Oscillation of second-order difference equations

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Abstract
We obtain new oscillation theorems for a class of second-order linear difference equations. Our criteria complement and improve related results reported in the literature. An illustrative example is given. ©2017 All rights reserved.

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1. Introduction
In this paper, we are concerned with the oscillation of a linear second-order difference equation
\[ \Delta^2 x_{n-1} + p_n x_n = 0, \quad n = 0, 1, 2, \ldots, \] (1.1)
where \( \Delta \) is the forward difference operator satisfying \( \Delta x_n = x_{n+1} - x_n \) and \( \{p_n\} \) is a sequence of non-negative real numbers. A solution \( \{x_n\} \) of (1.1) is termed oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its nontrivial solutions oscillate.

Oscillation and asymptotic behavior of various classes of difference equations have always attracted interest of researchers; see, e.g., the monograph [1], the papers [2–16], and the references cited therein. In particular, several interesting oscillation results for equation (1.1) were established in the papers by Erbe and Zhang [4], Jiang and Li [5], Lei [6], Sun [8], and Zhang and Cheng [14], some of which we present below for the convenience of the reader. In the following, we use the notation:

\[ u_n(\alpha) = n^{1-\alpha} \sum_{k=n+1}^{\infty} k^\alpha p_k, \quad \sum_{k=1}^{\infty} k^\alpha p_k < \infty, \quad p_*(\alpha) = \liminf_{n \to \infty} u_n(\alpha), \quad \text{and} \quad p^*(\alpha) = \limsup_{n \to \infty} u_n(\alpha). \]
Theorem 1.1 ([4]). If
\[ \liminf_{n \to \infty} n^2 p_n \geq 1/4, \]
then equation (1.1) is oscillatory.

Theorem 1.2 ([14]). If
\[ p_* (0) > 1/4, \]
then equation (1.1) is oscillatory.

Theorem 1.3 ([5, 6]). Let
\[ p_* (0) \leq 1/4. \]
If there exists a constant \( \alpha > 1 \) such that
\[ p^*(\alpha) > \frac{\alpha^2}{4(\alpha - 1)} - \frac{1}{2} \left( 1 - \sqrt{1 - 4p_* (0)} \right), \]
then equation (1.1) is oscillatory.

This study was strongly motivated by the research of Erbe and Zhang [4], Jiang and Li [5], Lei [6], and Zhang and Cheng [14]. Its purpose is to obtain new oscillation criteria for equation (1.1) that improve Theorems 1.1 and 1.2 and complement Theorem 1.3. It is not difficult to see that if there exists a constant \( \alpha < 1 \) such that
\[ \sum_{k=1}^{\infty} k^\alpha p_k = \infty, \]
then equation (1.1) is oscillatory. In the sequel, we assume that
\[ \sum_{k=1}^{\infty} k^\alpha p_k < \infty, \quad \alpha < 1. \]

As usual, all functional inequalities considered in this paper are supposed to hold eventually. Without loss of generality, we deal only with positive solutions of (1.1) since \( \{-x_n\} \) is also a solution of this equation provided that \( \{x_n\} \) is a solution.

2. Lemmas

To prove the main results, we need the following lemmas. For a compact presentation of our results, we adopt the notation:
\[ q = \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} k^2 p_k, \quad w_n = \frac{\Delta x_{n-1}}{x_{n-1}}, \quad \tau = \liminf_{n \to \infty} n w_{n+1}, \quad \text{and} \quad R = \limsup_{n \to \infty} n w_{n+1}. \]

Lemma 2.1. If \( \alpha \in [0, 1] \), then
\[ \sum_{k=n+1}^{\infty} \frac{(\Delta k^\alpha)^2}{k^\alpha} \leq \frac{\alpha^2}{1 - \alpha} n^{\alpha - 1} \quad (2.1) \]
and
\[ \sum_{k=n+1}^{\infty} k^{\alpha - 2} \leq \frac{n^{\alpha - 1}}{1 - \alpha}. \quad (2.2) \]

Proof. By virtue of the mean value theorem, there exist two numbers \( \xi_k \in (k, k+1) \) and \( \eta_k \in (k-1, k) \) such that
\[ \frac{(\Delta k^\alpha)^2}{k^\alpha} = \frac{\alpha^2 \xi_k^{2\alpha - 2}}{k^\alpha} \leq \frac{\alpha^2 k^{2\alpha - 2}}{k^\alpha} \quad \text{and} \quad \frac{(\Delta k - 1)^{1-\alpha}}{1 - \alpha} = \eta_k^{1-\alpha} \geq \frac{1}{k^{\alpha}}. \]
Hence, we deduce that
\[
\sum_{k=n+1}^{\infty} \frac{(\Delta k^\alpha)^2}{k^\alpha} \leq \frac{\alpha^2}{1-\alpha} \sum_{k=n+1}^{\infty} \frac{\Delta(k-1)^{1-\alpha}}{k^{2-2\alpha}}.
\]  \tag{2.3}

Define \( r(t) = (k-1)^{1-\alpha} + (t-k+1)\Delta(k-1)^{1-\alpha}, k-1 \leq t \leq k \). Then \( r(t) = \Delta(k-1)^{1-\alpha}, (k-1)^{1-\alpha} \leq r(t) \leq k^{1-\alpha}, k-1 \leq t \leq k \), and so
\[
\frac{\Delta(k-1)^{1-\alpha}}{k^{2-2\alpha}} = \int_{k-1}^{k} \frac{\Delta(k-1)^{1-\alpha}}{k^{2-2\alpha}} \, dt \leq \int_{k-1}^{k} \frac{r'(t)}{r^2(t)} \, dt = \frac{1}{(k-1)^{1-\alpha}} - \frac{1}{k^{1-\alpha}}.
\]

It follows from the latter inequality and (2.3) that (2.1) holds. Using the inequality
\[
\sum_{k=n+1}^{\infty} \frac{1}{k^{2-\alpha}} < \int_{n}^{\infty} \frac{1}{t^{2-\alpha}} \, dt,
\]
we have (2.2). The proof is complete.

**Lemma 2.2** ([5]). Let \( \{x_n\} \) be a nonoscillatory solution of equation (1.1) such that \( x_{n-1} > 0 \) for \( n \geq n_0 \). Then
\[
\Delta w_n + w_n w_{n+1} + p_n \leq 0, \quad n \geq n_0, \tag{2.4}
\]
\[
w_n \geq w_{n+1}, \quad 0 \leq (n-n_0)w_n < 1, \quad n \geq n_0, \tag{2.5}
\]
and
\[
p_+(0) \leq r - r^2, \quad q \leq R - R^2. \tag{2.6}
\]

3. Main Results

Let
\[
M_1 = \frac{1}{2} \left( 1 + \sqrt{1 - 4q} \right) \quad \text{and} \quad M_2 = \frac{1}{2} \left( 1 - \sqrt{1 - 4p_+(0)} \right).
\]

We give the following oscillation results for equation (1.1).

**Theorem 3.1.** Let \( q \leq 1/4 \). If there exists a constant \( \alpha \in [0, 1) \) such that
\[
p^+(\alpha) > \frac{\alpha^2}{4(1-\alpha)} + M_1, \tag{3.1}
\]
then equation (1.1) is oscillatory.

**Proof.** Let \( \{x_n\} \) be a nonoscillatory solution of equation (1.1) such that \( x_{n-1} > 0 \) for \( n \geq n_0 \). From (2.4) and (2.5), we conclude that
\[
p_k \leq -\Delta w_k - w_{k+1}^2. \tag{3.2}
\]
Multiplying (3.2) by \( k^\alpha \) and summing the resulting inequality from \( n+1 \) to \( \infty \), we get
\[
\sum_{k=n+1}^{\infty} k^\alpha p_k \leq -\sum_{k=n+1}^{\infty} k^\alpha \Delta w_k - \sum_{k=n+1}^{\infty} k^\alpha w_{k+1}^2
\]
\[
= (n+1)^\alpha w_{n+1} + \sum_{k=n+1}^{\infty} w_{k+1} \Delta k^\alpha - \sum_{k=n+1}^{\infty} k^\alpha w_{k+1}^2. \tag{3.3}
\]
Using (3.3), we have
\[
\sum_{k=n+1}^{\infty} k^\alpha p_k \leq (n+1)^\alpha w_{n+1} + \frac{1}{4} \sum_{k=n+1}^{\infty} \frac{(\Delta k^\alpha)^2}{k^\alpha} - \sum_{k=n+1}^{\infty} \left( k^\alpha w_{k+1} - \frac{1}{2} k^{\frac{\alpha-2}{\alpha}} \Delta k^\alpha \right)^2
\]
\[ \leq (n + 1)^\alpha w_{n+1} + \frac{1}{4} \sum_{k=n+1}^{\infty} \frac{(\Delta k^\alpha)^2}{k^\alpha}, \]

which yields
\[ \limsup_{n \to \infty} n^{1-\alpha} \sum_{k=n+1}^{\infty} k^\alpha p_k \leq \limsup_{n \to \infty} \left( \frac{n + 1}{n} \right)^\alpha n w_{n+1} + \limsup_{n \to \infty} \frac{1}{4} n^{1-\alpha} \sum_{k=n+1}^{\infty} \frac{(\Delta k^\alpha)^2}{k^\alpha}. \]

Hence, by (2.1) and (2.5), we deduce that
\[ p^*(\alpha) \leq \limsup_{n \to \infty} n w_{n+1} + \limsup_{n \to \infty} \frac{1}{4} n^{1-\alpha} n^{\alpha-1} \frac{\alpha^2}{1-\alpha} = R + \frac{\alpha^2}{4(1-\alpha)}. \]

On the other hand, we have
\[ R \leq M_1 \] (3.4)
due to (2.6). Therefore, we arrive at
\[ p^*(\alpha) \leq \frac{\alpha^2}{4(1-\alpha)} + \frac{1}{2} \left( 1 + \sqrt{1-4q} \right), \]

which contradicts (3.1). The proof is complete. \( \square \)

**Theorem 3.2.** Let \( p_\alpha(0) \leq 1/4 \) and \( q \leq 1/4 \). If there exists a constant \( \alpha \in [M_2, 1) \) such that
\[ p^*(\alpha) > \frac{M_1(1 - M_2)}{1-\alpha}, \] (3.5)

then equation (1.1) is oscillatory.

**Proof.** Assume that \( x_n \) is a positive solution of equation (1.1) such that \( x_{n-1} > 0 \) for \( n \geq n_0 \). By virtue of (2.6),
\[ r \geq M_2. \]

From the latter inequality and (3.4), we conclude that, for any \( \varepsilon > 0 \), there exists an \( n_1 \geq n_0 \) such that
\[ M_2 - \varepsilon < n w_{n+1} \leq \left( \frac{n + 1}{n} \right)^\alpha n w_{n+1} < M_1 + \varepsilon \]
for \( n \geq n_1 \). On the other hand, as in the proof of Theorem 3.1, we have (3.3). Using the fact that \( \Delta k^\alpha = (k+1)^\alpha - k^\alpha < ak^\alpha - 1 \) and multiplying (3.3) by \( n^{1-\alpha} \), we obtain
\[ n^{1-\alpha} \sum_{k=n+1}^{\infty} k^\alpha p_k \leq \left( \frac{n + 1}{n} \right)^\alpha n w_{n+1} + n^{1-\alpha} \sum_{k=n+1}^{\infty} k^\alpha \left[ k w_{k+1}(\alpha - k w_{k+1}) \right] \]
\[ \leq M_1 + \varepsilon + (M_1 + \varepsilon)(\alpha + \varepsilon - M_2) n^{1-\alpha} \sum_{k=n+1}^{\infty} k^{\alpha-2}. \]

Substituting (2.2) into the latter inequality, we deduce that
\[ n^{1-\alpha} \sum_{k=n+1}^{\infty} k^\alpha p_k \leq (M_1 + \varepsilon) \left[ 1 + n^{1-\alpha} n^{\alpha-1} \frac{1}{1-\alpha} (\alpha + \varepsilon - M_2) \right] = (M_1 + \varepsilon) \frac{1 - M_2 + \varepsilon}{1-\alpha}. \]

Since \( \varepsilon > 0 \) is arbitrary, we get
\[ p^*(\alpha) \leq \frac{M_1(1 - M_2)}{1-\alpha}, \]
which contradicts (3.5). This completes the proof. \( \square \)
Remark 3.3. Observe that $M_2 \in [0, 1/2]$ in the case when $p_*(0) \in [0, 1/4]$. Let all hypotheses of Theorem 3.2 be satisfied with condition $\alpha \in [M_2, 1]$ replaced by $\alpha \in [1/2, 1)$. Then equation (1.1) is oscillatory.

Remark 3.4. Note that $\alpha > 1$ is required in Theorem 1.3. Hence, Theorems 3.1 and 3.2 complement the results obtained in [5, 6].

4. Example

Example 4.1. Consider the difference equation

$$
\Delta^2 x_{n-1} + p_n x_n = 0, \quad n = 0, 1, 2, \ldots,
$$

(4.1)

where

$$
p_n = \begin{cases} 
\frac{1}{6^m}, & n = 6^m, \\
0, & n \neq 6^m,
\end{cases}
$$

$m = 0, 1, 2, \ldots$.

It is not difficult to verify that

$$
p_*(0) = \liminf_{n\to\infty} u_n(0) = \frac{1}{5} < \frac{1}{4} \quad \text{and} \quad q = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} k^2 p_k = \frac{1}{5} < \frac{1}{4}.
$$

Thus, we conclude that

$$
M_1 = \frac{1}{2} \left( 1 + \sqrt{1 - 4q} \right) = \frac{1}{2} \left( 1 + \sqrt{\frac{5}{5}} \right) \quad \text{and} \quad M_2 = \frac{1}{2} \left( 1 - \sqrt{1 - 4p_*(0)} \right) = \frac{1}{2} \left( 1 - \sqrt{\frac{5}{5}} \right).
$$

Let $\alpha = 1/2$. Then

$$
\frac{M_1 (1 - M_2)}{1 - \alpha} = \frac{1}{2} \left( \frac{6}{5} + \frac{2\sqrt{5}}{5} \right) \quad \text{and} \quad \frac{p^*(\alpha)}{1 - \alpha} = \limsup_{n\to\infty} u_n \left( \frac{1}{2} \right) = \limsup_{n\to\infty} n^{\frac{1}{2}} \sum_{k=n+1}^{\infty} k^2 p_k = \frac{6 + \sqrt{6}}{5},
$$

and so

$$
p^*(\alpha) > \frac{M_1 (1 - M_2)}{1 - \alpha}.
$$

Therefore, by Theorem 3.2, equation (4.1) is oscillatory. Observe that Theorems 1.1 and 1.2 cannot be applied to equation (4.1). Hence, Theorem 3.2 improves Theorems 1.1 and 1.2.

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