, and even make the system unstable. Therefore, the study is necessary and urgent on NCSs. At
present, many issues of optimal control performance under such network environment remain challenging for us, such as optimal attainable tracking performance of networked control systems in terms of the key factors of the communication channel.

In recent years, most works for the performance limitation of NCS are investigated communication network with channel noise, signal-to-noise ratio (SNR) constraints, packet loss, bandwidth constraint, and delay time. For instance, in [18], SNR fundamental limitations are investigated for discrete-time single-input and single-output (SISO) NCSs. A tight condition of communication SNR is obtained for the linear time invariant output feedback stabilization of a discrete-time. In [26], the optimal tracking problem is studied for SISO discrete-time systems over communication channel with network-induced delay in the feedback path. In [23], the problem of optimal tracking performance for SISO discrete-time NCSs with packet dropouts and channel noise is studied. [11] focuses on two kinds of network parameters for bandwidth and additive colored white Gaussian noise (ACGN), the optimal tracking performance are obtained for multi-input-multi-output (MIMO) NCSs. In [12], the stabilization and tracking performance issues are investigated for MIMO control system over additive white noise channels.

In this paper, optimal tracking performance issues are investigated for linear time invariant discrete-time system with quantized input and output. We are interested in the intrinsic limit on the tracking performance achievable via feedback. The tracking performance under consideration amounts to determine the minimal tracking error between the system output and the reference signals of a feedback system by output feedback stabilizing compensators. And, the tracking performance is defined in an square sense, and the reference signals under consideration are step signals. The quantization is considered in up-link and down-link communication channel. Logarithmic quantization law is employed in the quantizers. By using dynamic programming approach, discrete-time ARE is obtained. The optimal tracking performance is obtained by output feedback control, in terms of the space equation of the given system and the unique solution of the discrete-time ARE.

The notation used throughout this paper is fairly standard. For any complex number, denote its transpose by $(\cdot)^T$, and its Moore-Penrose pseudo inverse by $(\cdot)^\dagger$. Denote the expectation operator and the variance operator by $E\{\cdot\}$ and $D\{\cdot\}$, respectively.

2. Preliminaries and problem statements

We consider control over a communication link as illustrated in Fig. 1, where the sensor and plant are connected through a network, in which quantizer maybe be necessary and channel noise may also exist, as depicted in Fig. 1.

![Figure 1: Feedback control over communication channels.](image)

In Fig. 1, $G(z)$ denotes the plant that should be controlled, $Q$ denotes the quantizer, $K$ denotes the controller, and $K_0$ denotes the steady state part of the controller $K$. This steady state control signal and steady state system’s output signal are transmitted to the plant $G$ and the controller $K$ through the network with a sufficient accuracy at the initial time and are held by the storage $S_1$ and $S_2$, respectively. $u$ and $y$ denote output signal of the controller and the plant, respectively. $u_s$ and $y_s$ denote the steady
Thus, the system in tracking performance problem is quadratic form Lemma 2.1 where the magnitude $r_0$ of the reference signal is random variable with zero mean and variance $\sigma_r$. We consider a logarithmic quantization law of the quantizer $Q_1$ and $Q_2$ as [8], in which $u_t$, $y_t$ and $u_{t\text{q}}$, $y_{t\text{q}}$ are the input and output of quantizers $Q_1$ and $Q_2$, respectively, and have

$$
\begin{align*}
\{ u_{t\text{q}}(k) &= Q_1(u_t(k)) = u_t(k) + u_1(k)w_1(k), \\
y_{t\text{q}}(k) &= Q_2(y_t(k)) = y_t(k) + y_1(k)w_2(k),
\end{align*}
$$

where $w_1(k)$ and $w_2(k)$ are quantization error, and it holds that $\delta_1 = (1 - \rho_1)/(1 + \rho_1)$ and $\delta_2 = (1 - \rho_2)/(1 + \rho_2)$ where $\rho_i, i = 1, 2, (0 < \rho_1, \rho_2 < 1)$ are the quantization density. We assume that the quantization errors processes $w_1(k_1)$ and $w_1(k_2)$ are uncorrelated for any $k_1 \neq k_2$. The quantization errors processes $w_2$ have the same assumptions, and $w_1(k_1)$ and $w_2(k_1)$ are also uncorrelated. Furthermore, for any $k_1$ and $k_2$, it holds that

$$
E(w_1(k_1)w_1(k_2)) = \begin{cases} 
\sigma_1^2, & k_1 = k_2, \\
0, & k_1 \neq k_2,
\end{cases}
$$

$$
E(w_2(k_1)w_2(k_2)) = \begin{cases} 
\sigma_2^2, & k_1 = k_2, \\
0, & k_1 \neq k_2,
\end{cases}
$$

where $\sigma_1$ and $\sigma_2$ are the variances of $w_1$ and $w_2$, respectively. Additionally, the reference signal $r(k)$ and quantization errors $w_1(k_2)$ and $w_2(k_3)$ are uncorrelated for any $k_1$, $k_2$ and $k_3$.

**Lemma 2.1** ([1]). Let matrices $F = F^T$, $H$, and $G = G^T$ be given with appropriate sizes. Consider the following quadratic form

$$
q(x, u) = E(x^TFx + x^THu + u^THx + u^TGu),
$$

where $x$ and $u$ are random variables defined on a probability space $(\Omega, \mathcal{B}, P)$. Then the following conditions are equivalent:

(i) $G \geq 0$ and $H(I - GG^T) = 0$;

(ii) there exists a symmetric matrix $S = S^T$ such that $\inf_u q(x, u) = E(x^TSx)$ for any random variable $x$.

Assume that the plant is strictly proper and has a state space representation:

$$
\begin{align*}
x(k+1) &= Ax(k) + Bu_q(k), \\
y(k) &= Cx(k) + Du_q(k),
\end{align*}
$$

where $x(\cdot)$ is the system state with initial value $x(0)=0$, $u(\cdot)$ is the control input, $y(\cdot)$ is the system output, and assume that $(A, B)$ is stabilizable, $(A, C)$ is observable. Noting Fig. 1, we have

$$
u_q(k) = u_s(k) + u_{t\text{q}}(k) = u_s(k) + u_t(k) + u_1(k)\omega(k) = u(k) + u_t(k)\omega(k).
$$

Thus, the system in tracking performance problem is

$$
\begin{align*}
x(k+1) &= Ax(k) + Bu(k) + Bu_1\omega_1(k), \\
y(k) &= Cx(k) + Du(k) + Du_1\omega_1(k).
\end{align*}
$$

(2.1)
Denote the steady state values of the input, output, and state by $u_s$, $y_s$, and $x_s$, respectively. The transient part of the control input and the system’s state be denoted by $u_t := u - u_s$, $x_t := x - x_s$ and $y_t := y - y_s$, respectively. The random variables $\omega_1(k)$, $\omega_2(k)$ are defined on a given probability space $(\Omega, F, \mathbb{P})$. Owning to $y(k) = y_s + y_1(k)$, when the system achieves asymptotically tracking, the steady state value of the output $y_s = y(\infty)$, and the transient parts of the system’s output variable value is equal to $y_t(\infty) = 0$. Thus, the steady state value $y_s$ of the system’s output is equal to magnitude of the reference signal. The homeostatic control $u_s$ and status $x_s$ must satisfy equations

$$
\begin{align*}
    x_s &= Ax_s + Bu_s, \\
    y_s &= Cx_s + Du_s.
\end{align*}
$$

So, we have

$$
\begin{align*}
    x_s &= \frac{(I-A)^{-1}Br}{C(I-A)^{-1}B + D}, \\
    u_s &= \frac{r}{C(I-A)^{-1}B + D}.
\end{align*}
\tag{2.2}
$$

From (2.1), we can get the following equations:

$$
\begin{align*}
    x_t(k + 1) &= A_t x_t(k) + B u_t(k) + B u_t(k) \omega_1(k), \\
    y_t(k) &= C x_t(k) + D u_t(k) + D u_t(k) \omega_1(k).
\end{align*}
\tag{2.3}
$$

3. Optimal tracking performance

The performance index of the discrete-time networked control systems is defined as

$$
J = \lim_{N \to \infty} \inf_{u_0, u_1, \cdots, u_{N-1}} J(u_0, u_1, \cdots, u_{N-1}) = \inf_{u_t \in \mathcal{U}_{ad}} \mathbb{E} \sum_{k=0}^{\infty} [r - y(k)]^2. \tag{3.1}
$$

The admissible controller set $\mathcal{U}_{ad}$ is the set of all such controllers. The tracking problem under consideration is to find a controller sequence $(u_0, u_1, \cdots, u_{\infty})$ that minimizes $J$ over $\mathcal{U}_{ad}$. And the control sequences $u_0, u_1, \cdots, u_{\infty}, (u_t \in \mathbb{R}^n)$ are defined on a given probability space $(\Omega, F, \mathbb{P})$.

From the system (2.3), performance index (3.1) can be turned to

$$
J = \inf_{u_t \in \mathcal{U}_{ad}} \mathbb{E} \left\{ \sum_{k=0}^{\infty} [C x_t(k) + D u_t(k) + D u_t(k) \omega_1(k)]^2 \right\}.
$$

The optimal tracking performance is given as following theorem.

**Theorem 3.1.** For given discrete-time NCSs as depicted Fig. 1, considering the plant (2.3), optimal tracking controller can be designed as

$$
\begin{align*}
    u^* &= -\frac{1}{1 + \sigma_1^2} [B^T PB + DD^T]^{-1} [B^T PA + CD^T] C^\dagger y q \frac{1}{1 + \sigma_1^2} [B^T PB + DD^T]^{-1} [B^T PA + CD^T] C^\dagger r \\
    &\quad + \frac{C(I-A)^{-1}B}{C(I-A)^{-1}B + D}.
\end{align*}
$$

The optimal performance under uplink and downlink channels with quantization are given by

$$
J^* = \frac{B^T (I-A)^{-1} P (I-A)^{-1} B}{B^T (I-A)^{-1} C^\dagger P (I-A)^{-1} B} \frac{1}{1 + \sigma_1^2} \sigma_1^2,
$$

where, $P$ is the unique solution of discrete-time ARE

$$
P = C^T C + A^T PA - \frac{1 + \sigma_2^2}{1 + \sigma_1^2} [A^T PB + CD^T] \left( B^T PB + DD^T \right)^{-1} [B^T PA + CD^T].$$
Proof. Consider the following performance index:

\[
J_N = E \sum_{k=0}^{N} [C x_t(k) + D u_t(k) + D u_t(k) \omega_1(k)]^2,
\]

and let \( V_j = J_N - J_{j-1}, (j \in 1, \cdots, N) \). Then

\[
V_N = J_N - J_{N-1} = \text{E}[C x_t(N) + D u_t(N) + D u_t(N) \omega_1(N)]^2
\]

\[
= E\left\{ x_t^T(N) C x_t(N) + x_t(N)^T C^T D u_t(N) + u_t^T(N) D^T C x_t(N) + (1 + \sigma_t^2) u_t(N) D^T D u_t(N) \right\}.
\]

Following Lemma 2.1, we have

\[
\inf_{u_t(N) \in U_{ad}} V_N = E[x_t^T(N) P_N(N) x_t(N)],
\]

where \( P_N(N) \) is a symmetric matrix. From equations (2.1) and (3.2), we have

\[
\inf_{u_t(N) \in U_{ad}} V_N = E[x_t^T(N) P_N(N) x_t(N)]
\]

\[
= x_t^T(N) A^T P_N(N) A x_t(N) + u_t^T(N) B^T P_N(N) A x_t(N) - x_t^T(N) A^T P_N(N) B u_t(N) + x_t^T(N) A^T P_N(N) B u_t(N) - x_t^T(N) A^T P_N(N) B u_t(N)
\]

\[
\]
Therefore, when equations (3.3) and (3.4) are satisfied, we have
\[
\inf_{u_i \in U_{ad}} V_{N-1} = E[x^T(N-1)P_N(N-1)x(N-1)].
\]
We use the same treatment process, we can obtain
\[
P_N(j-1) = C^T C + A^T P_N(j) A - (1 + \sigma_2^2)(1 + \sigma_1^2) C^T K_{t} (j-1) [B^T P_N(N) B + DD^T] K_{t} (j-1) C,
\]
\[
K_{t} (j-1) = \frac{1}{1 + \sigma_1^2} [B^T P_N(j) B + DD^T]^{-1}[B^T P_N(j) A + CD^T] C^\dagger.
\]
Accordingly, we have
\[
\inf_{u_i \in U_{ad}} V_{j-1} = E[x^T(j-1)P_N(j-1)x(j-1)],
\]
where \( j = 2, \cdots, N - 1 \). It is implied that when
\[
u_{t}(j) = -K_{t}(j)y_{tq}(j),
\]
the cost function obtains the minimum. Therefore, the optimal tracking performance for system (2.3) is given by
\[
J_N = \inf_{u_i \in U_{ad}} V_{i} = E[x_{i}^T(0)P_{N}(0)x_{i}(0)],
\]
(3.5)
where \( P_N(0) > 0 \) is solution of the following discrete-time infinite ARE
\[
P_N(j-1) = C^T C + A^T P_N(j) A - \frac{(1 + \sigma_2^2)}{(1 + \sigma_1^2)} [A^T P_N(j) B + DC^T] (B^T P_N(j) B + DD^T)^{-1} [B^T P_N(j) A + CD^T].
\]
Thus, we know that the discrete-time finite ARE
\[
P_M(j-1) = C^T C + A^T P_M(j) A - \frac{(1 + \sigma_2^2)}{(1 + \sigma_1^2)} [A^T P_M(j) B + DC^T] (B^T P_M(j) B + DD^T)^{-1} [B^T P_M(j) A + CD^T],
\]
(3.6)
has a unique solution \( P_M(k) > 0, k \in \{0, 1, \cdots, M\} \). It is obvious that \( P_M(k) = P_{M-k}(0) \). If the output of system (2.1) asymptotically tracks the reference signal, the corresponding tracking performance limitation must exist, namely, \( P_M(0) \) in equation (3.6) must exist, and
\[
\lim_{M \to \infty} P_M(0) = \lim_{M \to \infty} P_{M-k}(0) = \lim_{M \to \infty} P_M(k) = P
\]
exists, and
\[
\lim_{j \to \infty} K_{t}(j-1) = \frac{1}{1 + \sigma_1^2} [B^T P_B + DD^T]^{-1}[B^T P_A + CD^T] C^\dagger \triangleq K_t^\dagger.
\]
Therefore, the discrete-time ARE can be converted to the following ARE
\[
P = C^T C + A^T P_A - \frac{(1 + \sigma_2^2)}{(1 + \sigma_1^2)} [A^T P_B + DC^T] (B^T P_B + DD^T)^{-1} [B^T P_A + CD^T].
\]
Noting equation (3.5) and the fact that \( x(0) = 0 \) and
\[
x_{t}(0) = x(0) - x_s = -x_s,
\]
therefore

\[ J^* = \lim_{N \to \infty} J^*_N = E\{x_s^T P x_s\}. \]

Noting the equation (2.2), the minimum tracking performance for system (2.3) can be given by

\[ J^* = \frac{B^T (I - A^T)^{-1} P (I - A)^{-1} B}{B^T (I - A^T)^{-1} C^T} \sigma_r^2. \]

Additionally, we have

\[ u^* = u^*_t + u_s = -\frac{1}{1 + \sigma^2_1} [B^T P B + D D^T]^{-1} [B^T P A + C D^T] C^T y_q \]

\[ + \frac{1}{1 + \sigma^2_1} [B^T P B + D D^T]^{-1} [B^T P A + C D^T] C^T r + \frac{r}{C (I - A)^{-1} B + D}. \]

Thus, the proof is completed.

\[ \Box \]

4. Conclusion

In this paper, the best attainable tracking performance of networked control systems in tracking step signal has been discussed for linear time-invariant unstable plants using output feedback control. By using dynamic programming approach, discrete-time algebraic Riccati equation (ARE) is obtained. Then the best attainable tracking performance is obtained, in terms of the space equation of given system and the unique solution of the discrete-time ARE.

Acknowledgment

This work is partially supported by the National Natural Science Foundation of China under Grant (61503133, 61374171, 51377057, 61672226), National key basic research development program (973) sub-project (61325309), the Province Natural Science Foundation of Hunan under Grant (15C0548), and the Postdoctoral Science Foundation of China under Grant 2016M592449.

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