Stability of additive-quadratic $\rho$-functional equations in Banach spaces: a fixed point approach

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Abstract

Let

$$M_1 f(x, y) := \frac{3}{4} f(x + y) - \frac{1}{4} f(-x - y) + \frac{1}{4} f(x - y) + \frac{1}{4} f(y - x) - f(x) - f(y),$$

$$M_2 f(x, y) := 2f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) + f\left(\frac{y - x}{2}\right) - f(x) - f(y).$$

We solve the additive-quadratic $\rho$-functional equations

$$M_1 f(x, y) = \rho M_2 f(x, y), \quad (1)$$

and

$$M_2 f(x, y) = \rho M_1 f(x, y), \quad (2)$$

where $\rho$ is a fixed nonzero number with $\rho \neq 1$.

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic $\rho$-functional equations (1) and (2) in Banach spaces. ©2017 All rights reserved.

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1. Introduction and preliminaries


The functional equation $f(x + y) = f(x) + f(y)$ is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. Hyers [6] gave a first affirmative partial

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answer to the question of Ulam for Banach spaces. Hyers’ Theorem was generalized by Aoki [1] for additive mappings and by Rassias [8] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruţa [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach. The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The stability of quadratic functional equation was proved by Skof [9] for mappings $f : E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group.

We recall a fundamental result in fixed point theory.

**Theorem 1.1** ([2, 4]). Let $(X, d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty,$$

for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that

1. $d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geq n_0$;
2. the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;
3. $y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
4. $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$, for all $y \in Y$.

In Section 2, we solve the additive-quadratic functional equation (1) and prove the Hyers-Ulam stability of the additive-quadratic functional equation (1) in Banach spaces.

In Section 3, we solve the additive-quadratic $\rho$-functional equation (2) and prove the Hyers-Ulam stability of the additive-quadratic $\rho$-functional equation (2) in Banach spaces.

Throughout this paper, assume that $X$ is a normed space and that $Y$ is a Banach space. Let $\rho$ be a nonzero number with $\rho \neq 1$.

2. Additive-quadratic $\rho$-functional equation (1) in Banach spaces

We solve and investigate the additive-quadratic $\rho$-functional equation (1) in normed spaces.

**Lemma 2.1.**

1. If a mapping $f : X \to Y$ satisfies $M_1 f(x, y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x) - f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x) + f(-x)}{2}$ is the quadratic mapping.
2. If a mapping $f : X \to Y$ satisfies $M_2 f(x, y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x) - f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x) + f(-x)}{2}$ is the quadratic mapping.

**Proof.**

1. (i) $M_1 f_o(x, y) = f_o(x + y) - f_o(x) - f_o(y) = 0,$

   for all $x, y \in X$. So $f_o$ is the Cauchy additive mapping.

   $M_1 f_e(x, y) = \frac{1}{2} f_e(x + y) + \frac{1}{2} f_e(x - y) - f_e(x) - f_e(y) = 0,$

   for all $x, y \in X$. So $f_o$ is the quadratic mapping.
Lemma 2.2. If a mapping $f : X \rightarrow Y$ satisfies $f(0) = 0$ and

\[ M_1 f(x, y) = \rho M_2 f(x, y), \tag{2.1} \]

for all $x, y \in X$, then $f : X \rightarrow Y$ is the sum of the Cauchy additive mapping $f_o$ and the quadratic mapping $f_e$. 

Proof. Letting $y = x$ in (2.1) for $f_o$, we get $f_o(2x) - 2f_o(x) = 0$ and so $f_o(2x) = 2f_o(x)$ for all $x \in X$. Thus

\[ f_o \left( \frac{x}{2} \right) = \frac{1}{2} f_o(x), \tag{2.2} \]

for all $x \in X$.

It follows from (2.1) and (2.2) that

\[ f_o(x + y) - f_o(x) - f_o(y) = \rho \left( 2f_o \left( \frac{x + y}{2} \right) - f_o(x) - f_o(y) \right) = \rho (f_o(x + y) - f_o(x) - f_o(y)), \]

and so

\[ f_o(x + y) = f_o(x) + f_o(y), \]

for all $x, y \in X$.

Letting $y = x$ in (2.1) for $f_e$, we get $\frac{1}{2} f_e(2x) - 2f_e(x) = 0$ and so $f_e(2x) = 4f_e(x)$ for all $x \in X$. Thus

\[ f_e \left( \frac{x}{2} \right) = \frac{1}{4} f_e(x), \tag{2.3} \]

for all $x \in X$.

It follows from (2.1) and (2.3) that

\[
\frac{1}{2} f_e(x + y) + \frac{1}{2} f_e(x - y) - f_e(x) - f_e(y) = \rho \left( 2f_e \left( \frac{x + y}{2} \right) + 2f_e \left( \frac{x - y}{2} \right) - f_e(x) - f_e(y) \right)
\]

\[ = \rho \left( \frac{1}{2} f_e(x + y) + \frac{1}{2} f_e(x - y) - f_e(x) - f_e(y) \right), \]

and so

\[ f_e(x + y) + f_e(x - y) = 2f_e(x) + 2f_e(y), \]

for all $x, y \in X$.

Therefore, the mapping $f : X \rightarrow Y$ is the sum of the Cauchy additive mapping $f_o$ and the quadratic mapping $f_e$. \qed
Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (2.1) in Banach spaces.

**Theorem 2.3.** Let \( \varphi : X^2 \to [0, \infty) \) be a function such that there exists an \( L < 1 \) with
\[
\varphi \left( \frac{x + y}{2}, \frac{x - y}{2} \right) \leq \frac{L}{4} \varphi(x, y),
\]
for all \( x, y \in X \). Let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) and
\[
\|M_1 f(x, y) - \rho M_2 f(x, y)\| \leq \varphi(x, y),
\]
for all \( x, y \in X \). Then there exist a unique additive mapping \( A : X \to Y \) and a unique quadratic mapping \( Q : X \to Y \) such that
\[
\|f_o(x) - A(x)\| \leq \frac{L}{4(1 - L)} \|\varphi(x, x) + \varphi(-x, -x)\|
\]
and
\[
\|f_e(x) - Q(x)\| \leq \frac{L}{4(1 - L)} \|\varphi(x, x) + \varphi(-x, -x)\|
\]
for all \( x \in X \).

**Proof.** Letting \( y = x \) in (2.5) for \( f_o \), we get
\[
\|f_o(2x) - 2f_o(x)\| \leq \frac{1}{2} \varphi(x, x) + \frac{1}{2} \varphi(-x, -x),
\]
for all \( x \in X \).

Consider the set
\[
S := \{ h : X \to Y, \ h(0) = 0 \},
\]
and introduce the generalized metric on \( S \):
\[
d(g, h) = \inf \{ \mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu(\varphi(x, x) + \varphi(-x, -x)), \ \forall x \in X \},
\]
where, as usual, \( \inf \varphi = +\infty \). It is easy to show that \((S, d)\) is complete (see [7]).

Now we consider the linear mapping \( J : S \to S \) such that
\[
Jg(x) := 2g \left( \frac{x}{2} \right),
\]
for all \( x \in X \).

Let \( g, h \in S \) be given such that \( d(g, h) = \epsilon \). Then
\[
\|g(x) - h(x)\| \leq \epsilon(\varphi(x, x) + \varphi(-x, -x)),
\]
for all \( x \in X \). Since \( \frac{1}{2} \varphi(x, y) \leq \frac{1}{2} \varphi(x, x) \) for all \( x, y \in X \),
\[
\|Jg(x) - Jh(x)\| = \left\|2g \left( \frac{x}{2} \right) - 2h \left( \frac{x}{2} \right) \right\| \leq 2\epsilon \left( \varphi \left( \frac{x}{2}, \frac{x}{2} \right) + \varphi \left( -\frac{x}{2}, -\frac{x}{2} \right) \right) \leq 2\epsilon \frac{L}{2} \varphi(x, x) \leq L\epsilon(\varphi(x, x) + \varphi(-x, -x)),
\]
for all \( x \in X \). So \( d(g, h) = \epsilon \) implies that \( d(Jg, Jh) \leq L\epsilon \). This means that
\[
d(Jg, Jh) \leq Ld(g, h),
\]
for all \( g, h \in S \).

It follows from (2.6) that
\[
\left\|f_o(x) - 2f_o \left( \frac{x}{2} \right) \right\| \leq \frac{1}{2} \varphi \left( \frac{x}{2}, \frac{x}{2} \right) + \frac{1}{2} \varphi \left( -\frac{x}{2}, -\frac{x}{2} \right) \leq \frac{L}{8} \varphi(x, x) + \varphi(-x, -x),
\]
for all \( x \in X \). So \( d(f_o, Jf_o) \leq \frac{L}{8} \leq \frac{1}{4} \).

By Theorem 1.1, there exists a mapping \( A : X \to Y \) satisfying the following:
A is a fixed point of $J$, i.e.,
$$A(x) = 2A\left(\frac{x}{2}\right),$$
for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set
$$M = \{g \in S : d(f, g) < \infty\}.$$
This implies that $A$ is a unique mapping satisfying (2.7) such that there exists a $\mu \in (0, \infty)$ satisfying
$$\|f_0(x) - A(x)\| \leq \mu(\varphi(x, x) + \varphi(-x, -x)),$$
for all $x \in X$;

$$d(J^lf_0, A) \to 0 \text{ as } l \to \infty.$$ This implies the equality
$$\lim_{l \to \infty} 2^n f_0\left(\frac{x}{2^n}\right) = A(x),$$
for all $x \in X$;

$$d(f_0, A) \leq \frac{1}{1 - L} d(f_0, Jf_0),$$ which implies
$$\|f_0(x) - A(x)\| \leq \frac{L}{4(1 - L)}(\varphi(x, x) + \varphi(-x, -x)),$$
for all $x \in X$.

It follows from (2.4) and (2.5) that
$$\left\|\lambda(x + y) - A(x) - A(y) - \rho\left(2A\left(\frac{x + y}{2}\right) - A(x) - A(y)\right)\right\|$$
$$= \lim_{n \to \infty} 2^n \left(f_0\left(\frac{x + y}{2^n}\right) - f_0\left(\frac{x}{2^n}\right) - f_0\left(\frac{y}{2^n}\right)\right) - 2^n \rho\left(2f_0\left(\frac{x + y}{2^{n+1}}\right) - f_0\left(\frac{x}{2^n}\right) - f_0\left(\frac{y}{2^n}\right)\right)$$
$$\leq \frac{1}{2} \lim_{n \to \infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + 2^n \varphi\left(-\frac{x}{2^n}, -\frac{y}{2^n}\right) = 0,$$
for all $x, y \in X$. So
$$A(x + y) - A(x) - A(y) = \rho\left(2A\left(\frac{x + y}{2}\right) - A(x) - A(y)\right),$$
for all $x, y \in X$. By Lemma 2.2, the mapping $A : X \to Y$ is additive.

Letting $y = x$ in (2.5) for $f_e$, we get
$$\left\|\frac{1}{2} f_e(2x) - 2f_e(x)\right\| \leq \frac{1}{2} \varphi(x, x) + \frac{1}{2} \varphi(-x, -x),$$
(2.8)
for all $x \in X$.

Now we consider the linear mapping $J : S \to S$ such that
$$Jg(x) := 4g\left(\frac{x}{2}\right),$$
for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then
$$\|g(x) - h(x)\| \leq \varepsilon(\varphi(x, x) + \varphi(-x, -x)),$$
It follows from (2.4) and (2.5) that
\[ \parallel f_e(x) - 4f_e \left( \frac{x}{2} \right) \parallel \leq \varphi \left( \frac{x}{2}, \frac{x}{2} \right) + \varphi \left( -\frac{x}{2}, -\frac{x}{2} \right) \leq \frac{L}{4} (\varphi (x, x) + \varphi (-x, -x)), \]
for all \( x \in X \). So \( d(f_e, Jf_e) \leq \frac{L}{4} \). By Theorem 1.1, there exists a mapping \( Q : X \to Y \) satisfying the following:

1. \( Q \) is a fixed point of \( J \), i.e.,
   \[ Q(x) = 4Q \left( \frac{x}{2} \right), \tag{2.9} \]
   for all \( x \in X \). The mapping \( Q \) is a unique fixed point of \( J \) in the set
   \[ M = \{ g \in S : d(f, g) < \infty \}. \]
   This implies that \( Q \) is a unique mapping satisfying (2.9) such that there exists a \( \mu \in (0, \infty) \) satisfying
   \[ \parallel f_e(x) - Q(x) \parallel \leq \mu (\varphi (x, x) + \varphi (-x, -x)), \]
   for all \( x \in X \);

2. \( d(f^l e, Q) \to 0 \) as \( l \to \infty \). This implies the equality
   \[ \lim_{n \to \infty} 4^n f_e \left( \frac{x}{2^n} \right) = Q(x), \]
   for all \( x \in X \);

3. \( d(f_e, Q) \leq \frac{1}{1-L} d(f_e, Jf_e) \), which implies
   \[ \parallel f_e(x) - Q(x) \parallel \leq \frac{L}{4(1-L)} (\varphi (x, x) + \varphi (-x, -x)), \]
   for all \( x \in X \).

It follows from (2.4) and (2.5) that
\[
\begin{align*}
\parallel \frac{1}{2} Q \left( \frac{x+y}{2} \right) + \frac{1}{2} Q \left( \frac{x-y}{2} \right) - Q(x) - Q(y) - \rho \left( 2Q \left( \frac{x+y}{2} \right) + 2Q \left( \frac{x-y}{2} \right) - Q(x) - Q(y) \right) \parallel \\
= \lim_{n \to \infty} \parallel 4^n \left( \frac{1}{2} f_e \left( \frac{x+y}{2^{n+1}} \right) + f_e \left( \frac{x-y}{2^n} \right) - f_e \left( \frac{x}{2^n} \right) - f_e \left( \frac{y}{2^n} \right) \right) \\
- 4^n \rho \left( 2f_e \left( \frac{x+y}{2^{n+1}} \right) + f_e \left( \frac{x-y}{2^n} \right) - f_e \left( \frac{x}{2^n} \right) - f_e \left( \frac{y}{2^n} \right) \right) \parallel \\
\leq \frac{1}{2} \lim_{n \to \infty} \left( 4^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right) + 4^n \varphi \left( -\frac{x}{2^n}, -\frac{y}{2^n} \right) \right) = 0,
\end{align*}
\]
for all \(x, y \in X\). So
\[
\frac{1}{2} Q \left( \frac{x+y}{2} \right) + \frac{1}{2} Q \left( \frac{x-y}{2} \right) - Q(x) - Q(y) = \rho \left( 2Q \left( \frac{x+y}{2} \right) + 2Q \left( \frac{x-y}{2} \right) - Q(x) - Q(y) \right),
\]
for all \(x, y \in X\). By Lemma 2.2, the mapping \(Q : X \rightarrow Y\) is quadratic.

**Corollary 2.4.** Let \(r > 2\) and \(\theta\) be nonnegative real numbers, and let \(f : X \rightarrow Y\) be a mapping satisfying \(f(0) = 0\) and
\[
\|M_1 f(x, y) - \rho M_2 f(x, y)\| \leq \theta(\|x\|^r + \|y\|^r),
\]
for all \(x, y \in X\). Then there exist a unique additive mapping \(A : X \rightarrow Y\) and a unique quadratic mapping \(Q : X \rightarrow Y\) such that
\[
\|f_o(x) - A(x)\| \leq \frac{20}{2^r - 2} \|x\|^r,
\]
\[
\|f_e(x) - Q(x)\| \leq \frac{4\theta}{2^r - 4} \|x\|^r,
\]
for all \(x \in X\).

**Proof.** The proof follows from Theorem 2.3 by taking \(\phi(x, y) = \theta(\|x\|^r + \|y\|^r)\) for all \(x, y \in X\). Then we can choose \(L = 2^{1-r}\) for \(f_o\) (respectively, \(L = 2^{2-r}\) for \(f_e\)) and we get the desired result. 

**Theorem 2.5.** Let \(\phi : X^2 \rightarrow [0, \infty)\) be a function such that there exists an \(L < 1\) with
\[
\phi(x, y) \leq 2L \phi \left( \frac{x+y}{2} \right),
\]
for all \(x, y \in X\). Let \(f : X \rightarrow Y\) be a mapping satisfying \(f(0) = 0\) and (2.5). Then there exist a unique additive mapping \(A : X \rightarrow Y\) and a unique quadratic mapping \(Q : X \rightarrow Y\) such that
\[
\|f_o(x) - A(x)\| \leq \frac{1}{4(1-L)} (\phi(x, x) + \phi(-x, -x)),
\]
\[
\|f_e(x) - Q(x)\| \leq \frac{1}{4(1-L)} (\phi(x, x) + \phi(-x, -x)),
\]
for all \(x \in X\).

**Proof.** Let \((S, d)\) be the generalized metric space defined in the proof of Theorem 2.3.

It follows from (2.6) that
\[
\left\| f_o(x) - \frac{1}{2} f_o(2x) \right\| \leq \frac{1}{4} \phi(x, x) + \frac{1}{4} \phi(-x, -x),
\]
for all \(x \in X\).

For \(f_o\), we consider the linear mapping \(J : S \rightarrow S\) such that
\[
Jg(x) := \frac{1}{2} g(2x),
\]
for all \(x \in X\).

It follows from (2.8) that
\[
\left\| f_e(x) - \frac{1}{4} f_e(2x) \right\| \leq \frac{1}{4} \phi(x, x) + \frac{1}{4} \phi(-x, -x),
\]
for all \(x \in X\).
For \( f_e \), we consider the linear mapping \( J : S \to S \) such that
\[
Jg(x) := \frac{1}{4}g(2x),
\]
for all \( x \in X \).

The rest of the proof is similar to the proof of Theorem 2.3.

**Corollary 2.6.** Let \( r < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) and (2.10). Then there exist a unique additive mapping \( A : X \to Y \) and a unique quadratic mapping \( Q : X \to Y \) such that
\[
\|f_\alpha(x) - A(x)\| \leq \frac{2\theta}{2 - 2^r}\|x\|^r, \\
\|f_e(x) - Q(x)\| \leq \frac{4\theta}{4 - 2^r}\|x\|^r,
\]
for all \( x \in X \).

**Proof.** The proof follows from Theorem 2.5 by taking \( \phi(x,y) = \theta(\|x\|^r + \|y\|^r) \) for all \( x, y \in X \). Then we can choose \( L = 2^{r-1} \) for \( f_\alpha \) (respectively, \( L = 2^{r-2} \) for \( f_e \)) and we get the desired result. \( \square \)

### 3. Additive-quadratic \( \rho \)-functional equation (2) in Banach spaces

We solve and investigate the additive-quadratic \( \rho \)-functional equation (2) in normed spaces.

**Lemma 3.1.** If a mapping \( f : X \to Y \) satisfies \( f(0) = 0 \) and
\[
M_2f(x,y) = \rho M_1f(x,y), \tag{3.1}
\]
for all \( x, y \in X \), then \( f : X \to Y \) is the sum of the Cauchy additive mapping \( f_\alpha \) and the quadratic mapping \( f_e \).

**Proof.** Letting \( y = 0 \) in (3.1) for \( f_\alpha \), we get
\[
f_\alpha \left( \frac{x}{2} \right) = \frac{1}{2}f_\alpha(x), \tag{3.2}
\]
for all \( x \in X \).

It follows from (3.1) and (3.2) that
\[
f_\alpha(x + y) - f_\alpha(x) - f_\alpha(y) = 2f_\alpha \left( \frac{x + y}{2} \right) - f_\alpha(x) - f_\alpha(y) = \rho(f_\alpha(x + y) - f_\alpha(x) - f_\alpha(y)),
\]
and so
\[
f_\alpha(x + y) = f_\alpha(x) + f_\alpha(y),
\]
for all \( x, y \in X \).

Letting \( y = 0 \) in (3.1) for \( f_e \), we get
\[
f_e \left( \frac{x}{2} \right) = \frac{1}{4}f_e(x), \tag{3.3}
\]
for all \( x \in X \).

It follows from (3.1) and (3.3) that
\[
\frac{1}{2}f_e(x + y) + \frac{1}{2}f_e(x - y) - f_e(x) - f_e(y) = 2f_e \left( \frac{x + y}{2} \right) + 2f_e \left( \frac{x - y}{2} \right) - f_e(x) - f_e(y)
\]
\[
= \rho \left( \frac{1}{2}f_e(x + y) + \frac{1}{2}f_e(x - y) - f_e(x) - f_e(y) \right),
\]
and so
\[
f_e(x + y) + f_e(x - y) = 2f_e(x) + 2f_e(y),
\]
for all \( x, y \in X \). \( \square \)
Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic \(\rho\)-functional equation (3.1) in Banach spaces.

**Theorem 3.2.** Let \(\varphi : X^2 \to [0, \infty)\) be a function such that there exists an \(L < 1\) with
\[
\varphi \left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{4} \varphi (x, y),
\]
for all \(x, y \in X\). Let \(f : X \to Y\) be a mapping satisfying \(f(0) = 0\) and
\[
\|M_2 f(x, y) - \rho M_1 f(x, y)\| \leq \varphi (x, y),
\]
for all \(x, y \in X\). Then there exist a unique additive mapping \(A : X \to Y\) and a unique quadratic mapping \(Q : X \to Y\) such that
\[
\|f_o(x) - A(x)\| \leq \frac{1}{2(1 - L)} (\varphi (x, 0) + \varphi (-x, 0)),
\]
\[
\|f_e(x) - Q(x)\| \leq \frac{1}{2(1 - L)} (\varphi (x, 0) + \varphi (-x, 0)),
\]
for all \(x \in X\).

**Proof.** Letting \(y = 0\) in (3.4) for \(f_o\), we get
\[
\left\| f_o(x) - 2f_o(\frac{x}{2}) \right\| = \left\| 2f_o(\frac{x}{2}) - f_o(x) \right\| \leq \frac{1}{2} \varphi (x, 0) + \frac{1}{2} \varphi (-x, 0),
\]
for all \(x \in X\).

Consider the set
\[
S := \{ h : X \to Y, \ h(0) = 0 \},
\]
and introduce the generalized metric on \(S\):
\[
d(g, h) = \inf \{ \mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu (\varphi (x, 0) + \varphi (-x, 0)), \ \forall x \in X \},
\]
where, as usual, \(\inf \varphi = +\infty\). It is easy to show that \((S, d)\) is complete (see [7]).

For \(f_o\), we consider the linear mapping \(J : S \to S\) such that
\[
Jg(x) := 2g \left(\frac{x}{2}\right),
\]
for all \(x \in X\).

Letting \(y = 0\) in (3.4) for \(f_e\), we get
\[
\left\| f_e(x) - 4f_e(\frac{x}{2}) \right\| = \left\| 4f_e(\frac{x}{2}) - f_e(x) \right\| \leq \frac{1}{2} \varphi (x, 0) + \frac{1}{2} \varphi (-x, 0),
\]
for all \(x \in X\).

For \(f_e\), we consider the linear mapping \(J : S \to S\) such that
\[
Jg(x) := 4g \left(\frac{x}{2}\right),
\]
for all \(x \in X\).

The rest of the proof is similar to the proof of Theorem 2.3. \(\square\)

**Corollary 3.3.** Let \(r > 2\) and \(\theta\) be nonnegative real numbers, and let \(f : X \to Y\) be a mapping satisfying \(f(0) = 0\) and
\[
\|M_2 f(x, y) - \rho M_1 f(x, y)\| \leq \theta (\|x\|^r + \|y\|^r),
\]
for all \(x, y \in X\). Then there exist a unique additive mapping \(A : X \to Y\) and a unique quadratic mapping \(Q : X \to Y\)
such that
\[ \| f_o(x) - A(x) \| \leq \frac{2r\theta}{2r - 2} \| x \| r, \]
\[ \| f_e(x) - Q(x) \| \leq \frac{2r\theta}{2r - 4} \| x \| r, \]
for all \( x \in X \).

Proof. The proof follows from Theorem 3.2 by taking \( \phi(x, y) = \theta(\| x \| r + \| y \| r) \) for all \( x, y \in X \). Then we can choose \( L = 2^{1-r} \) for \( f_o \) (respectively, \( L = 2^{2-r} \) for \( f_e \)) and we get the desired result. \( \square \)

**Theorem 3.4.** Let \( \varphi : X^2 \to [0, \infty) \) be a function such that there exists an \( L < 1 \) with
\[ \varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right), \]
for all \( x, y \in X \). Let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) and (3.4). Then there exist a unique additive mapping \( A : X \to Y \) and a unique quadratic mapping \( Q : X \to Y \) such that
\[ \| f_o(x) - A(x) \| \leq \frac{L}{2(1-L)} (\varphi(x, 0) + \varphi(-x, 0)), \]
\[ \| f_e(x) - Q(x) \| \leq \frac{L}{2(1-L)} (\varphi(x, 0) + \varphi(-x, 0)), \]
for all \( x \in X \).

Proof. Let \((S, d)\) be the generalized metric space defined in the proof of Theorem 3.2. It follows from (3.5) that
\[ \left\| f_o(x) - \frac{1}{2} f_o(2x) \right\| \leq \frac{1}{4} \varphi(2x, 0) + \frac{1}{4} \varphi(-2x, 0) \leq \frac{L}{2} \varphi(x, 0) + \frac{L}{2} \varphi(-x, 0), \]
for all \( x \in X \).

For \( f_o \), we consider the linear mapping \( J : S \to S \) such that
\[ Jg(x) := \frac{1}{2} g(2x), \]
for all \( x \in X \).

It follows from (3.6) that
\[ \left\| f_e(x) - \frac{1}{4} f_e(2x) \right\| \leq \frac{1}{8} \varphi(2x, 0) + \frac{1}{8} \varphi(-2x, 0) \leq \frac{L}{4} \varphi(x, 0) + \frac{L}{4} \varphi(-x, 0) \leq \frac{L}{2} \varphi(x, 0) + \frac{L}{2} \varphi(-x, 0), \]
for all \( x \in X \), since \( \frac{1}{4} \varphi(x, 0) + \frac{1}{4} \varphi(-x, 0) \leq \frac{1}{8} \varphi(x, 0) + \frac{1}{8} \varphi(-x, 0) \) for all \( x \in X \).

For \( f_e \), we consider the linear mapping \( J : S \to S \) such that
\[ Jg(x) := \frac{1}{4} g(2x), \]
for all \( x \in X \).

The rest of the proof is similar to the proof of Theorem 2.3. \( \square \)

**Corollary 3.5.** Let \( \tau < 1 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping satisfying (3.7). Then
there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$
\| f_o(x) - A(x) \| \leq \frac{2^r \theta}{2 - 2^r} \| x \|^r,
$$

$$
\| f_e(x) - Q(x) \| \leq \frac{2^r \theta}{4 - 2^r} \| x \|^r,
$$

for all $x \in X$.

**Proof.** The proof follows from Theorem 3.2 by taking $\varphi(x, y) = \theta(\| x \|^r + \| y \|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-1}$ for $f_o$ (respectively, $L = 2^{r-2}$ for $f_e$) and we get the desired result. \qed

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**References**