Integral inequalities of the Hermite–Hadamard type for \((\alpha, m)\)-GA-convex functions

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Abstract

In this paper, the authors introduce a notion “\((\alpha, m)\)-GA-convex function” and establish some Hermite–Hadamard type inequalities for this kind of convex functions. ©2017 All rights reserved.

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1. Introduction

The following definitions are well-known in the literature.

**Definition 1.1.** A function \(f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}\) is said to be convex, if

\[f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),\]

holds for all \(x, y \in I\) and \(t \in [0, 1]\).

**Definition 1.2.** A function \(f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}\) is said to be GA-convex, if

\[f(x^t y^{1-t}) \leq tf(x) + (1-t)f(y),\]

holds for all \(x, y \in I\) and \(t \in [0, 1]\).

**Definition 1.3** ([11]). For \(f : [0, b] \to \mathbb{R}\), \(b > 0\), and \(m \in (0, 1]\), if

\[f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y),\]

is valid for all \(x, y \in [0, b]\) and \(t \in [0, 1]\), then we say that \(f\) is an \(m\)-convex function on \([0, b]\).
Definition 1.4 ([6]). For \( f : [0, b] \to \mathbb{R}, b > 0 \), and \((\alpha, m) \in (0, 1]^2\), if
\[
f(\lambda x + m(1 - \lambda)y) \leq \lambda^\alpha f(x) + m(1 - \lambda^\alpha)f(y),
\]
is valid for all \( x, y \in [0, b] \) and \( \lambda \in [0, 1] \), then we say that \( f(x) \) is an \((\alpha, m)\)-convex function on \([0, b] \).

Now we recall some Hermite–Hadamard type inequalities for several kinds of convex functions.

Theorem 1.5 ([3, Theorem 2.2]). Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I \) and \( a, b \in I \) with \( a < b \). If \(|f'|\) is convex on \([a, b]\), then
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)|f'(a)| + |f'(b)|}{8}.
\]

Theorem 1.6 ([7, Theorems 1 and 2]). Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be differentiable on \( I \) and \( a, b \in I \) with \( a < b \). If \(|f'|^q\) is convex on \([a, b]\) and \( q \geq 1 \), then
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{|f'(a)| + |f'(b)|}{2} \right)^{1/q},
\]
and
\[
\left| \frac{f(a + b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q}.
\]

Theorem 1.7 ([4]). Let \( f : \mathbb{R} \to [0, \infty) \to \mathbb{R} \) be \( m \)-convex and \( m \in (0, 1] \). If \( f \in L_1([a, b]) \) for \( a, b \in \mathbb{R}_0 \) and \( a < b \), then
\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \min\left\{ \frac{f(a) + mf(b/m)}{2}, \frac{f(a/m) + f(b)}{2} \right\}.
\]

Theorem 1.8 ([2, Theorem 2.2]). Let \( f : \mathbb{R} \to \mathbb{R} \) be an \( m \)-convex function with \( m \in (0, 1] \). If \( a, b \in \mathbb{R}_0 \), \( a < b \), and \( f \in L_1([a, b]) \), then
\[
f\left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) + mf(x/m) \, dx \leq \frac{m+1}{4} \left[ \frac{f(a) + f(b)}{2} + mf(a/m) + f(b/m) \right].
\]

Theorem 1.9 ([5, Theorem 3.1]). Let \( I \supseteq \mathbb{R}_0 \) be an open real interval and let \( f : I \to \mathbb{R} \) be a differentiable function on \( I \) such that \( f' \in L([a, b]) \) for \( 0 \leq a < b < \infty \). If \(|f'|^q\) is \((\alpha, m)\)-convex on \([a, b]\) for some given numbers \( m, \alpha \in (0, 1] \) and \( q \geq 1 \), then
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2} \left( \frac{1}{2} \right)^{1-1/q} \times \min\left\{ \left[ v_1|f'(a)|^q + v_2m \left| f'\left( \frac{b}{m} \right) \right|^q \right]^{1/q}, \left[ v_2m \left| f'\left( \frac{a}{m} \right) \right|^q + v_1|f'(b)|^q \right]^{1/q} \right\},
\]
where
\[v_1 = \frac{1}{(\alpha + 1)(\alpha + 2)} \left( \alpha + \frac{1}{2\alpha} \right), \quad v_2 = \frac{1}{(\alpha + 1)(\alpha + 2)} \left( \frac{\alpha^2 + \alpha + 2}{2} - \frac{1}{\alpha^2} \right).
\]

For more information and recent developments on this topic, please refer to [1, 9, 10, 12–16] and the closely related references therein.

2. A definition and a lemma

Now we introduce the notion \( "(\alpha, m)\)-GA-convex function". 

Definition 2.1. For \( f : [0, b^*] \to \mathbb{R} \) and \((\alpha, m) \in [0, 1]^2\), a function \( f \) is said to be \((\alpha, m)\)-GA-convex on \( I \), if
\[
f(x^t y^{m(1-t)}) \leq t^\alpha f(x) + m(1-t^\alpha) f(y),
\]
for all \( x, y \in [0, b^*] \) and \( t \in [0, 1] \).
To establish some new Hermite–Hadamard type inequalities for \((\alpha, m)\)-GA-convex functions, we need the following lemma.

**Lemma 2.2** ([8]). Let \( f : I \subseteq \mathbb{R}^+ \rightarrow \mathbb{R} \) be a differentiable function on \( 1^\circ \) and \( a, b \in 1^\circ \) with \( 0 < a < b \). If \( f' \in L_1([a, b]) \), then

\[
\frac{f(a) + 4f(\sqrt{ab}) + f(b)}{6} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx = \frac{\ln b - \ln a}{4} \int_0^1 \left[ t - \frac{1}{3} \right] \left[ a^{1-t}b^{1/2}f'(a^{1-t}b^{1/2}) + a^{1-t}b^{1/2}f'(a^{1-t}b^{1/2}) \right] \, dt.
\]

**3. Some new integral inequalities of Hermite–Hadamard type**

Now we are in a position to establish some integral inequalities of the Hermite–Hadamard type for \((\alpha, m)\)-GA-convex functions.

**Theorem 3.1.** Let \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \) be differentiable on \( \mathbb{R}_+ \), \( a, b \in \mathbb{R}_+ \) with \( a < b \), and \( f' \in L_1([a, b]) \). If \( |f'|^q \) is \((\alpha, m)\)-GA-convex on \((0, \max\{b, b^{1/m}\})\) for \((\alpha, m) \in (0, 1]^2 \) and \( q \geq 1 \), then

\[
\left| \frac{f(a) + 4f(\sqrt{ab}) + f(b)}{6} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right| \leq \frac{\ln b - \ln a}{4} \left[ \frac{1}{2^{\alpha+2}\alpha+4(\alpha+1)(\alpha+2)(\alpha+3)} \right]^{1/q} \times \left\{ M^{(q-1)/q}(a, b) [N_1(a, b)|f'(b)|^q + mN_2(a, b)|f'(a^{1/m})|^q] \right\}^{1/q}
\]

where

\[
M(u, v) = 2\left[ u^{5/6}L(u^{1/6}, v^{1/6}) + u^{1/2}(2v^{1/2} - u^{1/2}) - 2u^{1/2}v^{1/6}L(u^{1/3}, v^{1/3}) \right],
\]

\[
N_1(u, v) = 12(5u + v)\alpha + 12(17u + v) + 6 \times 3^{\alpha+2}[(u + v)(2\alpha^2 + 3) + (9u + 5v)\alpha],
\]

\[
N_2(u, v) = 6^\alpha(\alpha + 1)(\alpha + 2)\alpha(\alpha + 3)(61u + 29v) - 6(10u + 2v)\alpha - 12(17u + v) - 6 \times 3^{\alpha+2}[(u + v)(2\alpha^2 + 3) + (9u + 5v)\alpha],
\]

and \( L(u, v) = u^{1/2}v^{1/6} \) is called the logarithmic mean.

**Proof.** From Lemma 2.2 and the well-known Hölder integral inequality, it follows that

\[
\left| \frac{f(a) + 4f(\sqrt{ab}) + f(b)}{6} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right| \leq \frac{\ln b - \ln a}{4} \int_0^1 \left[ t - \frac{1}{3} \right] \left[ a^{1-t}b^{1/2}f'(a^{1-t}b^{1/2}) + a^{1-t}b^{1/2}f'(a^{1-t}b^{1/2}) \right] \, dt
\]

\[
\leq \frac{\ln b - \ln a}{4} \left\{ \left( \int_0^1 \left[ t - \frac{1}{3} \right] a^{1-t}b^{1/2}f'(a^{1-t}b^{1/2}) \, dt \right)^{1/q} \int_0^1 \left[ t - \frac{1}{3} \right] a^{1-t}b^{1/2}f'(a^{1-t}b^{1/2}) \, dt \right\}^{1/q}
\]

\[
+ \left( \int_0^1 \left[ t - \frac{1}{3} \right] a^{1-t}b^{1/2}f'(a^{1-t}b^{1/2}) \, dt \right)^{1/q} \left\{ \left( \int_0^1 \left[ t - \frac{1}{3} \right] a^{1-t}b^{1/2}f'(a^{1-t}b^{1/2}) \, dt \right)^{1-q} \int_0^1 \left[ t - \frac{1}{3} \right] a^{1-t}b^{1/2}f'(a^{1-t}b^{1/2}) \, dt \right\}^{1/q},
\]

where

\[
\int_0^1 \left[ t - \frac{1}{3} \right] a^{1-t}b^{1/2} \, dt = M(a, b) \quad \text{and} \quad \int_0^1 \left[ t - \frac{1}{3} \right] a^{1/2}b^{1-t/2} \, dt = M(b, a).
\]
Since \( f'^q \) is \((\alpha, m)\)-GA-convex on \((0, \max\{b, b^{1/m}\})\), by the well-known GA-inequality, we have

\[
\int_0^1 t \frac{1}{3} \left| a^{1-t/2}b^{t/2} f'(a^{1-t/2}b^{t/2}) \right|^q \, dt \\
\leq \int_0^1 t \frac{1}{3} \left| a^{1-t/2}b^{t/2} \left[ \frac{a^{\alpha}}{2^\alpha} |f'(b)|^q + m \left( 1 - \frac{t^{\alpha}}{2^\alpha} \right) |f'(a^{1/m})|^q \right] \right| \, dt \\
\leq \frac{1}{3} \left( \left( 1 - \frac{t}{2} \right) a + \frac{t}{2} b \right) \left[ \frac{a^{\alpha}}{2^\alpha} |f'(b)|^q + m \left( 1 - \frac{t^{\alpha}}{2^\alpha} \right) |f'(a^{1/m})|^q \right] \, dt \\
= \frac{N_1(a, b)|f'(b)|^q + mN_2(a, b)|f'(a^{1/m})|^q}{2^{\alpha+2} \times 3^{\alpha+4}(\alpha+1)(\alpha+2)(\alpha+3)},
\]

and

\[
\int_0^1 t \frac{1}{3} a^{1-t/2}b^{1-t/2} |f'(a^{1-t/2}b^{1-t/2})|^q \, dt = \frac{N_1(b, a)|f'(a)|^q + mN_2(b, a)|f'(b^{1/m})|^q}{2^{\alpha+2} \times 3^{\alpha+4}(\alpha+1)(\alpha+2)(\alpha+3)}. \tag{3.4}
\]

Putting the equalities (3.3) and (3.4) into the inequality (3.2) leads to the inequality (3.1). The proof of Theorem 3.1 is complete. \( \square \)

**Corollary 3.2.** Under the assumptions of Theorem 3.1, if \( q = 1 \), then

\[
\left| f(a) + 4f(\sqrt{ab}) + f(b) \right| - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \\
\leq \frac{\ln b - \ln a}{6^{\alpha+6}(\alpha+1)(\alpha+2)(\alpha+3)} \\
\times \left\{ [N_1(b, a)|f'(a)| + N_1(a, b)|f'(b)|] + m[N_2(a, b)|f'(a^{1/m})| + N_2(b, a)|f'(b^{1/m})|] \right\}.
\]

**Corollary 3.3.** Under the assumptions of Theorem 3.1, if \( \alpha = m = 1 \), then

\[
\left| f(a) + 4f(\sqrt{ab}) + f(b) \right| - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \\
\leq \frac{\ln b - \ln a}{4} \left( \frac{1}{3 \times 6^4} \right)^{1/q} \\
\times \left\{ M^{(q-1)/q}(a, b)[(211a + 137b)|f'(b)|^q + (521a + 211b)|f'(a)|^q]^{1/q} \\
+ M^{(q-1)/q}(b, a)[(137a + 211b)|f'(a)|^q + (211a + 521b)|f'(b)|^q]^{1/q} \right\}.
\]

**Corollary 3.4.** Under the assumptions of Theorem 3.1, if \( \alpha = m = q = 1 \), then

\[
\left| f(a) + 4f(\sqrt{ab}) + f(b) \right| - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \\
\leq \frac{\ln b - \ln a}{6^5} \left\{ (329a + 211b)|f'(a)| + (211a + 329b)|f'(b)| \right\}.
\]

**Theorem 3.5.** Let \( f : \mathbb{R}_+ \to \mathbb{R} \) be differentiable on \( \mathbb{R}_+ \), \( a, b \in \mathbb{R}_+ \) with \( a < b \), and \( f' \in L_1([a, b]) \). If \( |f'^q| \) is \((\alpha, m)\)-GA-convex on \((0, \max\{b, b^{1/m}\})\) for \((\alpha, m) \in (0, 1]^2 \) and \( q > 1 \), then

\[
\left| f(a) + 4f(\sqrt{ab}) + f(b) \right| - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \\
\leq \frac{\ln b - \ln a}{4} \left( \frac{(q-1)[2^{(q-1)/(q-1)} + 1]}{2^{(2q-1)/(q-1)} + 1} \right)^{1-q}.
\]
Theorem 3.5 is thus proved.

Proof. Since \(|f'|^q \) is an \((\alpha, m)\)-GA-convex function on \((0, \max\{b, b^{1/m}\})\), by Lemma 2.2 and Hölder’s integral inequality, we have

\[
\begin{aligned}
&\left| f(a) + 4f(\sqrt{ab}) + f(b) \right| \leq \frac{1}{6} \ln b - \ln a \int_a^b \frac{f(x)}{x} \, dx \\
&\leq \left[ \frac{\ln b - \ln a}{4} \right] \left[ \int_0^1 t \frac{1}{3} \left[ a^{1-t} b^{t/2} |f'(a^{1-t} b^{t/2})| + a^{1/2} b^{1-t/2} |f'(a^{1/2} b^{1-t/2})| \right] \, dt \right]^{1/q} \\
&\leq \left[ \frac{\ln b - \ln a}{4} \right] \left[ \int_0^1 t \frac{1}{3} q/(q-1) \left[ \int_0^1 a^{q(1-t)2} b^{qt/2} |f'(a^{1-t} b^{t/2})|^q \, dt \right] \right]^{1/q} \\
&+ \left[ \frac{\ln b - \ln a}{4} \right] \left[ \int_0^1 q/(q-1) \left[ \int_0^1 a^{q(1-t)2} b^{qt/2} |f'(a^{1-t} b^{t/2})|^q \, dt \right] \right]^{1/q} \\
&\leq \ln b - \ln a \left[ \frac{(q-1)(2(q-1)/(q-1) + 1)}{(2q - 1)(3(q-1)/(q-1))} \right]^{1-1/q} \\
&\times \left\{ \left[ \int_0^1 (1 - t/2) a^{q(1-t)2} b^{qt/2} \left( t^{\alpha/2} |f'(b)|^q + m \left( 1 - t^{\alpha/2} \right) |f'(b)|^q \right) \, dt \right]^{1/q} \\
&+ \left[ \int_0^1 (1 - t/2) b^{q(1-t)2} \left( t^{\alpha/2} |f'(a)|^q + m \left( 1 - t^{\alpha/2} \right) |f'(a)|^q \right) \, dt \right]^{1/q} \\
&= \ln b - \ln a \left[ \frac{(q-1)(2(q-1)/(q-1) + 1)}{(2q - 1)(3(q-1)/(q-1))} \right]^{1-1/q} \\
&\times \left[ \frac{2((\alpha + 3)a^q + (\alpha + 1)b^q)|f'(b)|^q + m \left( 3 \times \frac{2^\alpha(\alpha + 1)(\alpha + 2)}{2(q-1)} - 2(\alpha + 3)a^q + (\alpha + 1)b^q \right) |f'(a)|^q + m \left( 5 \times \frac{2^\alpha(\alpha + 1)(\alpha + 2)}{2(q-1)} - 2(\alpha + 3)b^q \right) |f'(b)|^q \right]^{1/q} \\
&\times \left[ \frac{6a^q + 4b^q |f'(a)|^q + (28a^q + 8b^q) |f'(b)|^q \right]^{1/q} \\
&+ \left[ \frac{4a^q + 8b^q |f'(a)|^q + (8a^q + 28b^q) |f'(b)|^q \right]^{1/q} \\
\end{aligned}
\]

Theorem 3.5 is thus proved.

Corollary 3.6. Under the assumptions of Theorem 3.5, if \(\alpha = m = 1\), then

\[
\begin{aligned}
&\left| f(a) + 4f(\sqrt{ab}) + f(b) \right| \leq \frac{1}{6} \ln b - \ln a \int_a^b \frac{f(x)}{x} \, dx \\
&\leq \ln b - \ln a \left[ \frac{(q-1)(2(q-1)/(q-1) + 1)}{(2q - 1)(3(q-1)/(q-1))} \right]^{1-1/q} \\
&\times \left[ \frac{8a^q + 4b^q |f'(b)|^q + (28a^q + 8b^q) |f'(b)|^q \right]^{1/q} \\
&+ \left[ \frac{4a^q + 8b^q |f'(a)|^q + (8a^q + 28b^q) |f'(b)|^q \right]^{1/q} \\
\end{aligned}
\]

Theorem 3.7. Let \(f : \mathbb{R}_+ \to \mathbb{R}\) be differentiable on \(\mathbb{R}_+\), \(a, b \in \mathbb{R}_+\) with \(a < b\), and \(f' \in L_1([a, b])\). If \(|f'|^q \) is \((\alpha, m)\)-GA-convex on \((0, \max\{b, b^{1/m}\})\) for \((\alpha, m) \in (0, 1]^2\) and \(q > 1\), then

\[
\begin{aligned}
&\left| f(a) + 4f(\sqrt{ab}) + f(b) \right| \left( \frac{1}{6} \ln b - \ln a \int_a^b \frac{f(x)}{x} \, dx \right) \\
&\leq \ln b - \ln a \left[ \frac{(q-1)(2(q-1)/(q-1) + 1)}{(2q - 1)(3(q-1)/(q-1))} \right]^{1-1/q} \\
&\times \left[ \frac{8a^q + 4b^q |f'(b)|^q + (28a^q + 8b^q) |f'(b)|^q \right]^{1/q} \\
&+ \left[ \frac{4a^q + 8b^q |f'(a)|^q + (8a^q + 28b^q) |f'(b)|^q \right]^{1/q} \\
\end{aligned}
\]
Theorem 3.7 is thus proved.
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