Topological structures and the coincidence point of two mappings in cone b-metric spaces

Congjun Zhang, Sai Li*, Baoqing Liu

School of Applied Mathematics, Nanjing University of Finance and Economics, Nanjing, Jiangsu, 210023, China.

Communicated by Y. H. Yao

Abstract

Let \((X, d, K)\) be a cone b-metric space over a ordered Banach space \((E, \preceq)\) with respect to cone \(P\). In this paper, we study two problems:

1. We introduce a b-metric \(\rho_{c}\) and we prove that the b-metric space induced by b-metric \(\rho_{c}\) has the same topological structures with the cone b-metric space.

2. We prove the existence of the coincidence point of two mappings \(T, f: X \to X\) satisfying a new quasi-contraction of the type \(d(Tx, Ty) \preceq \Lambda \{d(fx, fy), d(fx, Ty), d(fy, Tx), d(fy, Tx)\}\), where \(\Lambda: E \to E\) is a linear positive operator and the spectral radius of \(K\Lambda\) is less than 1.


Keywords: Topological structures, cone b-metric spaces, quasi-contraction, points of coincidence, common fixed points.

2010 MSC: 47H10, 54H25.

1. Introduction

Cone metric spaces were introduced in [6]. In [3], they introduced a special metric \(\rho_{E}\) and they proved that the metric space induced by the metric \(\rho_{E}\) have the same topological spaces with the cone metric space. In [7], Hussain and Shah introduced the concept of cone b-metric spaces and they investigated topological properties of the cone b-metric spaces. In fact, the class of cone b-metric spaces is effectively larger than that of the ordinary cone metric spaces. That is, every cone metric space is a cone b-metric space. In [5], Czerwik first introduced the concept of b-metric spaces. Similarly, b-metric spaces are extensions of metric spaces. In the first part of this work, we introduce a special b-metric \(\rho_{c}\) and proves that the b-metric space induced by \(\rho_{c}\) has the same topological structures with the cone b-metric space.

The second part of this work involves coincidence points and common fixed points. In 1976, Jungck [9] extended the celebrated Banach contraction mapping principle to the common fixed theorem of two
commuting mappings. In this process, he introduced a new iteration process which was a generalization of the Picard iteration. The new iteration scheme can be defined as follows:

**Definition 1.1** ([8]). Let $T$ and $f$ be self-mappings of a set $X$, and let $(x_n)_{n=0}^{\infty}$ be a sequence in $X$ such that $f x_{n+1} = T x_n, n = 0,1,2,\ldots$.

Then the sequence $(f x_n)_{n=0}^{\infty}$ is said to be a $T$-$f$-sequence or Jungck iteration.

Let $f$ and $T$ be self-mappings of nonempty set $X, x \in X$ is called a coincidence point of $f$ and $T$ if $f x = T x$. A point $y \in X$ is called a point of coincidence of $f$ and $T$ if there exists a point $x \in X$ such that $y = f x = T x$. A point $z \in X$ is called a common fixed point of $f$ and $T$ if $z = f z = T z$.

**Definition 1.2** ([12]). Let $f$ and $g$ be self-mappings of a nonempty set $X$. Then $f$ and $g$ are called weakly compatible, if they commute at their coincidence points.

Let $(Y, \preceq)$ be an ordered vector space, $x \in X$ and $A \subset X$. We say that $x \preceq A$, if there exists at least one vector $y \in A$ such that $x \preceq y$. In 2010, Kadelburg and Radenović obtained the following result by using Jungck iteration.

**Theorem 1.3** ([10]). Let $(X, d)$ be a cone metric space over a Banach space $(Y, \preceq)$. And let $T, f : X \to X$ be mappings such that $T(X) \subset f(X)$ and $f(X)$ be a complete subspace of $X$. Supposing there exists $\lambda \in [0,1)$ such that for all $x, y \in X$,

\[ d(T x, T y) \preceq \lambda (d(f x, f y), d(f x, T y), d(f y, T y), d(f y, T x)). \]

Then $T$ and $f$ have a unique point of coincidence. Moreover, if $T$ and $f$ are weakly compatible, then every $T$-$f$-sequence $(f x_n)$ in $X$ converges to the unique common fixed point of $T$ and $f$.

In 2014, Cvetković and Rakočević [4] introduced notion of quasi-contraction of Perov type and partly extended Kadelburg's theorems to positive linear functional.

**Definition 1.4** ([4]). Let $(X, d)$ be a cone metric space over a Banach space $(E, \preceq)$. A map $T : X \to X$ such that for some bounded linear operator $\Lambda : E \to E$ whose spectral radius is less than 1 and for each $x, y \in X$,

\[ d(T x, T y) \preceq \Lambda (d(x, y), d(x, T x), d(y, T y), d(x, T y), d(y, T x)), \]

is called a quasi-contraction of Perov type.

**Theorem 1.5** ([4]). Let $(X, d)$ be a complete cone metric space with respect to cone $P$. If a mapping $T : X \to X$ is a quasi-contraction of Perov type and $\Lambda(P) \subset P$, then $f$ has a unique point $x^* \in X$ and, for any $x \in X$, the iterative sequence $(T^n x)_{n \in \mathbb{N}}$ converges to the fixed point of $T$.

In the second part of work, we study a new quasi-contraction, that is,

\[ d(T x, T y) \preceq \Lambda (d(f x, T x), d(f x, T y), d(f y, T y), d(f y, T x)), \]

where $\Lambda : E \to E$ is a linear positive operator and the spectral radius of $K \Lambda$ is less than 1. Our results can be considered as a further development of [10, Theorem 1.3] and [4, Theorem 1.5].

## 2. Preliminary and auxiliary results

In this section, we recall and provide some concepts and auxiliary results.

### 2.1. $b$-metric space

**Definition 2.1** ([5]). Let $X$ be a nonempty set, $K \geq 1$ and $D : X \times X \to [0, +\infty)$ is a function such that for all $x, y, z \in X$:

1. $D(x, y) = 0$ if and only if $x = y$;

(2) $D(x, y) = D(y, x)$;

(3) $D(x, z) \leq K[D(x, y) + D(y, z)]$.

Then $D$ is called a b-metric, and $(X, D, K)$ is called a b-metric space.

In b-metric spaces $(X, D, K)$, the sequence $\{x_n\}$ converges to $x \in X$, if and only if $\lim_{n \to \infty} D(x_n, x) = 0$ and the sequence $\{x_n\}$ is Cauchy, if and only if $\lim_{n, m \to \infty} D(x_n, x_m) = 0$. $(X, D, K)$ is complete if and only if any Cauchy sequence in $X$ is convergent. $B(a, \epsilon)$ denotes the subset $\{x \in X : D(a, x) < \epsilon\}$ of $X$, $a \in X$, $\epsilon > 0$.

**Definition 2.2** ([11]). Let $(X, D, K)$ be a b-metric space.

1. A subset $A \subset X$ is said to be open, if and only if for any $a \in A$, there exists $\epsilon > 0$ such that $B(a, \epsilon) \subset A$.

2. Let $B$ be a subset of $X$. An element $x \in X$ is called a limit point of $B$, whenever for any $\epsilon > 0$,

$$B(x, \epsilon) \cap (B \setminus \{x\}) \neq \emptyset.$$  

$B$ is called closed, whenever each limit point of $B$ belongs to $B$.

3. A subset $B \subset X$ is called bounded whenever, there exists $\epsilon > 0$ such that $D(x, y) < \epsilon$ for all $x, y \in B$.

4. A subset $B \subset X$ is called compact, whenever every open cover of $B$ has a finite subcover.

5. A subset $B$ is called sequentially compact, if and only if for any sequence $\{x_n\}$ in $B$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges, and $\lim_{k \to \infty} x_{n_k} \in B$.

6. A subset $B$ is called totally bounded, if and only if for any $\epsilon > 0$, there exist $x_1, x_2, x_3, \ldots, x_n \in B$ such that

$$B \subset B(x_1, \epsilon) \cup \cdots \cup B(x_n, \epsilon).$$

**Proposition 2.3** ([11]). Let $(X, D, K)$ be a b-metric space,

1. $A$ is closed, if and only if for any sequence $\{x_n\}$ in $X$ which converges to $x$, we have $x \in A$.

2. If we let $\overline{A}$ denote the intersection of all closed subset of $X$ which contains $A$, then for any $x \in \overline{A}$ and for any $\epsilon > 0$, we have $B(x, \epsilon) \cap A \neq \emptyset$.

3. $A$ is compact, if and only if $A$ is sequentially compact.

4. If $A$ is compact, then $A$ is totally bounded.

**Theorem 2.4.** Let $(X, D, K)$ be a b-metric space,

1. $A$ is closed, if and only if $A^c$ is open, where $A^c$ is the complement of $A$ in $X$.

2. $A$ is called relatively compact, whenever $\overline{A}$ is compact. If $(X, D, K)$ is complete, then $A$ is relatively compact, if and only if $A$ is totally bounded.

**Proof.**

1. Firstly assume that $A$ is closed. We show that $A^c$ is open. If $A^c$ is not open, then

$$\exists a \in A^c, \ \forall n \in \mathbb{N}, \ \exists x_n \in A \ \text{such that} \ D(a, x_n) < \frac{1}{n}.$$  

It contradict Proposition 2.3 (1). Conversely, assume that $A^c$ is open, we show that $A$ is closed. If $x \notin A$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset A^c$. Clearly $B(x, \epsilon) \cap (\overline{A} \setminus \{x\}) = \emptyset$, it implies that $A$ is closed.
(2) We start with that $A$ is relatively compact. So we have that $\overline{A}$ is totally bounded,

$$\forall \varepsilon > 0, \ \exists x_1, x_2, \ldots, x_n \in A, \ A \subset B(x_1, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon).$$

So there exist $y_1 \in B(x_i, \varepsilon) \cap A, \ i = 1, 2, \ldots, n$. Then $A \subset A \subset B(y_1, 2K\varepsilon) \cup \cdots \cup B(y_n, 2K\varepsilon)$, and it implies that $A$ is totally bounded. Conversely, assume that $A$ is totally bounded, then we show that $A$ is relatively compact. If $\{x_n\} \subset A$, then there exists $y_n$ such that $y_n \subset A \cap B(x_n, \frac{1}{n})$ for all $n \in N$, from Proposition 2.3 (2). Clearly,

$$\exists a_1 \in A, \ \{y^{(1)}_{n_k}\} \subset \{y_n\} \subset A, \ \{y^{(1)}_{n_k}\} \subset B(a_1, \frac{1}{n}),$$

where $\{y^{(1)}_{n_k}\}$ is the subsequence of $\{y_n\}$, from that $A$ is totally bounded. Similarly,

$$\exists a_n \in A, \ \{y^{(n)}_{n_k}\} \subset \{y^{(n-1)}_{n_k}\} \subset A, \ \{y^{(n)}_{n_k}\} \subset B(a_n, \frac{1}{n}), \ n \in N, \ n \geq 2,$$

where $\{y^{(n)}_{n_k}\}$ is the subsequence of $\{y^{(n-1)}_{n_k}\}$. So we can select $y_{n_m}$ such that $y_{n_m} \in \{y^{(m)}_{n_k}\}$ and $\{y_{n_m}\}$ is the subsequence of $\{y_n\}$. Since $y_{n_m} \subset B(a_1, \frac{1}{n}), m \geq 1$, then

$$D(y_{n_m}, y_{n_l}) \leq K[D(a_1, y_{n_m}) + D(a_1, y_{n_l})] \leq \frac{2K}{1}, \ m \geq 1.$$

It implies that $y_{n_m}$ is Cauchy. Since $(X, D, K)$ is complete, there exists $x \in \overline{A}$ and $\lim_{m \to \infty} y_{n_m} = x$ from Proposition 2.3 (1) (2). It is easy to check that $\lim_{m \to \infty} x_{n_m} = x$. It implies that $\overline{A}$ is sequentially compact and we have that $\overline{A}$ is compact from Proposition 2.3 (3). $\square$

2.2. Cone b-metric space

Let $E$ be a real Banach space. A subset $P$ of $E$ is called a cone whenever the following condition is satisfied:

1. $P$ is closed, nonempty and $P \neq \{\theta\}$, where $\theta$ is the zero vector in $E$.
2. $a, b \in R, a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$.
3. $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ on $E$ with respect to $P$ by $x \preceq y$, if and only if $y - x \in P$. We shall write $\prec$ to indicate that $x \preceq y$ but $x \not= y$, while $x \ll y$ stands for $y - x \in \text{int}P$ (interior of $P$).

**Definition 2.5 ([7]).** Let $X$ be a nonempty set and $(E, \preceq)$ an ordered Banach space with respect to cone $P$. A vector-valued function $d : X \times X \to E$ is said to be a cone b-metric function on $X$ with the constant $K \geq 1$, if the following conditions are satisfied:

1. $\theta \preceq d(x, y)$, for all $x, y \in X$, and $d(x, y) = \theta$, if and only if $x = y$;
2. $d(x, y) = d(y, x)$, for all $x, y \in X$;
3. $d(x, y) \preceq K[d(x, z) + d(y, z)]$, for all $x, y, z \in X$.

The pair $(X, d, K)$ is called the cone b-metric space over an ordered Banach space $(E, \preceq)$ with respect to cone $P$.

**Definition 2.6 ([7]).** Let $(X, d, K)$ be a cone b-metric space over the ordered Banach space $(E, \preceq)$ with respect to cone $P$. We say that $\{x_n\} \subset X$ is:
Proof. Let \( \lim_{n \to \infty} x_n = x \).

We say that the cone \( b \)-metric space \((X, d, K)\) is complete, if any cone-Cauchy is cone-convergent. Let \( A \subset X \), we claim that \( A \) is a complete subspace, if for every cone-Cauchy \( \{x_n\} \subset A \), cone \( \lim_{n \to \infty} x_n \in A \).

We claim that \( \{y_n\} \subset E \) norm-converges to \( y \), if for any \( \varepsilon > 0 \), there exists an \( m \in \mathbb{N} \) such that \( \|y_n - y\| < \varepsilon \), for all \( n > m \). Noting that if \( \{x_n\} \subset X \), \( \{y_n\} \subset E \), \( y_n \) norm-converges to \( \theta \) and \( d(x_n, x_m) \leq d(y_n, \lambda a) \), for all \( n, m \in \mathbb{N}, m > n \), then \( \{x_n\} \) is cone-Cauchy. We denote by \( \hat{B}(x, c) \) the cone-ball, given by \( \hat{B}(x, c) = \{y \in X : d(x, y) \ll c\} \).

Definition 2.7. Let \((X, d, K)\) be a cone \( b \)-metric space over the ordered Banach space \((E, \preceq)\) with respect to cone \( P \),

1. A subset \( A \subset X \) is said to be cone-open, if and only if for any \( a \in A \), there exists \( c \gg 0 \) such that the cone-ball \( \hat{B}(a, c) \subset A \).

2. An element \( x \in X \) is called a cone-limit point of \( B \) whenever for any \( c \gg 0 \), \( \hat{B}(x, c) \setminus \{\hat{B}(y, c)\} \neq \emptyset \). A subset \( B \subset X \) is called cone-closed, whenever each cone-limit point of \( B \) belongs to \( B \).

3. A subset \( B \subset X \) is called cone-bounded, whenever there exists \( c \gg 0 \) such that \( d(x, y) \ll c \) for all \( x, y \in B \).

4. A subset \( B \subset X \) is called cone-compact, whenever every cone-open cover of \( B \) has a finite subcover.

5. A subset \( B \) is called cone-sequentially compact, if and only if for any sequence \( \{x_n\} \) in \( B \), there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) which cone-converges, and cone \( \lim_{k \to \infty} x_{n_k} \in B \).

6. A subset \( B \) is called cone-totally bounded, if and only if for any \( c \gg 0 \), there exist \( x_1, x_2, x_3, \cdots, x_n \in B \) such that \( B \subset \hat{B}(x_1, c) \cup \cdots \cup \hat{B}(x_n, c) \).

Let \( A \subset X \), \( \bar{A} \) stands for the intersection of all cone-closed subsets of \( X \) including \( A \). We claim that \( A \) is cone-relatively compact, if \( \bar{A} \) is cone-compact.

Proposition 2.8. Let \((E, \preceq)\) be an ordered Banach space with respect to cone \( P \). Then the following properties are often used:

1. \( x, y, z \in E, x \ll y \ll z \) imply \( x \ll z \).

2. \( a \int P \subset \int P, \) for all \( a \in \mathbb{R} \), \( a > 0 \).

3. For any \( c \in \int P \), \( x \in E \), there exists an \( n \in \mathbb{N} \) such that \( \frac{x}{n} \ll c \).

4. If \( a \in P \), \( 0 < \lambda < 1 \) and \( a \leq \lambda a \), then \( a = 0 \).

5. \( c \in \int P \), \( \alpha, \beta \in \mathbb{R} \), \( \alpha > \beta \) imply \( \beta c \ll \alpha c \).

Lemma 2.9. Let \((E, \preceq)\) be an ordered Banach space with respect to cone \( P \). If \( x \ll y \), then exists \( n \in \mathbb{N} \) such that \( x \ll (1 - \frac{1}{n})y \).

Proof. Let \( \hat{B}(x, \varepsilon) = \{y \in E : \|y - x\| < \varepsilon\} \), \( x \in E, \varepsilon > 0 \). Since \( x \ll y \), then exists \( \varepsilon > 0 \) such that \( y - x + \hat{B}(0, \varepsilon) \subset P \). Clearly \( \hat{B}(0, \frac{x}{n}) + \hat{B}(0, \frac{y}{n}) \subset \hat{B}(0, \varepsilon) \) from the triangle inequality of the norm. We know that there exists \( n \in \mathbb{N} \) such that \(-\frac{y}{n} \in \hat{B}(0, \frac{x}{n}) \). So \( (1 - \frac{1}{n})y - x + \hat{B}(0, \frac{x}{n}) \subset P \). It implies that \( x \ll (1 - \frac{1}{n})y \). \( \square \)
2.3. The b-metric \( \rho_c \)

Let \((X,d,K)\) be a cone b-metric space over an ordered Banach space \((E,\leq)\) with respect to cone \(P\). Since \(P\) is closed, we have that the cone \(P\) is Archimedean (see \cite[page 63, lemma 2.3]{2}). Given \(a,b \in E\) with \(a \leq b\), we denote by \([a,b]\) the order interval, i.e.,

\[
[a,b] = \{x \in X : a \leq x \leq b\}.
\]

Let \(c \in \text{int}P\), then \(E = \bigcup_{n \in \mathbb{N}} [-c,c]\). We can define the Minkowski functional on \(E\) by setting

\[
\|x\|_c = \inf\{\lambda > 0 : x \in [-\lambda c,\lambda c]\},
\]

for all \(x \in E\). And furthermore, we have that \(-\|x\|_c \leq x \leq \|x\|_c\) (see \cite[page 104]{2}).

**Proposition 2.10** ([8]). Let \((E,\leq)\) be an ordered Banach space with respect to cone \(P\).

1. \(x,y \in E, \theta \leq x \leq y\) imply \(\|x\|_c \leq \|y\|_c\).
2. \(x,y \in E, \|x+y\|_c \leq \|x\|_c + \|y\|_c\).
3. \(\lambda \in \mathbb{R}, \lambda \geq 0, \|\lambda x\|_c = \lambda \|x\|_c\).

Now, we define b-metric \(\rho_c\) by setting \(\rho_c = \|d(x,y)\|_c\).

**Proposition 2.11.** \((X,\rho_c,K)\) is a b-metric space.

**Proof.** It is easy to check that \(\rho_c\) is a b-metric from Proposition 2.10. \(\square\)

We define \(B(x,\varepsilon) = \{y \in X : \rho_c(x,y) < \varepsilon\}, x \in X, \varepsilon > 0\) and we claim that \(\rho_c \star \lim_{n \to \infty} x_n = x, \{x_n\} \subset X\), if \(\lim_{n \to \infty} \rho_c(x_n,x) = 0\). Now we prove some basic results.

**Theorem 2.12.** Let \((X,d,K)\) be a cone b-metric space over the ordered Banach space \((E,\leq)\) with respect to cone \(P\). \(\hat{B}(x,rc) = B(x,r), \text{for all } x \in X, r \in \mathbb{R}, r > 0, c \gg \theta\).

**Proof.** Let \(y \in \hat{B}(x,rc)\), then \(d(x,y) \ll rc\). There exists an \(n \in \mathbb{N}\) such that \(d(x,y) \ll (1 - \frac{1}{n})rc\), from Lemma 2.9. It implies that

\[
-(1 - \frac{1}{n})rc \leq \theta \leq d(x,y) \leq (1 - \frac{1}{n})rc,
\]

so we have that \(\rho_c(x,y) = \|d(x,y)\|_c \ll (1 - \frac{1}{n})r < r\) from the definition of \(\|\|_c\). It implies that \(y \in B(x,r)\). Conversely, let \(y \in B(x,r)\), then \(\rho_c(x,y) < r\). So \(d(x,y) \ll \rho_c(x,y)c \ll rc\). It implies that \(y \in \hat{B}(x,rc)\). \(\square\)

**Theorem 2.13.** Let \((X,d,K)\) be a cone b-metric space over the ordered Banach space \((E,\leq)\) with respect to cone \(P\) and \(c \in \text{int}P\). \(\text{cone} \star \lim_{n \to \infty} x_n = x, \text{if and only if } \rho_c \star \lim_{n \to \infty} x_n = x, \{x_n\} \subset X\).

**Proof.** Assume that \(\text{cone} \star \lim_{n \to \infty} x_n = x, \text{we have that for any } c_1 \gg 0\) there exists an \(m \in \mathbb{N}\) such that \(x_n \in B(x,c_1)\) for all \(n > m\). Since \(\hat{B}(x,rc) = B(x,r)\) for all \(x \in X, r \in \mathbb{R}, r > 0, c \gg \theta\), then for any \(r > 0\), there exists an \(m \in \mathbb{N}\) such that \(x_n \in \hat{B}(x,rc) = B(x,r)\) for all \(n > m\). It implies that \(\rho_c \star \lim_{n \to \infty} x_n = x\).

Conversely, assume that \(\rho_c \star \lim_{n \to \infty} x_n = x, \text{then for any } r > 0, \text{there exists an } m \in \mathbb{N}\) such that \(x_n \in B(x,r)\) for all \(n > m\). We also have that for any \(c_1 \gg 0\) there exists a \(k \in \mathbb{N}\) such that \(\frac{c}{k} \ll c_1\).

So there exists an \(m \in \mathbb{N}\) such that \(x_n \in B(x,\frac{1}{k}) = \hat{B}(x,\frac{c}{k}) \subset \hat{B}(x,c_1)\) for all \(n > m\). It implies that \(\text{cone} \star \lim_{n \to \infty} x_n = x\). \(\square\)
2.4. The linear positive operator

Let \((X, d, K)\) be a cone b-metric space over the ordered Banach space \((E, \preceq)\) with respect to cone \(P\). We say that \(\Lambda : E \to E\) is a linear positive operator, if \(\Lambda\) is a linear operator and \(\Lambda(P) \subset P\). Clearly \(\Lambda\) is a linear positive operator, if and only if \(\Lambda\) is a linear operator and \(\Lambda(x) \preceq \Lambda(y)\) for all \(x, y \in E, x \preceq y\). In fact, if \(\Lambda\) is a linear positive operator, then \(\Lambda\) is continuous (see [2, page 84]). And furthermore, if \(\Lambda : E \to E\) is a linear continuous operator and there exists an \(m \in \mathbb{N}\) such that \(\|\Lambda^m\| < 1\), then \(\Lambda^m x\) norm-converges to \(0\) for any \(x \in E\) and \(I - \Lambda\) is invertible where \(I\) is the identity mapping of \(E\), that is, 
\[(I - \Lambda)^{-1} = \sum_{n=0}^{\infty} \Lambda^n.\]  Of course there exists an \(m \in \mathbb{N}\) such that \(\|\Lambda^m\| < 1\), if \(\Lambda : E \to E\) is a linear continuous operator and it’s spectral radius is less than one. It is inspired by Huang and Zhang [6], we say that \(P * \lim_{n \to \infty} x_n = x, \{x_n\} \subset E\), if for any \(c \gg 0\), there exists an \(m \in \mathbb{N}\) such that \(-c \ll x_n - x \ll c\) for all \(n > m\).

**Proposition 2.14.** Let \((X, d, K)\) be a cone b-metric space over the ordered Banach space \((E, \preceq)\) with respect to cone \(P\) and \(\Lambda : E \to E\) is a linear positive operator.

1. For any \(n \in \mathbb{N}\), there exists an \(m \in \mathbb{N}\) such that \(x - x_m \gg -\frac{c}{n}\). So we have that 
\[x = x - x_m + x_m \geq -\frac{c}{n} + y, \quad \forall n \in \mathbb{N}.\]

Let \(n \to \infty\), we obtain \(x \succeq y\) from \(P\) is closed in \(E\).

2. It is obvious.

3. For any \(n \in \mathbb{N}\) and \(c \in \text{int} P\), there exists an \(m \in \mathbb{N}\) such that \(x - x_m \gg -\frac{c}{n}\). So we have that 
\[\Lambda(x) = \Lambda(x - x_m) + \Lambda(x_m) \geq -\frac{\Lambda(c)}{n} + \Lambda(y)\] for any \(n \in \mathbb{N}\). Let \(n \to \infty\), we obtain \(\Lambda(x) \succeq \Lambda(y)\) from \(P\) is closed in \(E\).

4. For any \(c \gg 0\), there exists a \(j \in \mathbb{N}\) such that \(\frac{\Lambda c}{j} \ll c\). Since \(P * \lim_{n \to \infty} x_n = x\), there exists an \(m \in \mathbb{N}\) such that \(-\frac{\xi}{j} \ll x_n - x \ll \frac{\xi}{j}\) for all \(n \gg m\). It implies that \(-c \ll -\frac{\Lambda c}{j} \leq \Lambda x_n - \Lambda x \ll \frac{\Lambda c}{j} \ll c\) for all \(n > m\). So we have that \(P * \lim_{n \to \infty} \Lambda x_n = \Lambda x\).

3. Main results

**Theorem 3.1.** Let \((X, d, K)\) be a cone b-metric space over the ordered Banach space \((E, \preceq)\) with respect to cone \(P\) and \(A \subset X, c \in \text{int} P:\)

1. \(A\) is cone-open, if and only if \(A\) is open in b-metric space \((X, \rho_c, K)\).

2. \(A\) is cone-closed, if and only if \(A\) is closed in b-metric space \((X, \rho_c, K)\).

3. \(A\) is cone-compact, if and only if \(A\) is compact in b-metric space \((X, \rho_c, K)\).

4. \(A\) is cone-totally bounded, if and only if \(A\) is totally bounded in b-metric space \((X, \rho_c, K)\).

5. \(A\) is cone-sequentially compact, if and only if \(A\) is sequentially compact in b-metric space \((X, \rho_c, K)\).
(6) A is cone-relatively compact, if and only if A is relatively compact in b-metric space \((X, \rho_c, K)\).

Proof.

(1) Assume that A is cone-open. Then for any \(a \in A\), there exists a \(c_1 \gg \theta\) such that \(\hat{B}(a, c_1) \subset A\). There also exists an \(n \in \mathbb{N}\) such that \(\frac{c_1}{n} \ll c_1\). So we have that \(B(a, \frac{c_1}{n}) = \hat{B}(a, \frac{c_1}{n}) \subset B(a, c_1)\) from Theorem 2.12. It implies that A is open in b-metric space \((X, \rho_c, K)\).

Conversely, assume that A is open in b-metric space \((X, \rho_c, K)\). Then for any \(a \in A\), there exists an \(r > 0\) such that \(B(a, r) = \hat{B}(a, rc) \subset A\). It implies that A is cone-open.

(2) To prove the result, it is sufficient to show that \(a \in X\) is the cone-limit point of A, if and only if \(a \in X\) is the limit point of A in the b-metric space \((X, \rho_c, K)\). In fact, for any \(c_1 \gg \theta\), \(\hat{B}(a, c_1) \cap (A \setminus \{x\}) \neq \emptyset\), if and only if for any \(r > 0\), \(B(a, r) \cap (A \setminus \{x\}) \neq \emptyset\). So we complete the proof.

(3) It is obvious from (1).

(4) Assume first that A is cone-totally bounded, then for any \(c_1 \gg \theta\), there exist \(x_1, \ldots, x_n \in A\) such that \(A \subset \hat{B}(x_1, c_1) \cup \cdots \cup \hat{B}(x_n, c_1)\). So for any \(r > 0\) there exist \(x_1, \ldots, x_n \in A\) such that

\[
A \subset \hat{B}(x_1, rc) \cup \cdots \cup \hat{B}(x_n, rc) = B(x_1, r) \cup \cdots \cup B(x_n, r).
\]

It implies that A is totally bounded in b-metric space \((X, \rho_c, K)\).

Conversely, assume that A is totally bounded in b-metric space \((X, \rho_c, K)\), then for any \(r > 0\), there exist \(x_1, \ldots, x_n \in A\) such that \(A \subset B(x_1, r) \cup \cdots \cup B(x_n, r)\). We also know that for any \(c_1 \gg \theta\), there exists an \(m \in \mathbb{N}\) such that \(\frac{c_1}{m} \ll c_1\). So we have that there exist \(x_1, \ldots, x_n \in A\) such that

\[
A \subset B(x_1, \frac{1}{m}) \cup \cdots \cup B(x_n, \frac{1}{m}) \subset \hat{B}(x_1, c_1) \cup \cdots \cup \hat{B}(x_n, c_1).
\]

It implies that A is cone-totally bounded.

(5) It is obvious from Theorem 2.13.

(6) It is obvious from (2), (3).

\(\square\)

**Corollary 3.2.** Let \((X, d, K)\) be a cone b-metric space over the ordered Banach space \((E, \leq)\) with respect to cone \(P\) and \(A \subset X\), \(c \in \text{intP}\):

1. A is cone-closed, if and only if for any sequence \(\{x_n\}\) in X which cone-converges to \(x\), we have \(x \in A\).
2. A is cone-closed, if and only if \(A^c\) is cone-open where \(A^c\) is the complement of A in X.
3. If \(x \in \overset{\sim}{}A\), then for any \(c_1 \gg \theta\), \(\hat{B}(x, c_1) \cap A \neq \emptyset\).
4. A is cone-compact, if and only if A is cone-sequentially compact.
5. \((X, d, K)\) is complete, if and only if \((X, \rho_c, K)\) is complete.
6. If \((X, d, K)\) is complete, then A is cone-relatively compact, if and only if A is cone-totally bounded.

**Proof.** (1), (2), (4), (6) are obvious from Theorem 3.1, Theorem 2.13, Theorem 2.12, Theorem 2.4, Proposition 2.3. To get (5), it is sufficient to show that \(\{x_n\} \subset X\), \(x_n\) is cone-Cauchy if and only if \(x_n\) is Cauchy in b-metric space \((X, \rho_c, K)\). Assume first \(\{x_n\}\) is cone-Cauchy, then for any \(c_1 \gg \theta\), there exists a \(k \in \mathbb{N}\) such that \(d(x_n, x_m) \ll c_1\) for all \(n, m > k\). So for any \(r > 0\), there exists a \(k \in \mathbb{N}\) such that \(d(x_n, x_m) \ll rc\) for all \(n, m > k\). We also have that there exists a \(j \in \mathbb{N}\) that \(d(x_n, x_m) \ll (1 - \frac{1}{r})rc\). It
imply that $\rho_c(x_n, x_m) \subseteq (1 - \frac{1}{j})r < r$ for all $n, m > k$. So $\{x_n\}$ is Cauchy in b-metric space $(X, \rho_c, K)$. Conversely assume that $\{x_n\}$ is Cauchy in b-metric space $(X, \rho_c, K)$, then for any $r > 0$, there exists a $k \in \mathbb{N}$ such that $\rho_c(x_n, x_m) < r$ for all $n, m > k$. For any $c_1 > 0$, there also exists a $j \in \mathbb{N}$ such that $\frac{1}{j} < c_1$. So there exists a $k \in \mathbb{N}$ such that $\rho_c(x_n, x_m) < \frac{1}{j}$ for all $m, n > k$. It implies that $d(x_m, x_n) \leq \rho_c(x_m, x_n)c \leq \frac{1}{j} \leq c_1$ for all $n, m > k$. So $\{x_n\}$ is Cauchy.

**Remark 3.3.** In [7], they obtained Corollary 3.2 (1), (2), (3), (4) (see [7, Proposition 3.2, Proposition 3.6, Theorem 3.7, Theorem 3.9]). But our proof is completely different. And furthermore, we get an in-depth result, that is we can equate the cone b-metric space with the b-metric space, if we only discuss the topological properties.

**Lemma 3.4 ([1]).** Let $T$ and $f$ be weakly compatible self-mappings of a set $X$. If $T$ and $f$ have a unique point of coincidence $\xi \in X$, then $\xi$ is a unique common fixed point of $T$ and $f$.

**Theorem 3.5.** Let $(X, d, K)$ be a cone b-metric space over an ordered Banach space $(E, \preceq)$ with respect to cone $P$, and let two mappings $T, f$ be self-mappings of $X$ such that $TX \subset fX$ and $TX$ or $fX$ is a complete subspace of $X$ satisfying

$$d(Tx, Ty) \leq \Lambda(d(fx, fy), d(fx, Ty), d(fx, Tx), d(fy, Ty), d(fy, Tx)), $$

where $\Lambda : E \rightarrow E$ is a positive linear operator and $r(K\Lambda) < 1$. Then $T, f$ have a unique point of coincidence $\xi \in X$ and every $T$-f-sequence $(fx_n)_{n=0}^{\infty}$ converges to $\xi$. Moreover, if $T$ and $f$ are weakly compatible, then $\xi$ is a unique common fixed point of $T$ and $f$.

**Proof.** Since $fX \subset TX$, then for any $x_0 \in X$ there exists $T$-f-sequence $(fx_n)_{n=0}^{\infty}$. Now by induction, we show that

$$d(Tx_n, Tx_0) \leq (I - K\Lambda)^{-1}K\Lambda d(fx_0, fx_1), \quad \forall n \in \mathbb{N}. \quad (3.1)$$

If $n = 1$, then

$$d(Tx_1, Tx_0) \leq \Lambda[d(fx_1, fx_0), d(fx_1, Tx_0), d(fx_1, Tx_0), d(fx_0, Tx_0), d(fx_0, Tx_1)].$$

Note that $K \geq 1, r(K\Lambda) < 1$, $Tx_n = fx_{n+1}$, $n = 0, 1, 2 \cdots$.

When

$$d(Tx_1, Tx_0) \leq \Lambda[d(fx_1, fx_0), d(fx_1, Tx_0), d(fx_0, Tx_0)],$$

clearly $(3.1)$ holds.

When $d(Tx_1, Tx_0) \leq \Lambda d(fx_1, Tx_1) = \Lambda d(Tx_0, Tx_1)$, $(3.1)$ also holds.

When $d(Tx_1, Tx_0) \leq \Lambda d(fx_0, Tx_1)$ and using the triangle inequality,

$$d(Tx_1, Tx_0) \leq K\Lambda d(fx_0, fx_1) + K\Lambda d(fx_1, Tx_1),$$

$$(I - K\Lambda)d(Tx_1, Tx_0) \leq K\Lambda d(fx_0, fx_1).$$

Bearing in mind that $I - K\Lambda$ is invertible and positive, we have that

$$d(Tx_1, Tx_0) \leq (I - K\Lambda)^{-1}K\Lambda d(fx_0, fx_1).$$

The above discussion implies $(3.1)$ holds for $n = 1$.

Suppose $(3.1)$ holds for $m < n$. We show that $(3.1)$ holds for $n$. In fact

$$d(Tx_n, Tx_0) \leq \Lambda[d(fx_n, fx_0), d(fx_n, Tx_0)d(fx_n, Tx_0)d(fx_0, Tx_0)d(fx_0, Tx_n)].$$

We have to consider five different cases:

1. $d(Tx_n, Tx_0) \leq d(fx_n, fx_0)$. Using the triangle inequality,

$$d(Tx_n, Tx_0) \leq K\Lambda d(fx_n, Tx_0) + K\Lambda d(Tx_0, fx_0).$$

By assumption of the induction, we obtain that
\[ d(Tx_n, Tx_0) \leq (1 - K\Lambda)^{-1}(K\Lambda)^2d(fx_1, fx_0) + K\Lambda d(fx_1, fx_0) \]
\[ = (1 - K\Lambda)^{-1}K\Lambda d(fx_1, fx_0). \]

2. \( d(Tx_n, Tx_0) \leq \Lambda d(fx_n, Tx_0) = \Lambda d(Tx_{n-1}, Tx_0), \) then \( (3.1) \) holds.

3. \( d(Tx_n, Tx_0) \leq \Lambda d(fx_0, Tx_0) = \Lambda d(fx_0, fx_1), \) then \( (3.1) \) also holds.

4. \( d(Tx_n, Tx_0) \leq \Lambda d(fx_0, Tx_0). \) Using the triangle inequality, then
\[ d(Tx_n, Tx_0) \leq K\Lambda d(fx_0, Tx_0) + K\Lambda d(Tx_0, Tx_n), \]
\[ d(Tx_n, Tx_0) \leq (1 - K\Lambda)^{-1}K\Lambda d(fx_0, fx_1), \]

5. \( d(Tx_n, Tx_0) \leq \Lambda d(fx_n, Tx_0) = \Lambda d(Tx_{n-1}, Tx_n). \) We have that
\[ d(Tx_{n-1}, Tx_n) \leq \Lambda d(fx_{n-1}, Tx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), d(fx_n, Tx_{n-1}). \]

If \( d(Tx_{n-1}, Tx_n) \leq \Lambda d(fx_n, Tx_n), d(fx_n, Tx_{n-1}), \) then \( d(Tx_{n-1}, Tx_n) = 0. \) If
\[ d(Tx_{n-1}, Tx_n) \leq \Lambda d(fx_{n-1},Tx_n), d(fx_n, Tx_n), d(fx_n, Tx_{n-1}), \]
by continuing this process, we see that there exist \( p, m \in N, \) \( p \geq n, \) \( 0 \leq m < n \) such that \( d(Tx_n, Tx_0) \leq \Lambda^p d(Tx_m, Tx_0). \) By assumption of the induction, we obtain that
\[ d(Tx_n, Tx_0) \leq (1 - K\Lambda)^{-1}(K\Lambda)^{p+1}d(fx_0, fx_1) \]
\[ \leq (1 - K\Lambda)^{-1}(K\Lambda)^{p+1}d(fx_0, fx_1). \]

It implies that \( d(Tx_n, Tx_0) \leq (1 - K\Lambda)^{-1}K\Lambda d(fx_0, fx_1). \)

Hence, using the method of the mathematical induction, we have proved that inequality \( (3.1) \) holds for each \( n \in N. \) Now we shall prove that \( (fx_n)_{n=0}^{\infty} \) is Cauchy sequence. For \( m, n \in N \) and \( m > n, \) there exist \( 0 \leq i \leq n + 1, \) \( 0 \leq j \leq m + 1 \) such that
\[ d(fx_{n+2}, fx_{m+2}) = d(Tx_{n+1}, Tx_{m+1}) \leq \Lambda^{n+1}d(Tx_i, Tx_j) \]
\[ \leq \Lambda^{n+1}[Kd(Tx_i, Tx_0) + Kd(Tx_0, Tx_j)] \]
\[ \leq \Lambda^{n+2}[2(1 - K\Lambda)^{-1}K^2d(fx_i, fx_j)]. \]

We conclude that \( (fx_n)_{n=0}^{\infty} \) is Cauchy sequence. Let \( \lim_{n \to \infty} fx_n = x. \) Since \( fX \subset X \) and \( fX \) is complete subspace of \( X, \) then there exists \( x \in X \) such that \( fx = \xi. \) We shall show that \( \xi \) is a unique point of coincidence of \( T \) and \( f. \) Firstly, we prove the uniqueness. Let \( \xi_1, \xi_2 \) be point of coincidence of \( T \) and \( f, \) then there exist \( y_1, y_2 \in X \) such that \( Ty_1 = fy_1 = \xi_1, \) \( Ty_2 = fy_2 = \xi_2. \) Since
\[ d(T\xi_1, T\xi_2) \leq \Lambda d(f\xi_1, f\xi_2), d(f\xi_1, T\xi_2), d(f\xi_1, T\xi_1), d(f\xi_2, T\xi_2), d(f\xi_2, T\xi_1), \]
then \( \xi_1 = \xi_2. \) Secondly, we prove that \( \xi \) is a point of coincidence of \( T \) and \( f. \) Any given \( c \gg \theta, \) \( p \in N, \) there exists \( m \in N \) such that
\[ d(fx_n, \xi) \leq \frac{c}{p}, \quad d(Tx_n, \xi) \leq \frac{c}{p}, \quad d(fx_n, Tx_n) \leq \frac{c}{p}, \quad \forall n > m. \]

Since
\[ d(Tx_{n+1},Tx) \leq \Lambda d(fx_{n+1},fx), d(fx_{n+1},Tx), d(fx_{n+1},Tx_{n+1}), d(fx,Tx), d(fx,Tx_{n+1}), \]
then
\[ d(Tx_{n+1}, Tx) \preceq \Lambda d(fx_{n+1}, fx) + \Lambda d(fx_{n+1}, Tx) \]
\[ \leq \Lambda d(fx_{n+1}, fx) + K \Lambda d(Tx_{n+1}, Tx), \]
or
\[ d(Tx_{n+1}, Tx) \leq \Lambda d(fx_{n+1}, Tx_{n+1}) + \Lambda d(fx, Tx) + \Lambda d(fx, Tx_{n+1}) \]
\[ \leq \Lambda d(fx_{n+1}, Tx_{n+1}) + K \Lambda d(fx, Tx_{n+1}) + K \Lambda d(Tx_{n+1}, Tx) + \Lambda d(fx, Tx_{n+1}). \]

It implies that
\[ d(Tx_{n+1}, Tx) \leq (I - K\Lambda)^{-1} \frac{\Lambda c}{p} + (I - K\Lambda)^{-1} \frac{K \Lambda c}{p}, \]

or
\[ d(Tx_{n+1}, Tx) \leq (I - K\Lambda)^{-1} \frac{\Lambda c}{p} + (I - K\Lambda)^{-1} \frac{K \Lambda c}{p} + (I - K\Lambda)^{-1} \frac{\Lambda c}{p}. \]

Let \( p \to \infty \), \( Tx_{n+1} \to Tx \). It implies that \( fx = Tx \). So we conclude that \( \xi \) is a point of coincidence of \( T \) and \( f \). Every \( T\)-f-sequence \( (fx_n)_{n=0}^\infty \) converges to \( \xi \), from the uniqueness of \( \xi \). The latter part of Theorem 3.5 follows from Lemma 3.4.

**Corollary 3.6.** Let \( K = 1, \Lambda = \lambda I, 0 \leq \lambda < 1 \), we obtain \([10, \text{Theorem 1.3}]\) from Theorem 3.5.

**Corollary 3.7.** Let \( K = 1, f = I_x \), where \( I_x \) is the identity mapping on \( X \), we obtain \([4, \text{Theorem 1.5}]\) from Theorem 3.5.

**Acknowledgment**

The authors sincerely thank the anonymous reviewers and editor for their valuable suggestions, which improved the presentation of this paper. This work is supported by the National Natural Science Foundation of China (Grant No.11401296 and 11071109), and the Jiangsu Provincial Natural Science Foundation of China (Grant No.BK20141008).

**References**


