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Topological structures and the coincidence point of two mappings in cone b-metric spaces

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Abstract

Let (X, d, K) be a cone b-metric space over a ordered Banach space $(E, \preceq)$ with respect to cone P. In this paper, we study two problems:

(1) We introduce a b-metric ρ_c and we prove that the b-metric space induced by b-metric ρ_c has the same topological structures with the cone b-metric space.

(2) We prove the existence of the coincidence point of two mappings T , $f : X \to X$ satisfying a new quasi-contraction of the type $d(Tx, Ty) \preceq \Lambda\{d(fx, fy), d(fx, Ty), d(fx, Tx), d(fy, Ty), d(fy, Tx)\}$, where $\Lambda : E \to E$ is a linear positive operator and the spectral radius of $K\Lambda$ is less than 1.

Our results are new and extend the recent results of [N. Hussain, M. H. Shah, Comput. Math. Appl., **62** (2011), 1677–1684], [M. Cvetković, V. Rakočević, Appl. Math. Comput., **237** (2014), 712–722], [Z. Kadelburg, S. Radenović, J. Nonlinear Sci. Appl., **3** (2010), 193–202].

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1. Introduction

Cone metric spaces were introduced in [6]. In [3], they introduced a special metric ρ_ξ and they proved that the metric space induced by the metric ρ_ξ have the same topological spaces with the cone metric space. In [7], Hussain and Shah introduced the concept of cone b-metric spaces and they investigated topological properties of the cone b-metric spaces. In fact, the class of cone b-metric spaces is effectively larger than that of the ordinary cone metric spaces. That is, every cone metric space is a cone b-metric space. In [5], Czerwik first introduced the concept of b-metric spaces. Similarly, b-metric spaces are extensions of metric spaces. In the first part of this work, we introduce a special b-metric ρ_c and proves that the b-metric space induced by ρ_c has the same topological structures with the cone b-metric space.

The second part of this work involves coincidence points and common fixed points. In 1976, Jungck [9] extended the celebrated Banach contraction mapping principle to the common fixed theorem of two

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commuting mappings. In this process, he introduced a new iteration process which was a generalization of the Picard iteration. The new iteration scheme can be defined as follows:

Definition 1.1 ([8]). Let T and f be self-mappings of a set X, and let $(x_n)_{n=0}^{\infty}$ be a sequence in X such that $fx_{n+1} = Tx_n, n = 0, 1, 2 \cdots$.

Then the sequence $(fx_n)_{n=0}^{\infty}$ is said to be a T-f-sequence or Jungck iteration.

Let f and T be self-mappings of nonempty set X, $x \in X$ is called a coincidence point of f and T if $fx = Tx$. A point $y \in X$ is called a point of coincidence of f and T if there exists a point $x \in X$ such that $y = fx = Tx$. A point $z \in X$ is called a common fixed point of f and T if $z = fz = Tz$.

Definition 1.2 ([12]). Let f and g be self-mappings of a nonempty set X. Then f and g are called weakly compatible, if they commute at their coincidence points.

Let $(Y, \preceq)$ be an ordered vector space, $x \in X$ and $A \subset X$. We say that $x \preceq A$, if there exists at least one vector $y \in A$ such that $x \preceq y$. In 2010, Kadelburg and Radenović obtained the following result by using jungck iteration.

Theorem 1.3 ([10]). *Let (X, d) be a cone metric space over a Banach space $(Y, \preceq)$. And let $T, f : X \to X$ be mappings such that $T(X) \subset f(X)$ and $f(X)$ be a complete subspace of X. Supposing there exists $\lambda \in [0, 1)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \preceq \lambda\{d(fx, fy), d(fx, Ty), d(fx, Tx), d(fy, Ty), d(fy, Tx)\}.$$

Then T and f have a unique point of coincidence. Moreover, if T and f are weakly compatible, then every T-f-sequence (fx_n) in X converges to the unique common fixed point of T and f.

In 2014, Cvetkovič and Rakočevič [4] introduced notion of quasi-contraction of Perov type and partly extended Kadelburg's theorems to positive linear functional.

Definition 1.4 ([4]). Let (X, d) be a cone metric space over a Banach space $(E, \preceq)$. A map $T : X \to X$ such that for some bounded linear operator $\Lambda : E \to E$ whose spectral radius is less than 1 and for each $x, y \in X$,

$$d(Tx, Ty) \preceq \Lambda\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

is called a quasi-contraction of Perov type.

Theorem 1.5 ([4]). *Let (X, d) be a complete cone metric space with respect to cone P. If a mapping $T : X \to X$ is a quasi-contraction of Perov type and $\Lambda(P) \subset P$, then f has a unique point $x^* \in X$ and, for any $x \in X$, the iterative sequence $(T^n x)_{n \in N}$ converges to the fixed point of T.*

In the second part of work, we study a new quasi-contraction, that is,

$$d(Tx, Ty) \preceq \Lambda\{d(fx, Tx), d(fx, Ty), d(fx, fy), d(fy, Ty), d(fy, Tx)\},$$

where $\Lambda : E \to E$ is a linear positive operator and the spectral radius of $K\Lambda$ is less than 1. Our results can be considered as a further development of [10, Theorem 1.3] and [4, Theorem 1.5].

2. Preliminary and auxiliary results

In this section, we recall and provide some concepts and auxiliary results.

2.1. b-metric space

Definition 2.1 ([5]). Let X be a nonempty set, $K \geqslant 1$ and $D : X \times X \to [0, +\infty)$ is a function such that for all $x, y, z \in X$

(1) $D(x, y) = 0$ if and only if $x = y$;

(2) $D(x,y) = D(y,x)$;

(3) $D(x,z) \leqslant K[D(x,y) + D(y,z)]$.

Then D is called a b-metric, and (X, D, K) is called a b-metric space.

In b-metric spaces (X, D, K), the sequence $\{x_n\}$ converges to $x \in X$, if and only if $\lim_{n\to\infty} D(x_n, x) = 0$ and the sequence $\{x_n\}$ is Cauchy, if and only if $\lim_{n,m\to\infty} D(x_n, x_m) = 0$. (X, D, K) is complete if and only if any Cauchy sequence in X is convergent. $B(a, \varepsilon)$ denotes the subset $\{x \in X : D(x, a) < \varepsilon\}$ of X, $a \in X$, $\varepsilon > 0$.

Definition 2.2 ([11]). Let (X, D, K) be a b-metric space.

(1) A subset $A \subset X$ is said to be open, if and only if for any $a \in A$, there exists $\epsilon > 0$ such that $B(a, \varepsilon) \subset A$.

(2) Let B be a subset of X. An element $x \in X$ is called a limit point of B, whenever for any $\epsilon > 0$,

$$B(x, \varepsilon) \cap (B\backslash\{x\}) \neq \phi.$$

B is called closed, whenever each limit point of B belongs to B.

(3) A subset $B \subset X$ is called bounded whenever, there exists $\epsilon > 0$ such that $D(x, y) < \epsilon$ for all $x, y \in B$.

(4) A subset $B \subset X$ is called compact, whenever every open cover of B has a finite subcover.

(5) A subset B is called sequentially compact, if and only if for any sequence $\{x_n\}$ in B, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges, and $\lim_{k\to\infty} x_{n_k} \in B$.

(6) A subset B is called totally bounded, if and only if for any $\epsilon > 0$, there exist $x_1, x_2, x_3, \cdots, x_n \in B$ such that

$$B \subset B(x_1, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon).$$

Proposition 2.3 ([11]). *Let (X, D, K) be a b-metric space,*

(1) *A is closed, if and only if for any sequence $\{x_n\}$ in X which converges to x, we have $x \in A$.*

(2) *If we let $\overline{A}$ denote the intersection of all closed subset of X which contains A, then for any $x \in \overline{A}$ and for any $\epsilon > 0$, we have $B(x, \varepsilon) \cap A \neq \phi$.*

(3) *A is compact, if and only if A is sequentially compact.*

(4) *If A is compact, then A is totally bounded.*

Theorem 2.4. *Let (X, D, K) be a b-metric space,*

(1) *A is closed, if and only if A^c is open, where A^c is the complement of A in X.*

(2) *A is called relatively compact, whenever $\overline{A}$ is compact. If (X, D, K) is complete, then A is relatively compact, if and only if A is totally bounded.*

Proof.

(1) Firstly assume that A is closed. We show that A^c is open. If A^c is not open, then

$$\exists a \in A^c, \ \ \forall n \in N, \ \ \exists x_n \in A \quad \text{such that} \quad D(a, x_n) < \frac{1}{n}.$$

It contradict Proposition 2.3 (1). Conversely, assume that A^c is open, we show that A is closed. If $x \notin A$, there exists $\epsilon > 0$ such that $B(x, \varepsilon) \subset A^c$. Clearly $B(x, \varepsilon) \cap (A\backslash\{x\}) = \phi$, it implies that A is closed.

(2) We start with that A is relatively compact. So we have that $\overline{A}$ is totally bounded,

$$\forall \varepsilon > 0, \quad \exists x_1, x_2, \cdots, x_n \in \overline{A}, \quad \overline{A} \subset B(x_1, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon).$$

So there exist $y_i \subset B(x_i, \varepsilon) \cap A$, $i = 1, 2, \cdots, n$. Then $A \subset \overline{A} \subset B(y_1, 2K\varepsilon) \cup \cdots \cup B(y_n, 2K\varepsilon)$. It implies that A is totally bounded. Conversely, assume that A is totally bounded, then we show that A is relatively compact. If $\{x_n\} \subset \overline{A}$, then there exists y_n such that $y_n \subset A \cap B(x_n, \frac{1}{n})$ for all $n \in N$, from Proposition 2.3 (2). Clearly,

$$\exists a_1 \in A, \quad \{y_{n_k}^{(1)}\} \subset \{y_n\} \subset A, \quad \{y_{n_k}^{(1)}\} \subset B(a_1, \frac{1}{1}),$$

where $\{y_{n_k}^{(1)}\}$ is the subsequence of $\{y_n\}$, from that A is totally bounded. Similarly,

$$\exists a_n \in A, \quad \{y_{n_k}^{(n)}\} \subset \{y_{n_k}^{(n-1)}\} \subset A, \quad \{y_{n_k}^{(n)}\} \subset B(a_n, \frac{1}{n}), \quad n \in N, \quad n \geqslant 2,$$

where $\{y_{n_k}^{(n)}\}$ is the subsequence of $\{y_{n_k}^{(n-1)}\}$. So we can select y_{n_m} such that $y_{n_m} \in \left\{y_{n_k}^{(m)}\right\}$ and $\{y_{n_m}\}$ is the subsequence of $\{y_n\}$. Since $y_{n_m} \subset B(a_l, \frac{1}{l}), m \geqslant l$, then

$$D(y_{n_m}, y_{n_l}) \leqslant K[D(a_l, y_{n_m}) + D(a_l, y_{n_l})] \leqslant \frac{2K}{l}, \quad m \geqslant l.$$

It implies that y_{n_m} is Cauchy. Since (X, D, K) is complete, there exists $x \in \overline{A}$ and $\lim_{m\to\infty} y_{n_m} = x$ from Proposition 2.3 (1) (2). It is easy to check that $\lim_{m\to\infty} x_{n_m} = x$. It implies that $\overline{A}$ is sequentially compact and we have that $\overline{A}$ is compact from Proposition 2.3 (3). □

2.2. *Cone b-metric space*

Let E be a real Banach space. A subset P of E is called a cone whenever the following condition is satisfied:

(1) P is closed, nonempty and $P \neq \{\theta\}$, where θ is the zero vector in E.

(2) $a, b \in R$, $a, b \geqslant 0$ and $x, y \in P$ imply $ax + by \in P$.

(3) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$, if and only if $y - x \in P$. We shall write $\prec$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \text{int}P$ (interior of P).

Definition 2.5 ([7]). Let X be a nonempty set and $(E, \preceq)$ an ordered Banach space with respect to cone P. A vector-valued function $d : X \times X \to E$ is said to be a cone b-metric function on X with the constant $K \geqslant 1$, if the following conditions are satisfied:

(1) $\theta \preceq d(x, y)$, for all $x, y \in X$, and $d(x, y) = \theta$, if and only if $x = y$;

(2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(3) $d(x, y) \preceq K[d(x, z) + d(y, z)]$, for all $x, y, z \in X$.

The pair (X, d, K) is called the cone b-metric space over an ordered Banach space $(E, \preceq)$ with respect to cone P.

Definition 2.6 ([7]). Let (X, d, K) be a cone b-metric space over the ordered Banach space $(E, \preceq)$ with respect to cone P. We say that $\{x_n\} \subset X$ is:

(1) a cone-Cauchy, if for every $\theta \ll c$, there is a $k \in N$ such that for all $n, m > k$, $d(x_n, x_m) \ll c$;

(2) cone-convergent sequence, if for every $\theta \ll c$, there is an $m \in N$ such that for all $n > m$, $d(x_n, x) \ll c$, while we write $\text{cone} * \lim\limits_{n\to\infty} x_n = x$.

We say that the cone b-metric space (X, d, K) is complete, if any cone-Cauchy is cone-convergent. Let $A \subset X$, we claim that A is a complete subspace, if for every cone-Cauchy $\{x_n\} \subset A$, $\text{cone} * \lim\limits_{n\to\infty} x_n \in A$.

We claim that $\{y_n\} \subset E$ norm-converges to y, if for any $\epsilon > 0$, there exists an $m \in N$ such that $\|y_n - y\| < \varepsilon$, for all $n > m$. Noting that if $\{x_n\} \subset X$, $\{y_n\} \subset E$, y_n norm-converges to θ and $d(x_n, x_m) \preceq y_n$, for all $n, m \in N$, $m > n$, then $\{x_n\}$ is cone-Cauchy. We denote by $\hat{B}(x, c)$ the cone-ball, given by $\hat{B}(x, c) = \{y \in X : d(x, y) \ll c\}$, $c \in \text{int}P, x \in X$.

Definition 2.7 ([7]). Let (X, d, K) be a cone b-metric space over the ordered Banach space $(E, \preceq)$ with respect to cone P,

(1) A subset $A \subset X$ is said to be cone-open, if and only if for any $a \in A$, there exists $c \gg \theta$ such that the cone-ball $\hat{B}(a, c) \subset A$.

(2) An element $x \in X$ is called a cone-limit point of B whenever for any $c \gg \theta$, $\hat{B}(x, c) \backslash (B \backslash \{x\}) \neq \phi$. A subset $B \subset X$ is called cone-closed, whenever each cone-limit point of B belongs to B.

(3) A subset $B \subset X$ is called cone-bounded, whenever there exists $c \gg \theta$ such that $d(x, y) \ll c$ for all $x, y \in B$.

(4) A subset $B \subset X$ is called cone-compact, whenever every cone-open cover of B has a finite subcover.

(5) A subset B is called cone-sequentially compact, if and only if for any sequence $\{x_n\}$ in B, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which cone-converges, and $\text{cone} * \lim\limits_{k\to\infty} x_{n_k} \in B$.

(6) A subset B is called cone-totally bounded, if and only if for any $c \gg \theta$, there exist $x_1, x_2, x_3, \cdots, x_n \in B$ such that $B \subset \hat{B}(x_1, c) \cup \cdots \cup \hat{B}(x_n, c)$.

Let $A \subset X$, $\overset{\approx}{A}$ stands for the intersection of all cone-closed subsets of X including A. We claim that A is cone-relatively compact, if $\overset{\approx}{A}$ is cone-compact.

Proposition 2.8. *Let $(E, \preceq)$ be an ordered Banach space with respect to cone P. Then the following properties are often used:*

(1) $x, y, z \in E, x \preceq y \ll z$ *imply* $x \ll z$.

(2) $\alpha \text{int}P \subset \text{int}P$, *for all* $\alpha \in R$, $\alpha > 0$.

(3) *For any* $c \in \text{int}P, x \in E$, *there exists an* $n \in N$ *such that* $\frac{x}{n} \ll c$.

(4) *If* $a \in P$, $0 \leqslant \lambda < 1$ *and* $a \preceq \lambda a$, *then* $a = \theta$.

(5) $c \in \text{int}P$, $\alpha, \beta \in R$, $\alpha > \beta$ *imply* $\beta c \ll \alpha c$.

Lemma 2.9. *Let $(E, \preceq)$ be an ordered Banach space with respect to cone P. If $x \ll y$, then exists $n \in N$ such that $x \ll (1 - \frac{1}{n})y$.*

Proof. Let $\overset{\vee}{B}(x, \varepsilon) = \{y \in E : \|y - x\| < \varepsilon\}, x \in E, \varepsilon > 0$. Since $x \ll y$, then exists $\varepsilon > 0$ such that $y - x + \overset{\vee}{B}(\theta, \varepsilon) \subset P$. Clearly $\overset{\vee}{B}(\theta, \frac{\varepsilon}{2}) + \overset{\vee}{B}(\theta, \frac{\varepsilon}{2}) \subset \overset{\vee}{B}(\theta, \varepsilon)$ from the triangle inequality of the norm. We know that there exists $n \in N$ such that $-\frac{y}{n} \in \overset{\vee}{B}(\theta, \frac{\varepsilon}{2})$. So $(1 - \frac{1}{n})y - x + \overset{\vee}{B}(\theta, \frac{\varepsilon}{2}) \subset P$. It implies that $x \ll (1 - \frac{1}{n})y$. □

2.3. The b*-metric* ρ_c

Let (X, d, K) be a cone b-metric space over an ordered Banach space $(E, \preceq)$ with respect to cone P. Since P is closed, we have that the cone P is Archimedean (see [2, page 63, lemma 2.3]). Given $a, b \in E$ with $a \preceq b$, we denote by $[a, b]$ the order interval, i.e.,

$$[a, b] = \{x \in X : a \preceq x \preceq b\}.$$

Let $c \in \text{int}P$, then $E = \bigcup_{n \in N} [-c, c]$. We can define the Minkowski functional on E by setting

$$\|x\|_c = \inf\{\lambda > 0 : x \in [-\lambda c, \lambda c]\},$$

for all $x \in E$. And furthermore, we have that $-\|x\|_c c \preceq x \preceq \|x\|_c c$ (see [2, page 104]).

Proposition 2.10 ([8]). *Let* $(E, \preceq)$ *be an ordered Banach space with respect to cone* P*.*

(1) $x, y \in E$, $\theta \preceq x \preceq y$ *imply* $\|x\|_c \leqslant \|y\|_c$.

(2) $x, y \in E$, $\|x + y\|_c \leqslant \|x\|_c + \|y\|_c$.

(3) $x \in E$, $\lambda \in R$, $\lambda \geqslant 0$, $\|\lambda x\|_c = \lambda \|x\|_c$.

Now, we define b-metric ρ_c by setting $\rho_c = \|d(x, y)\|_c$.

Proposition 2.11. (X, ρ_c, K) *is a* b*-metric space.*

Proof. It is easy to check that ρ_c is a b-metric from Proposition 2.10. □

We define $B(x, \varepsilon) = \{y \in X : \rho_c(x, y) < \varepsilon\}$, $x \in X$, $\varepsilon > 0$ and we claim that $\rho_c * \lim_{n\to\infty} x_n = x, \{x_n\} \subset X$, if $\lim_{n\to\infty} \rho_c(x_n, x) = 0$. Now we prove some basic results.

Theorem 2.12. *Let* (X, d, K) *be a cone* b*-metric space over the ordered Banach space* $(E, \preceq)$ *with respect to cone* P*.* $\hat{B}(x, rc) = B(x, r)$*, for all* $x \in X$, $r \in R$, $r > 0$, $c \gg \theta$.

Proof. Let $y \in \hat{B}(x, rc)$, then $d(x, y) \ll rc$. There exists an $n \in N$ such that $d(x, y) \ll (1 - \frac{1}{n})rc$, from Lemma 2.9. It implies that

$$-(1 - \frac{1}{n})rc \preceq \theta \preceq d(x, y) \preceq (1 - \frac{1}{n})rc,$$

so we have that $\rho_c(x, y) = \|d(x, y)\|_c \leqslant (1 - \frac{1}{n})r < r$ from the definition of $\|\|_c$. It implies that $y \in B(x, r)$. Conversely, let $y \in B(x, r)$, then $\rho_c(x, y) < r$. So $d(x, y) \preceq \rho_c(x, y)c \ll rc$. It implies that $y \in \hat{B}(x, rc)$. □

Theorem 2.13. *Let* (X, d, K) *be a cone* b*-metric space over the ordered Banach space* $(E, \preceq)$ *with respect to cone* P *and* $c \in \text{int}P$*.* $\text{cone} * \lim_{n\to\infty} x_n = x$*, if and only if* $\rho_c * \lim_{n\to\infty} x_n = x$, $\{x_n\} \subset X$.

Proof. Assume that $\text{cone} * \lim_{n\to\infty} x_n = x$, we have that for any $c_1 \gg \theta$ there exists an $m \in N$ such that $x_n \in \hat{B}(x, c_1)$ for all $n > m$. Since $\hat{B}(x, rc) = B(x, r)$ for all $x \in X$, $r \in R$, $r > 0$, $c \gg \theta$, then for any $r > 0$, there exists an $m \in N$ such that $x_n \in \hat{B}(x, rc) = B(x, r)$ for all $n > m$. It implies that $\rho_c * \lim_{n\to\infty} x_n = x$.

Conversely, assume that $\rho_c * \lim_{n\to\infty} x_n = x$, then for any $r > 0$, there exists an $m \in N$ such that $x_n \in B(x, r)$ for all $n > m$. We also have that for any $c_1 \gg \theta$ there exists a $k \in N$ such that $\frac{c}{k} \ll c_1$. So there exists an $m \in N$ such that $x_n \in B(x, \frac{1}{k}) = \hat{B}(x, \frac{c}{k}) \subset \hat{B}(x, c_1)$ for all $n > m$. It implies that $\text{cone} * \lim_{n\to\infty} x_n = x$. □

2.4. The linear positive operator

Let (X, d, K) be a cone b-metric space over the ordered Banach space $(E, \preceq)$ with respect to cone P. We say that $\Lambda : E \to E$ is a linear positive operator, if Λ is a linear operator and $\Lambda(P) \subset P$. Clearly Λ is a linear positive operator, if and only if Λ is a linear operator and $\Lambda(x) \preceq \Lambda(y)$ for all $x, y \in E, x \preceq y$. In fact, if Λ is a linear positive operator, then Λ is continuous (see [2, page 84]). And furthermore, if $\Lambda : E \to E$ is a linear continuous operator and there exists an $m \in N$ such that $\|\Lambda^m\| < 1$, then $\Lambda^n x$ norm-converges to θ for any $x \in E$ and $I - \Lambda$ is invertible where I is the identity mapping of E, that is, $(I-\Lambda)^{-1} = \sum_{n=0}^{\infty} \Lambda^n$. Of course there exists an $m \in N$ such that $\|\Lambda^m\| < 1$, if $\Lambda : E \to E$ is a linear continuous operator and it's spectral radius is less than one. It is inspired by Huang and Zhang [6], we say that $P * \lim_{n\to\infty} x_n = x, \{x_n\} \subset E$, if for any $c \gg \theta$, there exists an $m \in N$ such that $-c \ll x_n - x \ll c$ for all $n > m$.

Proposition 2.14. *Let (X, d, K) be a cone b-metric space over the ordered Banach space $(E, \preceq)$ with respect to cone P and $\Lambda : E \to E$ is a linear positive operator.*

(1) $x, y \in E$, $\{x_n\} \subset E$, $P * \lim_{n\to\infty} x_n = x$, $y \preceq x_n$ *for all* n, *then* $y \preceq x$.

(2) $x \in X, \{x_n\} \subset X$, *then* $\mathrm{cone} * \lim_{n\to\infty} = x$, *if and only if* $P * \lim_{n\to\infty} d(x_n, x) = \theta$.

(3) $x, y \in E$, $\{x_n\} \subset E$, $P * \lim_{n\to\infty} x_n = x$, $y \preceq x_n$ *for all* n, *then* $\Lambda(y) \preceq \Lambda(x)$.

(4) *If* $\{x_n\} \subset E$, $P * \lim_{n\to\infty} x_n = x$, *then* $P * \lim_{n\to\infty} \Lambda x_n = \Lambda x$.

Proof.

(1) For any $n \in N$, there exists an $m \in N$ such that $x - x_m \gg -\frac{c}{n}$. So we have that

$$x = x - x_m + x_m \succeq -\frac{c}{n} + y, \quad \forall n \in N.$$

Let $n \to \infty$, we obtain $x \succeq y$ from P is closed in E.

(2) It is obvious.

(3) For any $n \in N$ and $c \in \mathrm{int}P$, there exists an $m \in N$ such that $x - x_m \gg -\frac{c}{n}$. So we have that $\Lambda(x) = \Lambda(x - x_m) + \Lambda(x_m) \succeq -\frac{\Lambda(c)}{n} + \Lambda(y)$ for any $n \in N$. Let $n \to \infty$, we obtain $\Lambda(x) \succeq \Lambda(y)$ from P is closed in E.

(4) For any $c \gg \theta$, there exists a $j \in N$ such that $\frac{\Lambda c}{j} \ll c$. Since $P * \lim_{n\to\infty} x_n = x$, there exists an $m \in N$ such that $-\frac{c}{j} \ll x_n - x \ll \frac{c}{j}$ for all $n > m$. It implies that $-c \ll -\frac{\Lambda c}{j} \preceq \Lambda x_n - \Lambda x \preceq \frac{\Lambda c}{j} \ll c$ for all $n > m$. So we have that $P * \lim_{n\to\infty} \Lambda x_n = \Lambda x$. □

3. Main results

Theorem 3.1. *Let (X, d, K) be a cone b-metric space over the ordered Banach space $(E, \preceq)$ with respect to cone P and $A \subset X$, $c \in \mathrm{int}P$:*

(1) *A is cone-open, if and only if A is open in b-metric space (X, ρ_c, K).*

(2) *A is cone-closed, if and only if A is closed in b-metric space (X, ρ_c, K).*

(3) *A is cone-compact, if and only if A is compact in b-metric space (X, ρ_c, K).*

(4) *A is cone-totally bounded, if and only if A is totally bounded in b-metric space (X, ρ_c, K).*

(5) *A is cone-sequentially compact, if and only if A is sequentially compact in b-metric space (X, ρ_c, K).*

(6) *A is cone-relatively compact, if and only if A is relatively compact in b-metric space (X, ρ_c, K).*

Proof.

(1) Assume that A is cone-open. Then for any $a \in A$, there exists a $c_1 \gg \theta$ such that $\hat{B}(a, c_1) \subset A$. There also exists an $n \in N$ such that $\frac{c}{n} \ll c_1$. So we have that $B(a, \frac{1}{n}) = \hat{B}(a, \frac{c}{n}) \subset \hat{B}(a, c_1)$ from Theorem 2.12. It implies that A is open in b-metric space (X, ρ_c, K).

Conversely, assume that A is open in b-metric space (X, ρ_c, K). Then for any $a \in A$, there exists an $r > 0$ such that $B(a, r) = \hat{B}(a, rc) \subset A$. It implies that A is cone-open.

(2) To prove the result, it is sufficient to show that $a \in X$ is the cone-limit point of A, if and only if $a \in X$ is the limit point of A in the b-metric space (X, ρ_c, K). In fact, for any $c_1 \gg \theta$, $\hat{B}(a, c_1) \cap (A \backslash \{x\}) \neq \phi$, if and only if for any $r > 0$, $B(a, r) \cap (A \backslash \{x\}) \neq \phi$. So we complete the proof.

(3) It is obvious from (1).

(4) Assume first that A is cone-totally bounded, then for any $c_1 \gg \theta$, there exist $x_1, \cdots, x_n \in A$ such that $A \subset \hat{B}(x_1, c_1) \cup \cdots \cup \hat{B}(x_n, c_1)$. So for any $r > 0$ there exist $x_1, \cdots, x_n \in A$ such that

$$A \subset \hat{B}(x_1, rc) \cup \cdots \hat{B}(x_n, rc) = B(x_1, r) \cup \cdots B(x_n, r).$$

It implies that A is totally bounded in b-metric space (X, ρ_c, K).

Conversely, assume that A is totally bounded in b-metric space (X, ρ_c, K), then for any $r > 0$, there exist $x_1, \cdots, x_n \in A$ such that $A \subset B(x_1, r) \cup \cdots B(x_n, r)$. We also know that for any $c_1 \gg \theta$, there exists an $m \in N$ such that $\frac{c}{m} \ll c_1$. So we have that there exist $x_1, \cdots, x_n \in A$ such that

$$A \subset B(x_1, \frac{1}{m}) \cup \cdots B(x_n, \frac{1}{m}) \subset \hat{B}(x_1, c_1) \cup \cdots \cup \hat{B}(x_n, c_1).$$

It implies that A is cone-totally bounded.

(5) It is obvious from Theorem 2.13.

(6) It is obvious from (2), (3). □

Corollary 3.2. *Let (X, d, K) be a cone b-metric space over the ordered Banach space $(E, \preceq)$ with respect to cone P and $A \subset X$, $c \in \mathrm{int}P$:*

(1) *A is cone-closed, if and only if for any sequence $\{x_n\}$ in X which cone-converges to x, we have $x \in A$.*

(2) *A is cone-closed, if and only if A^c is cone-open where A^c is the complement of A in X.*

(3) *If $x \in \overset{\simeq}{A}$, then for any $c_1 \gg \theta$, $\hat{B}(x, c_1) \cap A \neq \phi$.*

(4) *A is cone-compact, if and only if A is cone-sequentially compact.*

(5) *(X, d, K) is complete, if and only if (X, ρ_c, K) is complete.*

(6) *If (X, d, K) is complete, then A is cone-relatively compact, if and only if A is cone-totally bounded.*

Proof. (1), (2), (3), (4), (6) are obvious from Theorem 3.1, Theorem 2.13, Theorem 2.12, Theorem 2.4, Proposition 2.3. To get (5), it is sufficient to show that $\{x_n\} \subset X$, x_n is cone-Cauchy if and only if x_n is Cauchy in b-metric space (X, ρ_c, K). Assume first $\{x_n\}$ is cone-Cauchy, then for any $c_1 \gg \theta$, there exists a $k \in N$ such that $d(x_n, x_m) \ll c_1$ for all $n, m > k$. So for any $r > 0$, there exists a $k \in N$ such that $d(x_n, x_m) \ll rc$ for all $n, m > k$. We also have that there exists a $j \in N$ that $d(x_n, x_m) \ll (1 - \frac{1}{j})rc$. It implies that $\rho_c(x_n, x_m) \leqslant (1-\frac{1}{j})r < r$ for all $n, m > k$. So $\{x_n\}$ is Cauchy in b-metric space (X, ρ_c, K). Conversely assume that $\{x_n\}$ is Cauchy in b-metric space (X, ρ_c, K), then for any $r > 0$, there exists a $k \in \mathbb{N}$ such that $\rho_c(x_n, x_m) < r$ for all $n, m > k$. For any $c_1 \gg \theta$, there also exists a $j \in \mathbb{N}$ such that $\frac{c}{j} \ll c_1$. So there exists a $k \in \mathbb{N}$ such that $\rho_c(x_m, x_n) < \frac{1}{j}$ for all $m, n > k$. It implies that $d(x_m, x_n) \preceq \rho_c(x_m, x_n)c \ll \frac{c}{j} \ll c_1$ for all $n, m > k$. So $\{x_n\}$ is cone-Cauchy. □

Remark 3.3. In [7], they obtained Corollary 3.2 (1), (2), (3), (4) (see [7, Proposition 3.2, Proposition 3.6, Theorem 3.7, Theorem 3.9]). But our proof is completely different. And furthermore, we get an in-depth result, that is we can equate the cone b-metric space with the b-metric space, if we only discuss the topological properties.

Lemma 3.4 ([1]). *Let T and f be weakly compatible self-mappings of a set X. If T and f have a unique point of coincidence $\xi \in X$, then ξ is a unique common fixed point of T and f.*

Theorem 3.5. *Let (X, d, K) be a cone b-metric space over an ordered Banach space $(E, \preceq)$ with respect to cone P, and let two mappings T, f be self-mappings of X such that $TX \subset fX$ and TX or fX is a complete subspace of X satisfying*

$$d(Tx, Ty) \preceq \Lambda\{d(fx, fy), d(fx, Ty), d(fx, Tx), d(fy, Ty), d(fy, Tx)\},$$

where $\Lambda : E \to E$ is a positive linear operator and $r(K\Lambda) < 1$. Then T, f have a unique point of coincidence $\xi \in X$ and every T-f-sequence $(fx_n)_{n=0}^{\infty}$ converges to ξ. Moreover, if T and f are weakly compatible, then ξ is a unique common fixed point of T and f.

Proof. Since $fX \subset TX$, then for any $x_0 \in X$ there exists T-f-sequence $(fx_n)_{n=0}^{\infty}$. Now by induction, we show that

$$d(Tx_n, Tx_0) \preceq (I - K\Lambda)^{-1}K\Lambda d(fx_0, fx_1), \quad \forall\, n \in \mathbb{N}. \tag{3.1}$$

If $n = 1$, then

$$d(Tx_1, Tx_0) \preceq \Lambda\{d(fx_1, fx_0), d(fx_1, Tx_0), d(fx_1, Tx_1), d(fx_0, Tx_0), d(fx_0, Tx_1)\}.$$

Note that $K \geqslant 1, r(K\Lambda) < 1,\ Tx_n = fx_{n+1},\ n = 0, 1, 2 \cdots$.

When

$$d(Tx_1, Tx_0) \preceq \Lambda\{d(fx_1, fx_0), d(fx_1, Tx_0), d(fx_0, Tx_0)\},$$

clearly (3.1) holds.

When $d(Tx_1, Tx_0) \preceq \Lambda d(fx_1, Tx_1) = \Lambda d(Tx_0, Tx_1)$, (3.1) also holds.

When $d(Tx_1, Tx_0) \preceq \Lambda d(fx_0, Tx_1)$ and using the triangle inequality,

$$d(Tx_1, Tx_0) \preceq K\Lambda d(fx_0, fx_1) + K\Lambda d(fx_1, Tx_1),$$

$$(I - K\Lambda)d(Tx_1, Tx_0) \preceq K\Lambda d(fx_0, fx_1).$$

Bearing in mind that $I - K\Lambda$ is invertible and positive, we have that

$$d(Tx_1, Tx_0) \preceq (I - K\Lambda)^{-1}K\Lambda d(fx_0, fx_1).$$

The above discussion implies (3.1) holds for $n = 1$.

Suppose (3.1) holds for $m < n$. We show that (3.1) holds for n. In fact

$$d(Tx_n, Tx_0) \preceq \Lambda\{d(fx_n, fx_0), d(fx_n, Tx_0)d(fx_n, Tx_n)d(fx_0, Tx_0)d(fx_0, Tx_n)\}.$$

We have to consider five different cases:

1. $d(Tx_n, Tx_0) \preceq d(fx_n, fx_0)$. Using the triangle inequality,

$$d(Tx_n, Tx_0) \preceq K\Lambda d(fx_n, Tx_0) + K\Lambda d(Tx_0, fx_0).$$

By assumption of the induction, we obtain that

$$\begin{aligned} d(Tx_n, Tx_0) &\preceq (I - K\Lambda)^{-1}(K\Lambda)^2 d(fx_1, fx_0) + K\Lambda d(fx_1, fx_0) \\ &= (I - K\Lambda)^{-1} K\Lambda d(fx_1, fx_0). \end{aligned}$$

2. $d(Tx_n, Tx_0) \prec \Lambda d(fx_n, Tx_0) = \Lambda d(Tx_{n-1}, Tx_0)$, then (3.1) holds.
3. $d(Tx_n, Tx_0) \preceq \Lambda d(fx_0, Tx_0) = \Lambda d(fx_0, fx_1)$, then (3.1) also holds.
4. $d(Tx_n, Tx_0) \prec \Lambda d(fx_0, Tx_n)$. Using the triangle inequality, then

$$d(Tx_n, Tx_0) \preceq K\Lambda d(fx_0, Tx_0) + K\Lambda d(Tx_0, Tx_n),$$

$$d(Tx_n, Tx_0) \preceq (I - K\Lambda)^{-1} K\Lambda d(fx_0, fx_1),$$

5. $d(Tx_n, Tx_0) \preceq \Lambda d(fx_n, Tx_n) = \Lambda d(Tx_{n-1}, Tx_n)$. We have that

$$d(Tx_{n-1}, Tx_n) \preceq \Lambda\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), d(fx_n, Tx_{n-1})\}.$$

If $d(Tx_{n-1}, Tx_n) \preceq \Lambda\{d(fx_n, Tx_n), d(fx_n, Tx_{n-1})\}$, then $d(Tx_{n-1}, Tx_n) = \theta$. If

$$d(Tx_{n-1}, Tx_n) \preceq \Lambda\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_n)\}, d(fx_{n-1}, Tx_{n-1})\},$$

by continuing this process, we see that there exist $p, m \in N$, $p \geqslant n$, $0 \leqslant m < n$ such that $d(Tx_n, Tx_0) \preceq \Lambda^p d(Tx_m, Tx_0)$. By assumption of the induction, we obtain that

$$\begin{aligned} d(Tx_n, Tx_0) &\preceq (I - K\Lambda)^{-1} K\Lambda^{p+1} d(fx_0, fx_1) \\ &\preceq (I - K\Lambda)^{-1} (K\Lambda)^{p+1} d(fx_0, fx_1). \end{aligned}$$

Notice that

$$(I - K\Lambda)^{-1}(K\Lambda)^{p+1} = (I - K\Lambda)^{-1} K\Lambda - \sum_{i=1}^{p} (K\Lambda)^i.$$

It implies that $d(Tx_n, Tx_0) \preceq (I - K\Lambda)^{-1} K\Lambda d(fx_0, fx_1)$.

Hence, using the method of the mathematical induction, we have proved that inequality (3.1) holds for each $n \in N$. Now we shall prove that $(fx_n)_{n=0}^{\infty}$ is Cauchy sequence. For $m, n \in N$ and $m > n$, there exist $0 \leqslant i \leqslant n+1$, $0 \leqslant j \leqslant m+1$ such that

$$\begin{aligned} d(fx_{n+2}, fx_{m+2}) = d(Tx_{n+1}, Tx_{m+1}) &\preceq \Lambda^{n+1} d(Tx_i, Tx_j) \\ &\preceq \Lambda^{n+1}[K d(Tx_i, Tx_0) + K d(Tx_0, Tx_j)] \\ &\preceq \Lambda^{n+2}[2(I - K\Lambda)^{-1} K^2 d(fx_1, fx_0)]. \end{aligned}$$

We conclude that $(fx_n)_{n=0}^{\infty}$ is Cauchy sequence. Let $\lim_{n\to\infty} fx_n = \xi$. Since $fX \subset Tx$ and fX or TX is complete subspace of X, then there exists $x \in X$ such that $fx = \xi$. We shall show that ξ is a unique point of coincidence of T and f. Firstly, we prove the uniqueness. Let ξ_1, ξ_2 be point of coincidence of T and f, then there exist $y_1, y_2 \in X$ such that $Ty_1 = fy_1 = \xi_1$, $Ty_2 = fy_2 = \xi_2$. Since

$$d(T\xi_1, T\xi_2) \preceq \Lambda\{d(f\xi_1, f\xi_2), d(f\xi_1, T\xi_2), d(f\xi_1, T\xi_1), d(f\xi_2, T\xi_2), d(f\xi_2, T\xi_1)\},$$

then $\xi_1 = \xi_2$. Secondly, we prove that ξ is a point of coincidence of T and f. Any given $c \gg \theta$, $p \in N$, there exists $m \in N$ such that

$$d(fx_n, \xi) \ll \frac{c}{p}, \quad d(Tx_n, \xi) \ll \frac{c}{p}, \quad d(fx_n, Tx_n) \ll \frac{c}{p}, \quad \forall\, n > m.$$

Since

$$d(Tx_{n+1}, Tx) \preceq \Lambda\{d(fx_{n+1}, fx), d(fx_{n+1}, Tx), d(fx_{n+1}, Tx_{n+1}), d(fx, Tx), d(fx, Tx_{n+1})\},$$

then

$$\begin{aligned} d(Tx_{n+1}, Tx) &\preceq \Lambda d(fx_{n+1}, fx) + \Lambda d(fx_{n+1}, Tx) \\ &\preceq \Lambda d(fx_{n+1}, fx) + K\Lambda d(fx_{n+1}, Tx_{n+1}) + K\Lambda d(Tx_{n+1}, Tx), \end{aligned}$$

or

$$\begin{aligned} d(Tx_{n+1}, Tx) &\preceq \Lambda d(fx_{n+1}, Tx_{n+1}) + \Lambda d(fx, Tx) + \Lambda d(fx, Tx_{n+1}) \\ &\preceq \Lambda d(fx_{n+1}, Tx_{n+1}) + K\Lambda d(fx, Tx_{n+1}) + K\Lambda d(Tx_{n+1}, Tx) + \Lambda d(fx, Tx_{n+1}). \end{aligned}$$

It implies that

$$d(Tx_{n+1}, Tx) \prec (I - K\Lambda)^{-1}\frac{\Lambda c}{p} + (I - K\Lambda)^{-1}\frac{K\Lambda c}{p},$$

or

$$d(Tx_{n+1}, Tx) \preceq (I - K\Lambda)^{-1}\frac{\Lambda c}{p} + (I - K\Lambda)^{-1}\frac{K\Lambda c}{p} + (I - K\Lambda)^{-1}\frac{\Lambda c}{p}.$$

Let $p \to \infty$, $Tx_{n+1} \to Tx$. It implies that $fx = Tx$. So we conclude that ξ is a point of coincidence of T and f. Every T-f-sequence $(fx_n)_{n=0}^{\infty}$ converges to ξ from the uniqueness of ξ. The latter part of Theorem 3.5 follows from Lemma 3.4. □

Corollary 3.6. *Let* $K = 1$, $\Lambda = \lambda I$, $0 \leqslant \lambda < 1$, *we obtain* [10, Theorem 1.3] *from Theorem* 3.5.

Corollary 3.7. *Let* $K = 1$, $f = I_x$, *where* I_x *is the identity mapping on* X, *we obtain* [4, Theorem 1.5] *from Theorem* 3.5.

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