Differential equations for Daehee polynomials and their applications

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Abstract

Recently, differential equations for Changhee polynomials and their applications were introduced by Kim et al. and by using their differential equations, they derived some new identities on Changhee polynomials. Specially, they presented Changhee polynomials \( C_n^{N}(x) \) by sums of lower terms of Changhee polynomials \( C_n(x) \). Compare to the result, Kim et al. described Changhee polynomials \( C_n^{N}(x) \) via lower term of higher order Changhee polynomials by using non-linear differential equations arising from generating function of Changhee polynomials. In the first part of this paper, the author uses the idea of Kim et al. to apply to generating function for Daehee polynomials. From differential equations associated with the generating function of those polynomials, we derive some formulae and combinatorial identities.

Also, Kwon et al. developed the method of differential equations from the generating function of Daehee numbers and investigated new explicit identities of Daehee numbers. In the second part of the present paper, the author applies their methods to generating function of Daehee polynomials, and get the explicit representations of Daehee polynomials. And specially we put \( x = 0 \) in our results, we can get new representations of Daehee numbers compare to the above results. ©2017 All rights reserved.

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1. Introduction

It is common knowledge that the Bernoulli polynomials \( B_n(x) \) for \( n \geq 0 \) can be generated by

\[
\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},
\]

(see [2, 7, 11]).

With the viewpoint of deformed Bernoulli polynomials, the Daehee polynomials \( D_n(x) \) for \( n \geq 0 \) are defined by the generating function to be

\[
\frac{\log (1+t)}{t} (1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}.
\]

(1.1)

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It is easy to see that the generating function of the Daehee polynomials $D_n(x)$ can be reformed as

$$ \frac{\log (1+t)}{t} (1+t)^x = \frac{\log (1+t)}{e^{\log(1+t)} - 1} e^{x \log (1+t)}. $$

From (1.1), we note that

$$ \frac{\log (1+t)}{e^{\log(1+t)} - 1} e^{x \log (1+t)} = \sum_{n=0}^{\infty} B_n(x) \frac{1}{n!} (\log (1+t))^n $$

$$ = \sum_{n=0}^{\infty} B_n(x) \sum_{m=n}^{\infty} S_1(m,n) \frac{t^m}{m!} $$

$$ = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} B_n(x) S_1(m,n) \right) \frac{t^m}{m!}, \quad (1.2) $$

where $S_1(m,n)$ stands for the Stirling number of the first kind which is defined as

$$ (x)_0 = 1, \quad (x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (n \geq 1). $$

Combining (1.1) with (1.2) yields the following relation

$$ D_m(x) = \sum_{n=0}^{m} B_n(x) S_1(m,n), \quad (m \geq 0). $$

By replacing $t$ by $e^t - 1$ in (1.1), we can derive

$$ \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt} = \sum_{m=0}^{\infty} D_m(x) \frac{1}{m!} (e^t - 1)^m $$

$$ = \sum_{m=0}^{\infty} D_m(x) \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!} $$

$$ = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} D_n(x) S_2(n,m) \right) \frac{t^n}{n!}, \quad (1.3) $$

where $S_2(n,m)$ is the Stirling number of the second kind which is given by $x^n = \sum_{l=0}^{\infty} S_2(n,l)x^l, \quad (n \geq 0).$

Comparing the coefficients on the both sides of (1.3), we obtain

$$ B_n(x) = \sum_{m=0}^{n} D_m(x) S_2(n,m), \quad (n \geq 0). \quad (1.4) $$

In recent decades, many mathematicians have investigated some interesting extensions and modifications of Daehee polynomials related combinatorial identities and their applications (see [1, 2, 5, 6, 9, 11–13]).

In [4], differential equations for Changhee polynomials and their applications were introduced by Kim et al. By using their differential equations, they derived some new identities on Changhee polynomials. Specially, they presented Changhee polynomials $\mathcal{C}_{n+N}(x)$ by sums of lower terms of Changhee polynomials $\mathcal{C}_n(x)$. Compared to [4], Kim et al. described Changhee polynomials $\mathcal{C}_{n+N}(x)$ via lower term of higher order Changhee polynomials by using non-linear differential equations arising from generating function of Changhee polynomials in [8]. Both papers [4] and [8] treated the inversion problem for the results in Kim’s work (see [3]). In the first part of this paper, we use the idea of [4] to apply to generating function for Daehee polynomials. From differential equations associated with the generating function of
those polynomials, we derive some formulae and combinatorial identities.

Also in [10], Kwon et al. developed the method of differential equations from the generating function of Dahee numbers and investigated new explicit identities of Dahee numbers. In the second part of the present paper, we apply their methods to generating function of Dahee polynomials, and get the explicit representations of Dahee numbers. And specially we put $x = 0$ in our results, we can get new representations of Dahee numbers compare to the above results.

2. Differential equations for Dahee polynomials

Let

$$F = F(t, x) = \frac{\log (1 + t)}{t} (1 + t)^x. \quad (2.1)$$

By taking the derivative with respect to $t$ in (2.1), we can derive

$$F^{(1)} = \frac{d}{dt} F(t, x) = \left[ \left( x + \frac{1}{\log (1 + t)} \right) (t + 1)^{-1} - t^{-1} \right] F$$

and

$$F^{(2)} = \frac{d}{dt} F^{(1)} = \left[ 2t^{-2} - 2 \left( \frac{1}{\log (1 + t)} + x \right) (1 + t)^{-1} t^{-1} + \left( \frac{2x - 1}{\log (1 + t)} + x(x - 1) \right) (1 + t)^{-2} t^{-1} \right] F.$$

Similarly, we get

$$F^{(3)} = \frac{d}{dt} F^{(2)} = \left[ -6t^{-3} + 6 \left( \frac{1}{\log (1 + t)} + x \right) (1 + t)^{-1} t^{-2} - 3 \left( \frac{2x - 1}{\log (1 + t)} + x(x - 1) \right) (1 + t)^{-2} t^{-1} \right. \n\left. + \left( \frac{3x^2 - 6x + 2}{\log (1 + t)} + x(x - 1)(x - 2) \right) (1 + t)^{-3} \right] F.$$

Continuing this process, we are led to put

$$F^{(N)} = \left( \frac{d}{dt} \right)^N F(t, x) = \left[ \sum_{i=0}^{N} \left( a_i(N, x) + \frac{b_i(N, x)}{\log (1 + t)} \right) (t + 1)^{-i} t^{i-N} \right] F,$$

where $N = 0, 1, 2, \ldots$.

Moreover, from (2.3), we note that

$$F^{(N+1)} = \frac{d}{dt} F^{(N)} = - \sum_{i=0}^{N} \left( N - i \right) \left( a_i(N, x) + \frac{b_i(N, x)}{\log (1 + t)} \right) (t + 1)^{-i} t^{i-N-1} \right] F \n- \sum_{i=0}^{N} \left( a_i(N, x) + \frac{ib_i(N, x)}{\log (1 + t)} + \frac{b_i(N, x)}{\log (1 + t)} \right) (t + 1)^{-i} t^{i-N} \right] F \n+ \sum_{i=0}^{N} \left( a_i(N, x) + \frac{b_i(N, x)}{\log (1 + t)} \right) (t + 1)^{-i} t^{i-N} \right] F^{(N)} \n= - \sum_{i=0}^{N} \left( N - i \right) \left( a_i(N, x) + \frac{b_i(N, x)}{\log (1 + t)} \right) (t + 1)^{-i} t^{i-N-1} \right] F \n- \sum_{i=0}^{N} \left( a_i(N, x) + \frac{ib_i(N, x)}{\log (1 + t)} + \frac{b_i(N, x)}{\log (1 + t)} \right) (t + 1)^{-i} t^{i-N} \right] F \n+ \sum_{i=0}^{N} \left( a_i(N, x) + \frac{b_i(N, x)}{\log (1 + t)} + \frac{b_i(N, x)}{\log (1 + t)} \right) (t + 1)^{-i} t^{i-N} \right] F.$$
By comparing the coefficients on both sides of (2.4) and (2.5), we have

\[- \sum_{i=0}^{N} \left( a_i(N, x) + \frac{b_i(N, x)}{\log(1+t)} \right) (t+1)^{-i} t^{-i-1} \] \cdot \text{F}\]

\[- \sum_{i=0}^{N} (N-i) \left( a_i(N, x) + \frac{b_i(N, x)}{\log(1+t)} \right) (t+1)^{-i-1} t^{-i-1} \] \cdot \text{F}.

\[+ \sum_{i=1}^{N+1} \left( (i-1)a_{i-1}(N, x) + \frac{(i-1)b_{i-1}(N, x)}{\log(1+t)} \right) (t+1)^{-i} t^{-i-1} \] \cdot \text{F}.

\[+ \sum_{i=1}^{N+1} \left( \frac{i}{a_{i-1}(N, x)) + \frac{i}{xb_{i-1}(N, x)} + \frac{b_{i-1}(N, x)}{\log(1+t)} \right) (t+1)^{-i} t^{-i-1} \] \cdot \text{F}.

\[+ \sum_{i=0}^{N} \left( a_i(N, x) + \frac{b_i(N, x)}{\log(1+t)} \right) (t+1)^{-i} t^{-i-1} \] \cdot \text{F}.

On the other hand, by replacing N by N + 1 in (2.3), we obtain

\[F(N+1) = \left[ \sum_{i=0}^{N+1} \left( a_i(N+1, x) + \frac{b_i(N+1, x)}{\log(1+t)} \right) (t+1)^{-i} t^{-i-1} \] \cdot \text{F}. \quad (2.5)

By comparing the coefficients on both sides of (2.4) and (2.5), we have

\[a_0(N, x) + \frac{b_0(N, x)}{\log(1+t)} = -(N+1) \left( a_0(N, x) + \frac{b_0(N, x)}{\log(1+t)} \right), \quad \text{(2.6)}\]

\[a_{N+1}(N, x) + \frac{b_{N+1}(N+1, x)}{\log(1+t)} = (x-N) \left( a_0(N, x) + \frac{b_0(N, x)}{\log(1+t)} \right) + \frac{a_{N+1}(N, x)}{\log(1+t)}. \quad \text{(2.7)}\]

and

\[a_i(N+1, x) + \frac{b_i(N+1, x)}{\log(1+t)} = (x-i+1) \left( a_i(N+1, x) + \frac{b_i(N+1, x)}{\log(1+t)} \right) + \frac{a_{i-1}(N, x)}{\log(1+t)} \]

\[- (N-i+1) \left( a_i(N, x) + \frac{b_i(N, x)}{\log(1+t)} \right), \quad \text{(2.8)}\]

where 1 \leq i \leq N. We also note that

\[F = F^{(0)} = \left[ a_0(0, x) + \frac{b_0(0, x)}{\log(1+t)} \right] \cdot \text{F}. \quad \text{(2.9)}\]

Hence, by (2.9), we get

\[a_0(0, x) + \frac{b_0(0, x)}{\log(1+t)} = 1. \quad \text{(2.10)}\]

From (2.10), it follows that

\[a_0(0, x) = 1, \quad b_0(0, x) = 0. \quad \text{(2.11)}\]

In addition, (2.2) and (2.3) lead to

\[\left[ x + \frac{1}{\log(1+t)} \right] (t+1)^{-1} t^{-1} \] \cdot \text{F} = F^{(1)}

\[= \sum_{i=0}^{1} \left( a_i(1, x) + \frac{b_i(1, x)}{\log(1+t)} \right) (t+1)^{-i} t^{-i-1} \] \cdot \text{F}

\[= \left( a_0(1, x) + \frac{b_0(1, x)}{\log(1+t)} \right) t^{-1} + \left( a_1(1, x) + \frac{b_1(1, x)}{\log(1+t)} \right) (t+1)^{-1} \] \cdot \text{F}. \quad \text{(2.11)}

Comparing the coefficients on both sides of (2.11) results in

\[a_0(1, x) = -1, \quad b_0(1, x) = 0, \quad a_1(1, x) = x, \quad b_1(1, x) = 1. \]
Also by (2.6) and (2.7), we have

\[
a_0(N+1, x) + \frac{b_0(N+1, x)}{\log (1+t)} = -(N+1) \left( a_0(N, x) + \frac{b_0(N, x)}{\log (1+t)} \right)
\]

\[
= (-1)^2(N+1) \left( a_0(N-1, x) + \frac{b_0(N-1, x)}{\log (1+t)} \right)
\]

\[
\vdots
\]

\[
= (-1)^N(N+1) \sum_{k=0}^{N-1} (-1)^k(N)_k a_k(k, x)
\]

From (2.8), we can derive

\[
a_1(N+1, x) + \frac{b_1(N+1, x)}{\log (1+t)}
\]

\[
= x \left( a_0(N, x) + \frac{b_0(N, x)}{\log (1+t)} \right) + a_0(N, x) - N \left( a_1(N, x) + \frac{b_1(N, x)}{\log (1+t)} \right)
\]

\[
= x \left( a_0(N, x) + \frac{b_0(N, x)}{\log (1+t)} \right) - N \left( a_0(N-1, x) + \frac{b_0(N-1, x)}{\log (1+t)} \right)
\]

\[
+ a_0(N, x) - Na_0(N-1, x) \frac{1}{\log (1+t)} + (-1)^2N(N-1) \left( a_1(N-1, x) + \frac{b_1(N-1, x)}{\log (1+t)} \right)
\]

\[
\vdots
\]

\[
= \sum_{i=0}^{N-1} (-1)^i(N)_i \left[ x \left( a_0(N-i, x) + \frac{b_0(N-i, x)}{\log (1+t)} \right) + a_0(N-i, x) \right]
\]

\[
+ (-1)^NN! \left( a_1(1, x) + \frac{b_1(1, x)}{\log (1+t)} \right)
\]

\[
= \sum_{i=0}^{N} (-1)^i(N)_i \left[ x \left( a_0(N-i, x) + \frac{b_0(N-i, x)}{\log (1+t)} \right) + a_0(N-i, x) \right],
\]
Continuing in this fashion, we can find that

\[ a_2(N+1, x) + \frac{b_2(N+1, x)}{\log(1+t)} = (x-1) \left( a_1(N, x) + \frac{b_1(N, x)}{\log(1+t)} \right) + \frac{a_1(N, x)}{\log(1+t)} + (1-N) \left( a_2(N, x) + \frac{b_2(N, x)}{\log(1+t)} \right) \]

\[ = (x-1) \left( a_1(N, x) + \frac{b_1(N, x)}{\log(1+t)} \right) + (-1)(N-1) \left( a_1(N-1, x) + \frac{b_1(N-1, x)}{\log(1+t)} \right) + \frac{a_1(N, x) - (N-1)a_1(N-1, x)}{\log(1+t)} \]

\[ + (-1)^2(N-1)(N-2) \left( a_2(N-1, x) + \frac{b_2(N-1, x)}{\log(1+t)} \right) \]

\[ = \sum_{i=0}^{N-2} (-1)^i(N-1)_i \left[ (x-1) \left( a_1(N-i, x) + \frac{b_1(N-i, x)}{\log(1+t)} \right) + \frac{a_1(N-i, x)}{\log(1+t)} \right] \]

\[ + (N-1)N-1(N-1)! \left( a_2(2, x) + \frac{b_2(2, x)}{\log(1+t)} \right) \]

\[ = \sum_{i=0}^{N-1} (-1)^i(N-1)_i \left[ (x-1) \left( a_1(N-i, x) + \frac{b_1(N-i, x)}{\log(1+t)} \right) + \frac{a_1(N-i, x)}{\log(1+t)} \right] \]

and

\[ a_3(N+1, x) + \frac{b_3(N+1, x)}{\log(1+t)} = (x-2) \left( a_2(N, x) + \frac{b_2(N, x)}{\log(1+t)} \right) + \frac{a_2(N, x)}{\log(1+t)} + (2-N) \left( a_3(N, x) + \frac{b_3(N, x)}{\log(1+t)} \right) \]

\[ = (x-2) \left( a_2(N, x) + \frac{b_2(N, x)}{\log(1+t)} \right) + (-1)(N-2) \left( a_2(N-1, x) + \frac{b_2(N-1, x)}{\log(1+t)} \right) + \frac{a_2(N, x) - (N-2)a_2(N-1, x)}{\log(1+t)} \]

\[ + (-1)^2(N-2)(N-3) \left( a_3(N-1, x) + \frac{b_3(N-1, x)}{\log(1+t)} \right) \]

\[ = \sum_{i=0}^{N-3} (-1)^i(N-2)_i \left[ (x-2) \left( a_2(N-i, x) + \frac{b_2(N-i, x)}{\log(1+t)} \right) + \frac{a_2(N-i, x)}{\log(1+t)} \right] \]

\[ + (N-2)(N-2)! \left( a_3(3, x) + \frac{b_3(3, x)}{\log(1+t)} \right) \]

\[ = \sum_{i=0}^{N-2} (-1)^i(N-2)_i \left[ (x-2) \left( a_2(N-i, x) + \frac{b_2(N-i, x)}{\log(1+t)} \right) + \frac{a_2(N-i, x)}{\log(1+t)} \right] \]

Continuing in this fashion, we can find that

\[ a_j(N+1, x) + \frac{b_j(N+1, x)}{\log(1+t)} = \sum_{i=0}^{N-j+1} (-1)^i(N-j+1)_i \]

\[ \times \left[ (x+j) \left( a_{j-1}(N-i, x) + \frac{b_{j-1}(N-i, x)}{\log(1+t)} \right) + \frac{a_{j-1}(N-i, x)}{\log(1+t)} \right] \]

(2.16)
where \( i \leq j \leq N \).

We remark that the \((N+1) \times (N+1)\) matrix with the \((i, j)\) entry given by \((a_i(j, x))_{0 \leq i, j \leq N}\) is given by

\[
\begin{pmatrix}
0 & 1 & -1 & \cdots & \frac{(-1)^{2}}{2!} & \cdots & \frac{(-1)^{N} N!}{N!} \\
1 & 0 & x & \cdots & \cdots & \cdots & \cdots \\
2 & 0 & 0 & \ddots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \\
N & 0 & 0 & 0 & \cdots & \cdots & (x)ᴺ \\
\end{pmatrix}
\]

Now, we give explicit expressions for \(a_i(j, x)\). From (2.12), (2.13), (2.14), (2.15), and (2.16), we can derive

\[
a_1(N + 1, x) = x \sum_{i = 0}^{N} (-1)^i(N)_ia_0(N - i, x) = x(-1)^N(N + 1)!,
\]

\[
a_2(N + 1, x) = (x - 1) \sum_{i_1 = 0}^{N - 1} (-1)^{i_1}(N - 1)_{i_1}a_1(N - i_1, x) = (x)²(-1)^{N - 1}(N - 1)! \sum_{i_1 = 0}^{N - 1} (N - i_1),
\]

\[
a_3(N + 1, x) = (x - 2) \sum_{i_2 = 0}^{N - 2} (-1)^{i_2}(N - 2)_{i_2}a_2(N - i_2, x) = (x)³(-1)^{N - 2}(N - 2)! \sum_{i_2 = 0}^{N - 2} (N - i_2 - i_1 - 1),
\]

and

\[
a_4(N + 1, x) = (x - 3) \sum_{i_3 = 0}^{N - 3} (-1)^{i_3}(N - 3)_{i_3}a_3(N - i_3, x)
\]

\[
= (x)⁴(-1)^{N - 3}(N - 3)! \sum_{i_3 = 0}^{N - 3} \sum_{i_2 = 0}^{N - 3 - i_3} \sum_{i_1 = 0}^{N - 3 - i_3 - i_2} (N - i_3 - i_2 - i_1 - 2).
\]

Continuing in this manner, we get

\[
a_j(N + 1, x) = (x)_j(-1)^{N - j + 1}(N - j + 1)!
\]

\[
\times \sum_{i_{j-1} = 0}^{N - j + 1} \sum_{i_{j-2} = 0}^{N - j + 1 - i_{j-1}} \cdots \sum_{i_1 = 0}^{N - j + 1 - i_{j-2} \cdots - i_2} (N - i_{j-1} \cdots - i_1 - j + 2),
\]

(2.17)

where \( 1 \leq j \leq N + 1 \).

On the other hand, we now turn our attention to \(b_i(j, x)\). From (2.12), (2.13), (2.14), (2.15), and (2.16), we can obtain

\[
b_1(N + 1, x) = \sum_{i = 0}^{N - 1} (-1)^i(N)_i \left[ x b_0(N - i, x) + a_0(N - i, x) \right] = (-1)^N(N + 1)!,
\]

\[
b_2(N + 1, x) = \sum_{i_1 = 0}^{N - 1} (-1)^{i_1}(N - 1)_{i_1} \left[ (x - 1) b_1(N - i_1, x) + a_1(N - i_1, x) \right]
\]

\[
= (x - 1)(-1)^{N - 1}(N - 1)! \sum_{i_1 = 0}^{N - 1} (N - i_1) + x(-1)^{N - 1}(N - 1)! \sum_{i_1 = 0}^{N - 1} (N - i_1)
\]

where

By continuing this process, we get

Therefore, by combining (2.17) and (2.18), we obtain the following theorem.

By continuing this process, we get

where $1 \leq j \leq N + 1$.

Therefore, by combining (2.17) and (2.18), we obtain the following theorem.

**Theorem 2.1.** For $N = 0, 1, 2, \cdots$, let us consider the following family of differential equations:

$$F^{(N)} = \left[ \sum_{i=0}^{N} \left( a_i(N, x) + \frac{b_i(N, x)}{\log (1 + t)} \right)(t + 1)^{-i} t^{-N} \right] F,$$

where

$$a_0(N, x) = (-1)^N N!, \quad b_0(N, x) = 0, \quad \text{for all } N,$$

$$a_j(N, x) = (x)^j (-1)^{N-j} (N-j)! \times \sum_{i_{j-1}=0}^{N-j} \sum_{i_{j-2}=0}^{N-j-i_{j-1}} \cdots \sum_{i_1=0}^{N-j-i_{j-1}\cdots-i_2} (N-i_{j-1}\cdots-i_1-j+1),$$

$$b_j(N, x) = \sum_{k=0}^{j-1} \prod_{i=0}^{j-1} (x-i) (x-k)^{-1} (-1)^{N-j} (N-j)! \times \sum_{i_{j-1}=0}^{N-j} \sum_{i_{j-2}=0}^{N-j-i_{j-1}} \cdots \sum_{i_1=0}^{N-j-i_{j-1}\cdots-i_2} (N-i_{j-1}\cdots-i_1-j+1), \quad (1 \leq j \leq N + 1).$$

Then the above family of differential equations has a solution

$$F = F(t, x) = \frac{\log (1 + t)}{t} (1 + t)^x.$$
From (2.1), we obtain
\[
 F^{(N)} = \sum_{k=N}^\infty D_k(x)(k)_N \frac{t^{k-N}}{k!} = \sum_{k=0}^\infty D_{N+k}(x) \frac{t^k}{k!}. \tag{2.19}
\]

Recall from [7] that the Bernoulli numbers of the second kind \( b_n \) for \( n \geq 0 \) are generated by
\[
\frac{t}{\log(1+t)} = \sum_{n=0}^\infty b_n \frac{t^n}{n!}. \tag{2.20}
\]
The first few Bernoulli numbers \( b_n \) of the second kind are
\[
b_0 = 1, \quad b_1 = \frac{1}{2}, \quad b_2 = -\frac{1}{12}, \quad b_3 = \frac{1}{24}, \quad b_4 = -\frac{19}{720}, \ldots.
\]
Furthermore (2.20), we observe that
\[
\frac{1}{\log(1+t)} = \sum_{n=0}^\infty b_n \frac{t^{n-1}}{n!} = \frac{b_0}{t} + \sum_{n=0}^\infty b_{n+1} \frac{t^n}{(n+1)!} = \frac{1}{t} + \sum_{n=0}^\infty \frac{b_{n+1} t^n}{n+1 n!}.
\]
Therefore, by Theorem 2.1, we acquire that
\[
 F^{(N)} = \left[ \sum_{i=0}^N \left( a_i(N,x) + \frac{b_i(N,x)}{\log(1+t)} \right)(t+1)^{i-N} \right] F
 = \sum_{i=0}^N \left( a_i(N,x) + b_i(N,x) \sum_{n=0}^\infty \frac{b_{n+1} t^n}{n+1 n!} + \frac{b_i(N,x)}{t} \right)
 \times \left( \sum_{i=0}^\infty (-1)^i \binom{i+1-l}{l} t^l \right) t^{l-N} \sum_{p=0}^\infty D_p(x) \frac{t^p}{p!}
 = \sum_{i=0}^N a_i(N,x) \sum_{k=0}^\infty \sum_{l+p+i-N=k} k!(-1)^l \binom{i+1-l}{l} \frac{1}{p!} D_p(x) \frac{t^k}{k!}
 + \sum_{i=0}^N b_i(N,x) \sum_{k=0}^\infty \sum_{l+s+i-N+p=k} k!(-1)^l \binom{i+1-l}{l} \frac{b_{s+1} t^s}{p!(s+1)!} D_p(x) \frac{t^k}{k!}
 + \sum_{i=0}^N b_i(N,x) \sum_{k=0}^\infty \sum_{l+p+i-N-1=k} k!(-1)^l \binom{i+1-l}{l} \frac{1}{p!} D_p(x) \frac{t^k}{k!}
 = \sum_{k=0}^\infty \left[ k! \sum_{i=0}^N a_i(N,x) \sum_{l+p+i-N=k} (-1)^l \binom{i+1-l}{l} \frac{1}{p!} D_p(x) \right] \frac{t^k}{k!}
 + \sum_{k=0}^\infty \left[ k! \sum_{i=0}^N b_i(N,x) \sum_{l+s+i-N+p=k} (-1)^l \binom{i+1-l}{l} \frac{1}{p!(s+1)!} b_{s+1} D_p(x) \right] \frac{t^k}{k!}
 + \sum_{k=0}^\infty \left[ k! \sum_{i=0}^N b_i(N,x) \sum_{l+p+i-N-1=k} (-1)^l \binom{i+1-l}{l} \frac{1}{p!} D_p(x) \right] \frac{t^k}{k!}.
\]
By equating coefficients of (2.19) and (2.21), we finally arrive at the following theorem.
Theorem 2.2. For \( N = 0, 1, 2, \cdots \) and \( k = 0, 1, 2, \cdots \), we have

\[
D_{k+N}(x) = k! \sum_{i=0}^{N} a_i(N, x) \sum_{l+p+i-N=k} (-1)^l \binom{i+l-1}{l} \frac{1}{p!} D_p(x) \\
+ k! \sum_{i=0}^{N} b_i(N, x) \sum_{l+s+i-N+p=k} (-1)^l \binom{i+l-1}{l} \frac{1}{p!(s+1)!} b_{s+1} D_p(x) \\
+ k! \sum_{i=0}^{N} b_i(N, x) \sum_{l+p+i-N-1=k} (-1)^l \binom{i+l-1}{l} \frac{1}{p!} D_p(x),
\]

where \( a_i(N, x)'s \) and \( b_i(N, x)'s \) are as in Theorem 2.1.

3. Dahee numbers associated with differential equations

Now we consider

\[
G = G(t, x) = \log(1 + t)(1 + t)^x.
\]  

(3.1)

Then, by (3.1), we have

\[
G^{(1)} = \frac{d}{dx} G(t, x) = (1 + t)^{x-1} + x \log(1 + t)(1 + t)^{x-1} = (1 + t)^{x-1} + xG(t, x - 1)
\]

and

\[
G^{(2)} = \frac{d}{dx} G^{(1)} = (x - 1)(1 + t)^{x-2} + xG'(t, x - 1) = (x - 1)(1 + t)^{x-2} + x(1 + t)^{x-2} + x(x - 1)G(t, x - 2).
\]

Continuing this differentiation \( N \)-times, we derive

\[
G^{(N)} = \left( \frac{d}{dx} \right)^{(N)} G(t, x) = \sum_{i=0}^{N-1} \left( \prod_{j=0}^{N-1} (x-j) \right) (1 + t)^{x-N} + (x)_N G(t, x - N)
\]

(3.2)

\[
= \sum_{i=0}^{N-1} \left( \prod_{j=0}^{N-1} (x-j) \right) (1 + t)^{x-N} + (x)_N \log(1 + t)(1 + t)^{x-N}.
\]

On the other hand, the left hand side of (3.2) has the presentation as follows:

\[
G^{(N)} = \left( \frac{d}{dx} \right)^{(N)} \log(1 + t)(1 + t)^x
\]

\[
= \left( \frac{d}{dx} \right)^{(N)} \log(1 + t) \frac{(1 + t)^x}{t}
\]

\[
= \left( \frac{d}{dx} \right)^{(N)} \sum_{m=0}^{\infty} D_m(x) \frac{t^{m+1}}{m!}
\]

\[
= \left( \frac{d}{dx} \right)^{(N)} \sum_{m=0}^{\infty} \frac{mD_{m-1}(x) t^{m}}{m!}
\]

\[
= \sum_{m=0}^{\infty} (m+N)D_{m+N-1}(x) \frac{t^{m}}{m!}.
\]  

(3.3)
For the right hand side of (3.3), we observe that

\[(x)_N \log(1 + t)(1 + t)^{x-N} t = (x)_N \sum_{m=0}^{\infty} D_m(x-N) \frac{t^{m+1}}{m!} \]

\[= \sum_{m=0}^{\infty} (x)_N mD_{m-1}(x-N) \frac{t^m}{m!}. \quad (3.4)\]

In addition, we note that

\[(1 + t)^{x-N} = \frac{t \log(1 + t)}{\log(1 + t)} (1 + t)^{x-N} = \left( \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \right) \left( \sum_{l=0}^{\infty} \frac{D_l(x-N)}{l!} \right) \]

\[= \sum_{m=0}^{\infty} \left[ \sum_{n=0}^{m} \left( \begin{array}{c} m \\ n \end{array} \right) b_n D_l(x-N) \right] \frac{t^m}{m!}. \quad (3.5)\]

Also we can expand \((1 + t)^{x-N}\) as follows:

\[(1 + t)^{x-N} = \sum_{m=0}^{\infty} (x-N)_m \frac{t^m}{m!}. \quad (3.6)\]

Now by (3.2), (3.3), (3.4), and (3.5), we have the following results

\[(m+N)D_{m+N-1}(x) = \sum_{i=0}^{N-1} \left( \prod_{j=0}^{N-1} (x-j) \right) \sum_{n=0}^{m} \left( \begin{array}{c} m \\ n \end{array} \right) b_n D_l(x-N) \quad (3.7)\]

and

\[(m+N)D_{m+N-1}(x) = \sum_{i=0}^{N-1} \left( \prod_{j=0}^{N-1} (x-j) \right) (x-N)_m + (x)_N mD_{m-1}(x-N). \quad (3.8)\]

Therefore, we obtain the following theorem.

**Theorem 3.1.** For \(N \in \mathbb{N}\) and \(m \geq 0\), we have

\[(m+N)D_{m+N-1}(x) = \sum_{i=0}^{N-1} \left( \prod_{j=0}^{N-1} (x-j) \right) \sum_{n=0}^{m} \left( \begin{array}{c} m \\ n \end{array} \right) b_n D_l(x-N) \]

\[= \sum_{i=0}^{N-1} \left( \prod_{j=0}^{N-1} (x-j) \right) (x-N)_m + (x)_N mD_{m-1}(x-N). \]

When \(x = 0\) in (3.7), we have the following identity on Daehee numbers.

**Corollary 3.2.** For \(N \in \mathbb{N}\) and \(m \geq 0\), we have

\[D_{m+N-1} = \frac{(-1)^{N-1}(N-1)!}{m+N} \sum_{n=0}^{m} \left( \begin{array}{c} m \\ n \end{array} \right) b_n D_l(-N). \]

The above corollary can be compared to Theorem 2.1 of [10], which presents \(D_{m+N-1}\) via higher order Daehee numbers, we record the result here.
Theorem 3.3 ([10, Theorem 2.1]). For \( N \in \mathbb{N} \) and \( m \geq 0 \), we have

\[
D_{m+N-1} = \frac{(-1)^{N-1}(N-1)!}{m+N} \sum_{n=0}^{m} \binom{m}{n} (-1)^n N^n D_{m-n}^{(n)}.
\]

When \( x = 0 \) in (3.8), we get the same identity on Dahee numbers, which appears in [10].

Corollary 3.4 ([10, Theorem 1]). For \( N \in \mathbb{N} \) and \( m \geq 0 \), we have

\[
D_{m+N-1} = \frac{(-1)^{N-1}(N-1)!}{m+N} \sum_{n=0}^{m} (-1)^n N^n S_1(m, n).
\]

From (3.5), we replace \( t \) by \( e^t - 1 \), we get

\[
(e^t)^{x-N} = \frac{e^t - 1}{t} t e^{t-1} (e^t)^{x-N} = \left( \sum_{n=0}^{\infty} b_n \frac{(e^t - 1)^n}{n!} \right) \left( \sum_{k=0}^{\infty} B_k(x-N) \frac{t^k}{k!} \right)
\]

\[
= \left( \sum_{n=0}^{\infty} b_n \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!} \right) \left( \sum_{k=0}^{\infty} B_k(x-N) \frac{t^k}{k!} \right)
\]

\[
= \left( \sum_{m=0}^{\infty} \sum_{n=0}^{m} b_n S_2(m, n) \frac{t^m}{m!} \right) \left( \sum_{k=0}^{\infty} B_k(x-N) \frac{t^k}{k!} \right)
\]

\[
= \sum_{m=0}^{\infty} \left[ \sum_{l=0}^{m} \binom{m}{l} \sum_{n=0}^{l} b_n S_2(l, n) B_{m-l}(x-N) \right] \frac{t^m}{m!}.
\]

On the other hand,

\[
(e^t)^{x-N} = \sum_{m=0}^{\infty} (x-N)^m \frac{t^m}{m!}.
\]

Comparing the coefficients of (3.9) and (3.10), we have

\[
(x-N)^m = \sum_{l=0}^{m} \binom{m}{l} \sum_{n=0}^{l} b_n S_2(l, n) B_{m-l}(x-N).
\]

Thus if we take \( x = 0 \) in (3.11), we can obtain

\[
(-N)^m = \sum_{l=0}^{m} \sum_{n=0}^{l} \binom{m}{l} b_n S_2(l, n) B_{m-l}(-N).
\]

By (1.4), we observe

\[
B_{m-l}(-N) = \sum_{k=0}^{m-l} D_k(-N) S_2(m-l, k).
\]

From (3.12) and (3.13), we know that

\[
(-N)^m = \sum_{l=0}^{m} \sum_{n=0}^{l} \sum_{k=0}^{m-l} \binom{m}{l} b_n S_2(l, n) D_k(-N) S_2(m-l, k).
\]

Theorem 3.5. For \( N \in \mathbb{N} \) and \( m \geq 0 \), we have

\[
N^m = \sum_{l=0}^{m} \sum_{n=0}^{l} \sum_{k=0}^{m-l} \binom{m}{l} (-1)^m b_n S_2(l, n) D_k(-N) S_2(m-l, k).
\]
From (3.6), we replace $t$ by $e^t - 1$, we have

$$(e^t)^{x-N} = \sum_{m=0}^{\infty} (x-N)_m \frac{(e^t - 1)^m}{m!} = \sum_{m=0}^{\infty} [\sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!}]
$$

Thus we have

$$(x-N)^n = \sum_{m=0}^{n} (x-N)_m S_2(n, m).$$

If we take $x = 0$ in (3.14), we have the following result, which is the same that of (2.16) in [10]

$$e^{-Nt} = \sum_{n=0}^{\infty} (-1)^n N^n \frac{t^n}{n!}.$$

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References