Estimates of initial coefficients for certain subclasses of bi-univalent functions involving quasi-subordination

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Abstract

The object of the present paper is to introduce and investigate new subclasses of the function class $\Sigma$ of bi-univalent functions defined in the open unit disk $U$, involving quasi subordination. The coefficients estimate $|a_2|$ and $|a_3|$ for functions in these new subclasses are also obtained. ©2017 All rights reserved.

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1. Introduction

Let $\mathcal{A}$ be the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C}; |z| < 1\}$.

Suppose $\mathcal{S}$ denotes the class of all functions of $\mathcal{A}$ which satisfy normalized condition $f(0) = 0$ and $f'(0) = 1$ which are univalent in $U$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $U$. In fact, the Koebe-one quarter theorem [1] ensures that the image of $U$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $1/4$. Thus, every univalent function $f$ has an inverse $f^{-1}$ satisfying $f^{-1}(f(z)) = z$, ($z \in U$), and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f), r_0(f) \geq 1/4).$$

In fact, the inverse function $f^{-1}$ is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_3^2 - 5a_2 a_3 + a_4) w^4 + \cdots.$$
A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. Let $\Sigma$ denote the class of bi-univalent functions defined in $\mathbb{U}$.

An analytic function $f(z)$ is subordinate to an analytic function $g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ satisfying $w(0) = 0$ and $|w(z)| < 1 (z \in \mathbb{U})$ satisfying $f(z) = g(w(z))$. We denote this subordination by, cf. [5, p. 226],

$$f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

Further, a function $f(z)$ is said to be quasi-subordinate to $g(z)$ in the open unit disk $\mathbb{U}$ if there exists an analytic function $\varphi(z)$ such that $f(z)/\varphi(z)$ is analytic in $\mathbb{U}$,

$$\frac{f(z)}{\varphi(z)} \prec g(z) \quad (z \in \mathbb{U})$$

and $|\varphi(z)| \leq 1 (z \in \mathbb{U})$. We also denote this quasi-subordination by

$$f(z) \prec_q g(z) \quad (z \in \mathbb{U}). \quad (1.2)$$

Note that the quasi-subordination (1.2) is equivalent to

$$f(z) = \varphi(z)g(w(z)) \quad (z \in \mathbb{U}), \quad (1.3)$$

where $|\varphi(z)| \leq 1 (z \in \mathbb{U})$.

In the quasi-subordination (1.3), if $\varphi(z) \equiv 1$, then (1.3) becomes the subordination (1.3).

For analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, we say $f(z)$ is majorized by $g(z)$ if there exists an analytic function $\varphi(z)$ in $\mathbb{U}$ satisfying $|\varphi(z)| \leq 1$ and $f(z) = \varphi(z)g(z)$ ($z \in \mathbb{U}$). See [7, 11, 12] for works related to quasi-subordination.

Lewin [8] investigated the bi-univalent function class $\Sigma$ and showed that $|a_2| < 1.51$. Subsequently, Brannan et al. [1] conjectured that $|a_2| < \sqrt{2}$. Netanyahu [10], on the other hand, showed that $\max_{f \in \Sigma} |a_2| = \frac{4}{3}$. Brannan and Taha [2] obtained initial coefficient bounds for certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. Later, Srivastava et al. [13] introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients. Many researchers (see [3, 4, 6, 13, 14]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class $\Sigma$ and they have found non-sharp estimates of initial coefficient bounds $|a_2|$ and $|a_3|$.

In the present paper, the coefficient bounds of $|a_2|$ and $|a_3|$ for certain classes involving the quasi-subordination are obtained. The subclasses in this paper are motivated essentially by corresponding subclasses investigated in [7].

2. Coefficient estimates

Let us assume that there exists a function $f(z)$ analytic in the open unit disk $\mathbb{U}$ and $|\varphi(z)| \leq 1$ s.t.

$$\varphi(z) = A_0 + A_1 z + A_2 z^2 + \cdots (|z| < 1). \quad (2.1)$$

Since $\varphi(z)$ is analytic and bounded in $\mathbb{U}$, we have [9, page 172]

$$|A_n| \leq 1 - |A_0|^2 \leq 1 \quad (n > 0). \quad (2.2)$$

Also, let $h(z)$ be an analytic function with positive real part in $\mathbb{U}$, satisfying $h(0) = 1$, $h'(0) > 0$ and $h(\mathbb{U})$ is symmetric with respect to the real axis. Such a function has a Taylor series of the form

$$h(z) = 1 + B_1 z + B_2 z^2 + \cdots , B_1 > 0.$$

Suppose that $u(z)$ and $v(z)$ are analytic in $\mathbb{U}$ with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(z)| < 1$, and suppose that

$$u(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, \quad v(z) = c_1 z + \sum_{n=2}^{\infty} c_n z^n \quad (|z| < 1).$$
It is well-known that
\[
|b_1| \leq 1, \quad |b_2| \leq 1 - |b_1|^2, \quad |c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2. \tag{2.3}
\]
By a simple calculation, we have
\[
h(u(z)) = 1 + B_1 b_1 z + (B_1 b_2 + B_2 b_1^2) z^2 + \cdots, |z| < 1,
\]
and
\[
h(v(w)) = 1 + B_1 c_1 w + (B_1 c_2 + B_2 c_1^2) w^2 + \cdots, |w| < 1.
\]
We define a subclass of \( \Sigma \) as follows:

**Definition 2.1.** A function \( f \in \Sigma \) is said to be in the class \( H_q(\Sigma, h) \) if and only if:
\[
f'(z) - 1 <_q h(z) - 1, \quad g'(w) - 1 <_q h(w) - 1,
\]
or
\[
f'(z) - 1 = \phi(z) (h(u(z)) - 1), \quad g'(w) - 1 = \phi(w) (h(v(w)) - 1),
\]
where \( |\phi(z)| \leq 1 \) \((z \in U)\) and \( g(w) = f^{-1}(w) \).

On taking \( \phi(z) \equiv 1 \) in Definition 2.1, we get the following subordination class:
\[
f'(z) < h(z), \quad g'(w) < h(w).
\]
We name this class as \( H(\Sigma, h) \).

Now, we first derive following coefficient estimates for the subclass \( H_q(\Sigma, h) \):

**Theorem 2.2.** If \( f \) given by (1.1) is in the subclass \( H_q(\Sigma, h) \), then
\[
|a_2| \leq \min \left\{ \frac{B_1}{2}, \sqrt{\frac{B_1 + |B_2|}{3}} \right\}, \quad |a_3| \leq \min \left\{ \frac{B_1^2}{4}, \frac{2}{3} B_1, \frac{|B_2|}{3} + B_1 \right\}.
\]

**Proof.** Let \( f \in H_q^c(h) \) and \( g = f^{-1} \). Then, there are analytic functions \( u, v : U \to U \) given by (2.2) and a function \( \phi(z) \) in \( U \) defined by (2.1) satisfying
\[
f'(z) - 1 = \phi(z) (h(u(z)) - 1), \quad g'(w) - 1 = \phi(w) (h(v(w)) - 1).
\]
Since
\[
f'(z) = 1 + 2a_2 z + 3a_3 z^2 + \cdots, \quad g'(w) = 1 - 2a_2 w + 3 (2a_2^2 - a_3) w^2 + \cdots,
\]
and
\[
\phi(z) (h(u(z)) - 1) = A_0 B_1 b_1 z + \left[ A_1 B_1 b_1 + A_0 (B_1 b_2 + B_2 b_1^2) \right] z^2 + \cdots, \tag{2.4}
\]
with
\[
\phi(w) (h(v(w)) - 1) = A_0 B_1 c_1 w + \left[ A_1 B_1 c_1 + A_0 (B_1 c_2 + B_2 c_1^2) \right] w^2 + \cdots. \tag{2.5}
\]
It follows that
\[
2a_2 = A_0 B_1 b_1, \tag{2.6}
\]
\[
3a_3 = A_1 B_1 b_1 + A_0 (B_1 b_2 + B_2 b_1^2), \tag{2.7}
\]
\[
-2a_2 = A_0 B_1 c_1, \tag{2.8}
\]
\[
3 (2a_2^2 - a_3) = A_1 B_1 c_1 + A_0 (B_1 c_2 + B_2 c_1^2). \tag{2.9}
\]
From (2.6) and (2.8), we have
\[
b_1 = -c_1. \tag{2.10}
\]
Squaring and adding (2.6) and (2.8), and then using (2.10), we have

\[4a_2^2 = A_0^2 B_1^2 c_1^2.\]  

(2.11)

Adding (2.7) and (2.9), we have

\[6a_2^2 = A_0 [B_1 (b_2 + c_2) + 2B_2 c_1^2].\]  

(2.12)

By using (2.2) and (2.3), we have from (2.11) and (2.12) that

\[|a_2| \leq \frac{B_1}{2} \quad \text{and} \quad |a_2| \leq \sqrt{\frac{B_1 + B_2}{3}},\]  

(2.13)

respectively. So we get the desired estimate on the coefficient \(|a_2|\) as asserted in (2.1).

Next, in order to find the bound on the coefficient \(|a_3|\), we subtract (2.9) from (2.7). Thus, we get

\[6a_3 - 6a_2^2 = A_1 B_1 (b_1 - c_1) + A_0 [B_1 (b_2 - c_2) + B_2 (b_1^2 - c_1^2)].\]  

Therefore (2.10), (2.11), and (2.13) yield

\[6a_3 = \frac{3}{2} A_0^2 B_1^2 c_1^2 - 2c_1 A_1 B_1 + A_0 B_1 (b_2 - c_2).\]  

Thus, we find by using (2.2) and (2.3) that

\[|a_3| \leq \frac{1}{4} B_1^2 + \frac{2}{3} B_1.\]  

Also from (2.10), (2.11), and (2.13), we have

\[6a_3 = A_0 [B_1 (b_2 + c_2) + 2B_2 c_1^2] - 2c_1 A_1 B_1 + A_0 B_1 (b_2 - c_2).\]  

Thus, we find by (2.2) and (2.3) that

\[|a_3| \leq B_1 + \frac{|B_2|}{3}.\]  

This evidently completes the proof of Theorem 2.2. \(\square\)

**Theorem 2.3.** If \(f\) given by (1.1) is in the subclass \(H(\Sigma, h)\) then

\[|a_2| \leq \min \left\{ \frac{B_1}{2}, \sqrt{\frac{B_1 + |B_2|}{3}} \right\},\]  

and

\[|a_3| \leq \min \left\{ \frac{B_1^2}{4} + \frac{B_1 |B_2| + 2B_1}{6}, \frac{|B_2| + 2B_1}{3} \right\}.\]  

**Proof.** The result is obvious, by taking \(\phi(z) \equiv 1\) \((z \in U)\) and using the similar procedure as in Theorem 2.2. \(\square\)

**Definition 2.4.** A function \(f \in \Sigma\) is said to be in the class \(S_\phi^q(\Sigma, h)\) if and only if:

\[\frac{zf'(z)}{f(z)} - 1 \prec_q h(z) - 1, \quad \frac{wg'(z)}{g(w)} - 1 \prec_q h(w) - 1,\]  

where \(|\phi(z)| \leq 1\) \((z \in U)\) and \(g(w) = f^{-1}(w)\).
On taking $\phi(z) \equiv 1$ in Definition 2.1, we get the following subordination class:

$$\frac{zf'(z)}{f(z)} < h(z), \quad \frac{wg'(z)}{g(w)} < h(w).$$

We name this class as $S^*(\Sigma, h)$, where $g(w) = f^{-1}(w)$.

**Theorem 2.5.** If $f$ given by (1.1) is in the subclass $S^*_q(\Sigma, h)$ then

$$|a_2| \leq \min \left\{ B_1, \sqrt{B_1 + |B_2|} \right\} \tag{2.14}$$

and

$$|a_3| \leq \min \{ B_1(1 + B_1), 3B_1 \}.$$  

**Proof.** Let $f \in S^*_q(\Sigma, h)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \mathbb{U} \to \mathbb{U}$ given by (2.2) and a function $\phi(z)$ in $\mathbb{U}$ defined by (2.1) satisfying

$$\frac{zf'(z)}{f(z)} - 1 = \phi(z) (h(u(z) - 1), \quad \frac{wg'(w)}{g(w)} - 1 = \phi(w) (h(v(w) - 1). \tag{2.15}$$

Since

$$\frac{zf'(z)}{f(z)} = 1 + a_2z + \left(2a_3 - a_2^2\right)z^2 + \cdots, \quad \frac{wg'(w)}{g(w)} = 1 - 2a_2w + (3a_2^2 - 2a_3)w^2 + \cdots. \tag{2.16}$$

By using (2.4), (2.5), (2.15), and (2.16), we have

$$a_2 = A_0B_1b_1, \tag{2.17}$$

$$2a_3 - a_2^2 = A_1B_1b_1 + A_0 \left(B_1b_2 + B_2b_1^2\right), \tag{2.18}$$

$$-a_2 = A_0B_1c_1, \tag{2.19}$$

$$3a_2^2 - 2a_3 = A_1B_1c_1 + A_0 \left(B_1c_2 + B_2c_1^2\right). \tag{2.20}$$

From (2.17) and (2.19), we have

$$b_1 = -c_1. \tag{2.21}$$

Squaring and adding (2.17) and (2.19), and then using (2.21), we have

$$a_2^2 = A_0^2B_1^2c_1^2. \tag{2.22}$$

Adding (2.18) and (2.20), and then using (2.21), we have

$$2a_3^2 = A_0 \left[B_1(b_2 + c_2) + 2B_2c_1^2\right]. \tag{2.23}$$

By using (2.2) and (2.3), we have from (2.22) and (2.23) that

$$|a_2| \leq B_1 \quad \text{and} \quad |a_2| \leq \sqrt{B_1 + |B_2|},$$

respectively. So we get the desired estimate on the coefficient $|a_2|$ as asserted in (2.14).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (2.20) from (2.18). Thus, we get

$$4a_3 - 4a_2^2 = A_1B_1(b_1 - c_1) + A_0 \left[B_1(b_2 - c_2) + B_2(b_1^2 - c_1^2)\right]. \tag{2.24}$$

Therefore, (2.21), (2.22), and (2.24) yield

$$4a_3 = 4A_0^2B_1^2c_1^2 - 2c_1A_1B_1 + A_0B_1(b_2 - c_2).$$
Hence, we find by using (2.2) and (2.3) that
\[ |a_3| \leq B_1(1 + B_1). \]

Also from (2.21), (2.22), and (2.24), we have
\[ 4a_3 = 2A_0 \left[ B_1(b_2 + c_2) + 2B_2c_1^2 \right] - 2c_1A_1B_1 + A_0B_1(b_2 - c_2). \]

Thus, we find by (2.2) and (2.3) that
\[ |a_3| \leq 3B_1. \]

This evidently completes the proof of Theorem 2.5.

\[ \square \]

**Theorem 2.6.** If \( f \) given by (1.1) is in the subclass \( S^*(\Sigma, h) \), then
\[ |a_2| \leq \min \left\{ B_1, \sqrt{B_1 + |B_2|} \right\}, \]
and
\[ |a_3| \leq \min \left\{ B_1^2 + \frac{B_1}{2}, \frac{3}{2}B_1 + |B_2| \right\}. \]

**Proof.** The result is obvious, by taking \( \phi(z) \equiv 1 \) \((z \in U)\) and using the similar procedure as in Theorem 2.5.

\[ \square \]

**Definition 2.7.** A function \( f \in \Sigma \) is said to be in the class \( C_q(\Sigma, h) \) if and only if:
\[ \frac{zf''(z)}{f'(z)} < q h(z) - 1, \quad \frac{wg''(z)}{g'(w)} < q h(w) - 1, \]
where \( |\phi(z)| \leq 1 \) \((z \in U)\) and \( g(w) = f^{-1}(w) \).

On taking \( \phi(z) \equiv 1 \) in Definition 2.1, we get the following subordination class:
\[ 1 + \frac{zf''(z)}{f'(z)} < h(z), \quad 1 + \frac{wg''(z)}{g'(w)} < h(w). \]

We name this class as \( C(\Sigma, h) \).

**Theorem 2.8.** If \( f \) given by (1.1) is in the subclass \( C_q(\Sigma, h) \), then
\[ |a_2| \leq \min \left\{ \frac{B_1}{2}, \sqrt{B_1 + |B_2|} \right\}, \tag{2.25} \]
and
\[ |a_3| \leq \min \left\{ \frac{1}{4}B_1^2 + \frac{1}{3}B_1 + \frac{5}{6}B_1 + \frac{1}{2}|B_2| \right\}. \]

**Proof.** Let \( f \in K_q^*(h) \) and \( g = f^{-1} \). Then, there are analytic functions \( u, v : U \to U \) given by (2.2) and a function \( \phi(z) \) in \( \mathcal{U} \) defined by (2.1) satisfying
\[ \frac{zf''(z)}{f'(z)} = \phi(z)(h(u(z)) - 1), \quad \frac{wg''(w)}{g'(w)} = \phi(w)(h(v(w)) - 1). \tag{2.26} \]

Since
\[ \frac{zf''(z)}{f'(z)} = 2a_2z + (6a_3 - 4a_2^2)z^2 + \cdots, \quad \frac{wg''(w)}{g'(w)} = -2a_2w + (8a_2^2 - 6a_3)w^2 + \cdots. \tag{2.27} \]
By using (2.4), (2.5), (2.26), and (2.27), we have
\[ 2a_2 = A_0 B_1 b_1, \] (2.28)
\[ 6a_3 - 4a_2^2 = A_1 B_1 b_1 + A_0 \left( B_1 b_2 + B_2 b_2^2 \right), \] (2.29)
\[ -2a_2 = A_0 B_1 c_1, \] (2.30)
\[ 8a_2^2 - 6a_3 = A_1 B_1 c_1 + A_0 \left( B_1 c_2 + B_2 c_2^2 \right). \] (2.31)
From (2.28) and (2.30), we have
\[ b_1 = -c_1. \] (2.32)
Squaring and adding (2.28) and (2.30), and then using (2.32), we have
\[ 4a_2^2 = A_2^0 B_2^0 c_2^2. \] (2.33)
Adding (2.29) and (2.31), and then using (2.32), we have
\[ 4a_2^2 = A_0 \left[ B_1 (b_2 + c_2) + 2B_2 c_2^2 \right]. \] (2.34)
By using (2.2) and (2.3), we have from (2.33) and (2.34) that
\[ |a_2| \leq \frac{B_1}{2} \text{ and } |a_2| \leq \frac{\sqrt{B_1 + |B_2|}}{2}, \]
respectively. So we get the desired estimate on the coefficient $|a_2|$ as asserted in (2.25).

Next, in order to find the bound on the coefficient $|a_3|$ we subtract (2.31) from (2.29). Thus, we get
\[ 12a_3 - 12a_2^2 = A_1 B_1 (b_1 - c_1) + A_0 \left[ B_1 (b_2 - c_2) + B_2 (b_2^2 - c_2^2) \right]. \] (2.35)
Therefore the equations (2.32), (2.33), and (2.34) yield
\[ 12a_3 = 3A_0^2 B_1^2 c_1^2 - 2c_1 A_1 B_1 + A_0 B_1 (b_2 - c_2). \]
Thus, we find by using (2.2) and (2.3) that
\[ |a_3| \leq \frac{1}{4} B_1^2 + \frac{1}{3} B_1. \]
Also from (2.32), (2.34), and (2.35), we have
\[ 12a_3 = 3A_0 \left[ B_1 (b_2 + c_2) + 2B_2 c_2^2 \right] - 2c_1 A_1 B_1 + A_0 B_1 (b_2 - c_2). \]
Hence, we find by (2.2) and (2.3) that
\[ |a_3| \leq \frac{5}{6} B_1 + \frac{1}{2} |B_2|. \]
This evidently completes the proof of Theorem 2.8.

**Theorem 2.9.** If $f$ given by (1.1) is in the subclass $C(\Sigma, h)$, then
\[ |a_2| \leq \min \left\{ \frac{B_1}{2}, \sqrt{\frac{B_1 + |B_2|}{2}} \right\}, \]
and
\[ |a_3| \leq \min \left\{ \frac{B_1^2}{4} + \frac{B_1}{6}, \frac{2}{3} B_1 + \frac{|B_2|}{2} \right\}. \]

**Proof.** The result is obvious, by taking $\phi(z) \equiv 1$ ($z \in U$) and using the similar procedure as in Theorem 2.8.\]
3. Conclusion

In the paper, classes of analytic bi-univalent are introduced with the help of quasi-subordination. Further, coefficient estimates of initial Maclaurin coefficients are also obtained.

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