Majorization by starlike functions

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Abstract

The main object of this paper is to investigate some majorization problems involving the subclass \( S(\alpha, A, B) \) of starlike functions in the open unit disk \( U \). Relevant connections of the results presented here with those given by earlier workers on the subject are also indicated. ©2017 All rights reserved.

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1. Introduction

Let \( A \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in the open unit disk

\[
U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}.
\]

**Definition 1.1.** For two functions \( f \) and \( g \), which are analytic in \( U \), the function \( f \) is said to be subordinate to \( g \), written as

\[
f \prec g \quad \text{or} \quad f(z) \prec g(z)
\]

if there exists a Schwarz function \( w \) analytic in \( U \), with

\[
\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in U)
\]

and such that

\[
f(z) = g(\omega(z)) \quad (z \in U).
\]

In particular, if the function \( g \) is univalent in \( U \), the above subordination is equivalent to

\[
f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).
\]

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Definition 1.2. For two functions \( f \) and \( g \), which are analytic in \( U \), the function \( f \) is said to be majorized to \( g \), written as
\[
 f << g \quad \text{or} \quad f(z) << g(z)
\]
if there exists a function \( \varphi \) analytic in \( U \), with
\[
 |\varphi(z)| < 1 \quad (z \in U)
\]
and such that
\[
 f(z) = \varphi(z) g(z) \quad (z \in U),
\]
(see MacGregor [6]).

The majorization is closely related to the concept of quasi-subordination between analytic functions, which was considered recently by (for example) Altıntış and Owa [3]. Some majorization problems were studied by Altıntaş et al. in [4, 5]. Therefore, various subclasses of univalent functions in \( U \) were studied by Akgul in [1, 2].

We purpose to investigate the majorization problems associated with the class \( S(\alpha, A, B) \) of starlike functions.

Definition 1.3. We denote by \( S(\alpha, A, B) \) the class of functions satisfying the condition
\[
 \frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf'(z)}{f(z)} \right)' < \frac{1 + Az}{1 + Bz},
\]
(\( z \in U, \ f \in A, \ 0 \leq \alpha \leq 1, \ -1 \leq B < A \leq 1 \)).

Clearly, we have the following relationships:

- \( S(0,1,-1) = S^{*} \) is the class of starlike functions;
- \( S(0,0,-1) = C \) is the class of convex functions;
- \( S(0,1-2\alpha,-1) = S^{*}(\alpha) \) is the class of starlike functions of order \( \alpha \), \( 0 \leq \alpha < 1 \);
- \( S(0,1-\alpha,-1) = C(\alpha) \) is the class of convex functions of order \( \alpha \), \( 0 \leq \alpha < 1 \).

2. Majorization problems for the class \( S(\alpha, A, B) \)

We first state and prove the following Lemma 2.1.

Lemma 2.1 ([9]). If the function \( h(z) = 1 + \sum_{n=1}^{\infty} c_{n} z^{n} \) is analytic in \( U \) and satisfies the condition
\[
 h(z) < \frac{1 + Az}{1 + Bz} \quad (z \in U, \ -1 < B < A \leq 1),
\]
then
\[
 \text{Re} \ h(z) > \frac{1 - A}{1 - B} = \beta. \tag{2.2}
\]

Proof. Using (1.1) and (2.1) we have
\[
 h(z) = \frac{1 + A \omega(z)}{1 + B \omega(z)} \quad (\omega(0) = 0, \ |\omega(z)| < 1)
\]
and
\[
 |\omega(z)| = \left| \frac{h(z) - 1}{A - Bh(z)} \right|,
\]
for \( h(z) = u + iv \).
Since $|h(z)|^2 \geq [\text{Re } h(z)]^2$, we have
\[(1 - B^2) u^2 - 2(1 - AB) u + 1 - A^2 < 0,\]
which implies that
\[
\frac{1 - A}{1 - B} < u = \text{Re } h(z) < \frac{1 + A}{1 + B}.
\]

Lemma 2.2 ([8]). If the function $p(z) = 1 + \sum_{n=1}^\infty p_n z^n$ is analytic in $U$ and satisfies the condition
\[
\text{Re } (p(z) + \alpha z p'(z)) > \beta, \quad (2.3)
\]
then
\[
\text{Re } p(z) > \frac{\alpha + 2\beta}{\alpha + 2} \quad (0 \leq \alpha \leq 1, \ 0 \leq \beta < 1). \quad (2.4)
\]

Theorem 2.3. Let the function $f(z)$ be in the class $A$ and suppose that $g \in S(\alpha, A, B)$. If $f(z)$ is majorized by $g(z)$ in $U$, then
\[
|f'(z)| \leq |g'(z)| \quad (|z| \leq r_1),
\]
where
\[
r_1 = r_1(\alpha, A, B) = \frac{3 + |1 - 2\gamma| - \sqrt{|1 - 2\gamma|^2 + 2|1 - 2\gamma| + 9}}{2|1 - 2\gamma|} \quad (2.5)
\]
and
\[
\gamma = \frac{\alpha(1 - B) + 2(1 - A)}{(\alpha + 2)(1 - B)} \quad (0 \leq \alpha \leq 1, -1 \leq B < A < 1). \quad (2.6)
\]

Proof. Since $g \in S(\alpha, A, B)$, if we let
\[
\frac{zg'(z)}{g(z)} = p(z) \quad \text{and} \quad (p(z) + \alpha z p'(z)) = h(z)
\]
and $\beta = \frac{1 - A}{1 - B}$, then using (1.2), (2.2), (2.3), and (2.4) we find
\[
\text{Re } \frac{zg'(z)}{g(z)} > \frac{\alpha + 2\beta}{\alpha + 2}.
\]
Letting $\gamma = \frac{\alpha + 2\beta}{\alpha + 2}$, we obtain
\[
\frac{zg'(z)}{g(z)} = \frac{1 - (1 - 2\gamma) \omega(z)}{1 + \omega(z)},
\]
where $\omega(0) = 0$ and $|\omega(z)| < 1$.

Hence we find the inequality
\[
|g(z)| \leq \left( \frac{(1 + |z|)|z|}{1 - |1 - 2\gamma||z|} \right) |g'(z)| \quad (z \in U). \quad (2.7)
\]

Since $f(z)$ is majorized by $g(z)$ in $U$, from (1.1) we have
\[
f'(z) = \varphi(z) g'(z) + \varphi'(z) g(z). \quad (2.8)
\]

We know that $\varphi(z)$ satisfies the inequality (Nehari, [7, p.168])
\[
|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \quad (z \in U), \quad (2.9)
\]
and using (2.7) and (2.9) in (2.8), we get
\[ |f'(z)| \leq \left| \frac{\varphi(z)}{1-|z|^2} \frac{(1+|z|^2)}{1-|1-2\gamma|} \right| |g'(z)|, \]
which upon setting
\[ |z| = r \quad \text{and} \quad |\varphi(z)| = \mu \quad (0 \leq \mu \leq 1) \]
we have the inequality
\[ |f'(z)| \leq \frac{\Theta(\mu)}{(1-r)(1-|1-2\gamma|)} |g'(z)| \quad (z \in U), \tag{2.10} \]
where the function \( \Theta(\mu) \) defined by
\[ \Theta(\mu) = -r\mu^2 + (1-r)(1-|1-2\gamma|) \mu + r \quad (0 \leq \mu \leq 1) \]
takes the maximum value at \( \mu = 1 \) with \( r = r_1(\gamma) \) given by (2.5).
Furthermore, if \( 0 \leq q \leq r_1(\gamma) \) is given by (2.5), then we have
\[ \Lambda(\mu) \leq \Lambda(1) = (1-r)(1-|1-2\gamma|) \quad (0 \leq \mu \leq 1, \ 0 \leq q \leq r_1(\gamma)). \]
Hence, upon setting \( \mu = 1 \) in (2.10), we conclude that the inequality in (2.5) holds true for \( |z| \leq r_1(\gamma) \) and is given by (2.6). The proof of Theorem 2.4 is based on Lemma 1 in [4],
\[ f \in C(\gamma) \implies f \in S\left(\frac{1}{2}\gamma\right), \]
\[ \square \]

**Theorem 2.4.** Let the function \( f(z) \) be analytic in \( U \) and suppose that \( g \in C(\gamma) \). If \( f(z) \) is majorized by \( g(z) \) in \( U \), then
\[ |f'(z)| \leq |g'(z)| \quad (|z| \leq r_2), \]
where
\[ r_2 = r_2(\alpha, A, B) = \frac{3 + |1-\gamma| - \sqrt{|1-\gamma|^2 + 2|1-\gamma| + 9}}{2|1-\gamma|} \]
and
\[ \gamma = \frac{\alpha(1-B) + 2(1-A)}{(\alpha+2)(1-B)} \quad (0 \leq \alpha \leq 1, -1 \leq B < A \leq 1). \]

**Proof.** Upon replacing \( \gamma \) in Theorem 2.3 by \( \frac{1}{2}\gamma \), the conclusion follows. \[ \square \]

Letting special values for \( \alpha, A, B \) we have the following corollaries.

**Corollary 2.5.** If \( g \in S(\alpha, 1, -1) \) and \( f(z) \) is majorized by \( g(z) \) in \( U \), then
\[ |f'(z)| \leq |g'(z)| \quad (|z| \leq r), \]
where
\[ |z| \leq r = \frac{8 + 2\alpha - \sqrt{8\alpha^2 + 32\alpha + 48}}{2(2-\alpha)} \quad (0 \leq \alpha \leq 1). \]

**Proof.** We let \( A = 1, B = -1 \) in (2.6) and \( \gamma = \frac{\alpha}{\alpha+2} \) in Theorem 2.3. \[ \square \]

**Corollary 2.6.** If \( g \in S(0, 1, -1) \) and \( f(z) \) is majorized by \( g(z) \) in \( U \), then
\[ |f'(z)| \leq |g'(z)| \quad (|z| \leq r), \]
where
\[ |z| \leq r = 2 - \sqrt{3}. \]
Proof. We let $\alpha = 0, A = 1, B = -1$ in (2.6) and $\gamma = 0$ in Theorem 2.3.

**Corollary 2.7.** If $g \in S(\alpha, 0, -1)$ and $f(z)$ is majorized by $g(z)$ in $U$, then

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq r),$$

where

$$|z| \leq r = \frac{2\alpha + 3 - \sqrt{3\alpha^2 + 10\alpha + 9}}{\alpha} \quad (0 \leq \alpha \leq 1).$$

Proof. We let $A = 0, B = -1$ in (2.6) and $\gamma = \frac{\alpha + 1}{\alpha + 2}$ in Theorem 2.3.

**Corollary 2.8.** If $g \in S(1, 1, -1)$ and $f(z)$ is majorized by $g(z)$ in $U$, then

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq r),$$

where

$$|z| \leq r = 5 - \sqrt{22}.$$

Proof. We let $A = 1, B = -1$ in (2.6) and $\gamma = \frac{1}{3}$ in Theorem 2.3.

**Corollary 2.9.** If $g \in C(\alpha, 1, -1)$ and $f(z)$ is majorized by $g(z)$ in $U$, then

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq r),$$

where

$$|z| \leq r = \frac{8 + 3\alpha - \sqrt{9\alpha^2 + 40\alpha + 48}}{4} \quad (0 \leq \alpha \leq 1).$$

Proof. We let $A = 1, B = -1$ in (2.6) and $\gamma = \frac{\alpha + 2}{\alpha + 2}$ in Theorem 2.4.

**Corollary 2.10.** If $g \in C(0, 1, -1)$ and $f(z)$ is majorized by $g(z)$ in $U$, then

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq r),$$

where

$$|z| \leq r = 2 - \sqrt{3}.$$

Proof. We let $A = 0, B = -1$ in (2.6) and $\gamma = 0$ in Theorem 2.4.

**Corollary 2.11.** If $g \in C(\alpha, 0, -1)$ and $f(z)$ is majorized by $g(z)$ in $U$, then

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq r),$$

where

$$|z| \leq r = \frac{7 + 3\alpha - \sqrt{9\alpha^2 + 38\alpha + 41}}{2} \quad (0 \leq \alpha \leq 1).$$

Proof. We let $A = 0, B = -1$ in (2.6) and $\gamma = \frac{\alpha + 1}{\alpha + 2}$ in Theorem 2.4.

**Corollary 2.12.** If $g \in C(1, 1, -1)$ and $f(z)$ is majorized by $g(z)$ in $U$, then

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq r),$$

where

$$|z| \leq r = \frac{11 - \sqrt{97}}{4}.$$

Proof. We let $\alpha = 1, A = 1, B = -1$ in (2.6) and $\gamma = \frac{1}{3}$ in Theorem 2.4.
Corollary 2.13. If $g \in S(0, 0, -1)$ and $f(z)$ is majorized by $g(z)$ in $U$, then

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq r),$$

where

$$|z| \leq r = \frac{1}{3}.$$  

Remark 2.14. $S(0, 0, -1) = S^*\left(\frac{1}{2}\right)$ and $C \subset S^*\left(\frac{1}{2}\right)$.

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