Fixed point theorems for \((L)\)-type mappings in complete CAT(0) spaces

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Abstract

In this paper, fixed point properties for a class of more generalized nonexpansive mappings called \((L)\)-type mappings are studied in geodesic spaces. Existence of fixed point theorem, demiclosed principle, common fixed point theorem of single-valued and set-valued are obtained in the third section. Moreover, in the last section, \(\Delta\)-convergence and strong convergence theorems for \((L)\)-type mappings are proved. Our results extend the fixed point results of Suzuki’s results in 2008 and Llorens-Fuster’s results in 2011. ©2017 All rights reserved.

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1. Introduction

Let \(D\) be a nonempty subset of a metric space \((X, d)\). A mapping \(T : D \rightarrow D\) is said to be

1. nonexpansive if \(d(Tx, Ty) \leq d(x, y)\) for all \(x, y \in D\);
2. quasi-nonexpansive if \(d(Tx, p) \leq d(x, p)\) for all \(x \in D\) and \(p \in F(T)\), where \(F(T) = \{x \in D : Tx = x\}\) denotes the set of fixed points of \(T\).

We can find in the literature research about more general classes of mappings than the nonexpansive ones and quasi-nonexpansive ones. For instance, in 2008, Suzuki [28] defined a class of generalized nonexpansive mappings, which he called \((C)\)-type mappings, whose set-valued version was defined and studied in [1, 2, 26, 30]. In 2011, Garcia-Falset et al. [14] introduced two classes of single-valued generalized nonexpansive mappings called \((C_\lambda)\)-type mappings and \((E_\mu)\)-type mappings, respectively, which both enlarged the family of \((C)\)-type mappings. Again these new classes were generalized to the set-valued case in [3, 9, 12, 17].

**Definition 1.1.** Let \(D\) be a nonempty subset of a metric space \((X, d)\). A mapping \(T : D \rightarrow D\) is said to
1. satisfy condition (C), (or be a (C)-type mapping) if
   \[ \frac{1}{2} d(x, Tx) \leq d(x, y) \quad \text{implies} \quad d(Tx, Ty) \leq d(x, y), \]  
   for all \( x, y \in D \);
2. satisfy condition (\( C_\lambda \)), (or be a (\( C_\lambda \))-type mapping) if
   \[ \lambda d(x, Tx) \leq d(x, y) \quad \text{implies} \quad d(Tx, Ty) \leq d(x, y), \]  
   for all \( x, y \in D \) and \( \lambda \in (0, 1) \);
3. satisfy condition (\( E_\mu \)), (or be a (\( E_\mu \))-type mapping) if
   \[ d(x, Ty) \leq \mu d(x, Tx) + d(x, y), \]  
   for all \( x, y \in D \) and \( \mu \geq 1 \).

In 2011 [23], fixed point results for a class of single-valued generalized nonexpansive mappings called (L)-type mappings were proved by Llorens-Fuster and Moreno-Gálvez. This class properly contains Suzuki’s (C)-type mappings as (1.1) and several of its generalizations such as (\( C_\lambda \))-type mappings as (1.2) and (\( E_\mu \))-type mappings as (1.3) mentioned before. The set-valued case for (L)-type mappings were discussed in [13] and more results in [24]. Their results closely depend upon geometric characteristics of the Banach space under consideration. In this paper, we shall prove the fixed point property for (L)-type mappings in a metric space without notion of a “topology” and “weak topology”.

The aim of this paper is to prove fixed point property for (L)-type mappings in a special kind of metric spaces, namely CAT(0) spaces, which will be defined in the next section. Firstly, we prove the existence theorem of fixed point and demiclosed principle for (L)-type mappings in complete CAT(0) spaces. Furthermore, two common fixed point theorems are also obtained. Finally, we prove that a sequence defined by a three-step iteration \( \Delta \)-converges (even on some condition strongly converges) to a fixed point of these kind of mappings. Our results extend and improve some results in [23] and [13].

2. Preliminaries

Let \((X, d)\) be a metric space and \(x, y \in X\) with \(d(x, y) = 1\). A geodesic path joining \(x\) to \(y\) is an isometric map \(c\) from a closed interval \([0, 1] \subset \mathbb{R}\) to \(X\) such that \(c(0) = x\), \(c(1) = y\). The image of \(c\) is called a geodesic (or metric) segment joining \(x\) and \(y\) denoted by \([x, y]\) whenever it is unique. The space \((X, d)\) is said to be a (uniquely) geodesic space if every two points of \(X\) are joined by (exactly) one geodesic segment. A geodesic triangle \(\Delta(x_1, x_2, x_3)\) in a geodesic space \(X\) consists of three points \(x_1, x_2, x_3\) of \(X\) and three geodesic segments joining each pair of vertices. A comparison triangle of a geodesic triangle \(\Delta(x_1, x_2, x_3)\) is the triangle \(\hat{\Delta}(x_1, \bar{x}_2, \bar{x}_3)\) in the Euclidean space \(\mathbb{E}^2\) such that
   \[ d(x_i, x_j) = d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j), \quad \forall i, j = 1, 2, 3. \]

A geodesic space is a CAT(0) space, if for each geodesic triangle \(\Delta(x_1, x_2, x_3)\) in \(X\) and its comparison triangle \(\hat{\Delta} := \hat{\Delta}(\bar{x}_1, \bar{x}_2, \bar{x}_3)\) in \(\mathbb{E}^2\), the CAT(0) inequality
   \[ d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}), \]
holds for all \(x, y \in \Delta\) and \(\bar{x}, \bar{y} \in \hat{\Delta}\).

A thorough discussions of these spaces are given in [4]. The following lemma plays an important role in our paper.

**Lemma 2.1** ([11]). Let \((X, d)\) be a CAT(0) space.
Lemma 2.2. (see [16]), Hadamard manifolds, and many others. R-trees (see [18]), Euclidean buildings (see [6]), the complex Hilbert ball with a hyperbolic metric CAT(0) spaces arise in a wide variety of contexts. Some examples of CAT(0) spaces are pre-Hilbert spaces uniform convexity and orthogonal projection onto convex subsets, etc. Also, because of their generality, (CN) inequality holds.

The following lemma is a consequence of [25, Lemma 2.5].

**Lemma 2.2.** Let \( \{x_n\} \) and \( \{y_n\} \) be bounded sequences in a CAT(0) space \( X \) and \( r \in [0, 1) \). Suppose that \( x_{n+1} = r y_n \oplus (1-r)x_n \) and \( d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n) \) for all \( n \in \mathbb{N} \). Then \( \lim_{n \to \infty} d(x_n, y_n) = 0 \).

Firstly the definition of \((L)\)-type mappings in the single-valued case will be given in a metric space as follows.

**Definition 2.3.** Let \( D \) be a nonempty bounded closed convex subset of a CAT(0) space \( X \). A mapping \( T : D \to D \) is said to satisfy condition \((L)\) (or it is an \((L)\)-type mapping) on \( D \) provided that it fulfills the following two conditions.

1. If a set \( K \subset D \) is nonempty, closed, convex, and \( T \)-invariant, (i.e., \( T(K) \subset K \)), then there exists an a.f.p.s. for \( T \) in \( K \) (i.e., \( d(x_n, T x_n) \to 0 \) for a sequence \( \{x_n\} \) in \( K \)).

2. For any a.f.p.s. \( \{x_n\} \) of \( T \) in \( D \) and each \( x \in D \),

\[
\limsup_{n \to \infty} d(x_n, T x) \leq \limsup_{n \to \infty} d(x_n, x).
\]

**Proposition 2.4.** Let \( D \) be a nonempty bounded closed convex subset of a CAT(0) space \( X \) and \( T : D \to D \) be a mapping satisfying condition \((L)\) with a nonempty fixed point set, then \( T \) is a quasi-nonexpansive mapping.

**Proof.** Let \( p \in F(T) \). Taking \( x_n = p \) for every positive integer \( n \), it is obvious that \( \{x_n\} \) is an a.f.p.s. for \( T \). From condition \((L)\), we have for each \( x \in D \),

\[
d(p, T x) = \limsup_{n \to \infty} d(x_n, T x) \leq \limsup_{n \to \infty} d(x_n, x) = d(p, x).
\]

In other words, \( T \) is a quasi-nonexpansive mapping. \( \square \)

Next, in order to define the set-valued case for \((L)\)-type mappings, we introduce some elementary concepts. Let \( D \) be a nonempty subset of a metric space \( X \). We denote by \( B(D) \) the collection of all nonempty bounded closed subsets of \( D \) and \( C(D) \) the collection of all nonempty compact subsets of \( D \). Suppose \( H \) is the Hausdorff metric with respect to \( d \), that is,

\[
H(U, V) := \max \left\{ \sup_{u \in U} \text{dist}(u, V), \sup_{v \in V} \text{dist}(v, U) \right\}, \quad U, V \in B(X),
\]
where \( \text{dist}(u, V) = \inf_{v \in V} d(u, v) \) is the distance from the point \( u \) to the set \( V \).

Let \( T: X \to 2^X \) be a set-valued mapping. If an element \( x \in X \) satisfies \( x \in Tx \), then \( x \) is called a fixed point of \( T \). The set of fixed points of \( T \) is denoted by \( F(T) \). If a sequence \( \{x_n\} \) in \( D \) satisfies \( \text{dist}(x_n, Tx_n) \to 0 \) as \( n \to \infty \), then \( \{x_n\} \) is called an a.f.p.s. for \( T \).

**Definition 2.5.** Let \( D \) be a nonempty bounded closed convex subset of a CAT(0) space \( X \). A set-valued mapping \( T: D \to B(D) \) is said to satisfy condition (L), (or it is an \((L)\)-type set-valued mapping), on \( D \) provided that it fulfills the following two conditions.

1. If a set \( K \subset D \) is nonempty, closed, convex, and \( T \)-invariant, then there exists an a.f.p.s. for \( T \) in \( K \).
2. For any a.f.p.s. \( \{x_n\} \) of \( T \) in \( D \) and each \( x \in D \),

\[
\limsup_{n \to \infty} \text{dist}(x_n, Tx) \leq \limsup_{n \to \infty} d(x_n, x).
\]

Along with Definition 2.3 and the above two lemmas, we can obtain the following propositions which show the inclusion relations between \((L)\)-type mappings and other generalized nonexpansive mappings in CAT(0) spaces.

**Proposition 2.6.** Let \( D \) be a nonempty, bounded, and convex subset of a CAT(0) space \( X \) and \( T: D \to D \) be a mapping satisfying condition (C), then \( T \) satisfies condition (L).

**Proof.** Recall that if \( T: D \to D \) is a mapping satisfying condition (C), then there exists an a.f.p.s \( \{x_n\} \) for \( T \) in \( D \) by [25, Lemma 3.6]. Moreover, in view of [25, Lemma 3.5], we have that, for every \( x, y \in D \),

\[
d(x, Ty) \leq 3d(Tx, x) + d(x, y).
\]

Hence, for the a.f.p.s. \( \{x_n\} \) and each \( x \in D \),

\[
\limsup_{n \to \infty} d(x_n, Tx) \leq \limsup_{n \to \infty} (3d(x_n, Tx_n) + d(x_n, x)) = \limsup_{n \to \infty} d(x_n, x),
\]

which means such mappings satisfy condition (L). \( \square \)

**Proposition 2.7.** Let \( D \) be a nonempty, bounded, and convex subset of a CAT(0) space \( X \) and \( T: D \to D \) be a mapping satisfying condition \((E_\mu)\) for some \( \mu \geq 0 \), then \( T \) satisfies condition (L) provided that it satisfies assumption 1 of Definition 2.3.

**Proof.** Replace 3 with \( \mu \) in the proof of Proposition 2.6. Therefore, the desired conclusion is obtained. \( \square \)

**Proposition 2.8.** Let \( D \) be a nonempty, bounded and convex subset of a CAT(0) space \( X \) and \( T: D \to D \) be a continuous mapping satisfying condition \((C_\lambda)\) for some \( \lambda \in (0, 1) \), then \( T \) satisfies condition (L).

**Proof.** Define a sequence \( \{x_n\} \) in \( D \) by taking \( x_1 \in D \) and

\[
x_{n+1} = rTx_n \oplus (1 - r)x_n,
\]

for \( n \geq 1 \) and \( r \in [\lambda, 1] \). It follows from Lemma 2.1 (1) that

\[
\lambda d(x_n, Tx_n) \leq r d(x_n, Tx_n) = d(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N}.
\]

By condition \((C_\lambda)\), we have

\[
d(Tx_{n+1}, Tx_n) \leq d(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N}.
\]

Hence, \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \) by Lemma 2.2.
Case 1. If for some \( x \in D \), there is a subsequence \( \{ x_n \} \) of \( \{ x_n \} \) converging to \( x \). Since \( \{ x_n \} \) is an a.f.p.s., then it is obvious that the sequence \( \{ T x_n \} \) has the same limit as \( \{ x_n \} \), and therefore by the continuity of \( T \), \( x = T x \). Thus, for \( \{ x_n \} \) in \( D \) and \( x \in D \), we have

\[
\limsup_{n \to \infty} d(x_n, T x) \leq \limsup_{n \to \infty} d(x_n, x)
\]

holds, i.e., \( T \) satisfies condition (L).

Case 2. Suppose that for every \( x \in D \), the sequence \( \{ x_n \} \) does not have any subsequence converging to \( x \). Noticing that \( \{ x_n \} \) is an a.f.p.s., for any \( \varepsilon > 0 \), there exists some \( n_0 \in \mathbb{N} \) such that \( d(x_n, T x_n) < \varepsilon \) for all \( n \geq n_0 \). Since \( \{ x_n \} \) does not converge to \( x \), we can put \( \varepsilon := \frac{1}{2} \liminf_n d(x_n, x) > 0 \). Therefore,

\[
\lambda d(x_n, T x_n) \leq d(x_n, T x_n) < \varepsilon < d(x_n, x).
\]

By condition \( (C_{\lambda}) \), we have

\[
d(T x_n, T x) \leq d(x_n, x),
\]

which implies

\[
\limsup_{n \to \infty} d(x_n, T x) \leq \limsup_{n \to \infty} (d(x_n, T x_n) + d(T x_n, T x)) \leq \limsup_{n \to \infty} d(x_n, x).
\]

So \( T \) satisfies condition (L).

We now give the notion of \( \Delta \)-convergence and collect some of its basic properties. Let \( \{ x_n \} \) be a bounded sequence in a \( \text{CAT}(0) \) space \( X \). For \( z \in X \), we set

\[
r(z, \{ x_n \}) = \limsup_{n \to \infty} d(z, x_n).
\]

The asymptotic radius \( r(\{ x_n \}) \) of \( \{ x_n \} \) is given by

\[
r(\{ x_n \}) = \inf \{ r(z, \{ x_n \}) : z \in X \}.
\]

The asymptotic radius \( r_D(\{ x_n \}) \) of \( \{ x_n \} \) with respect to \( D \subset X \) is given by

\[
r_D(\{ x_n \}) = \inf \{ r(z, \{ x_n \}) : z \in D \}.
\]

The asymptotic center \( A(\{ x_n \}) \) of \( \{ x_n \} \) is the set

\[
A(\{ x_n \}) = \{ z \in X : r(z, \{ x_n \}) = r(\{ x_n \}) \}.
\]

And the asymptotic center \( A_D(\{ x_n \}) \) of \( \{ x_n \} \) with respect to \( D \subset X \) is the set

\[
A_D(\{ x_n \}) = \{ z \in D : r(z, \{ x_n \}) = r(\{ x_n \}) \}.
\]

It follows from [10, Proposition 7] that \( A(\{ x_n \}) \) consists of exactly one point in a \( \text{CAT}(0) \) space. In 1976, Lim [21] introduced the concept of \( \Delta \)-convergence in a general metric space. In 2008, Kirk and Panyanak [19] brought in \( \Delta \)-convergence to \( \text{CAT}(0) \) spaces and proved that there is an analogy between \( \Delta \)-convergence and weak convergence.

**Definition 2.9 ([19]).** A sequence \( \{ x_n \} \) in a \( \text{CAT}(0) \) space \( X \) is said to \( \Delta \)-converge to \( x \in X \) if \( x \) is the unique asymptotic center of \( \{ u_n \} \) for every subsequence \( \{ u_n \} \) of \( \{ x_n \} \). In this case, we write \( \Delta \lim_{n \to \infty} x_n = x \) and call \( x \) the \( \Delta \)-limit of \( \{ x_n \} \).

**Lemma 2.10 ([19]).** If \( D \) is a closed convex subset of a complete \( \text{CAT}(0) \) space and if \( \{ x_n \} \) is a bounded sequence in \( D \), then the asymptotic center asymptotic center of \( \{ x_n \} \) is in \( D \).

**Lemma 2.11 ([19]).** Every bounded sequence in a complete \( \text{CAT}(0) \) space always has a \( \Delta \)-convergent subsequence.

**Lemma 2.12 ([11]).** If \( \{ x_n \} \) is a bounded sequence in a complete \( \text{CAT}(0) \) space with \( A(\{ x_n \}) = \{ p \} \), \( \{ u_n \} \) is a subsequence of \( \{ x_n \} \) with \( A(\{ u_n \}) = \{ u \} \), and the sequence \( \{ d(x_n, u) \} \) is convergent, then \( p = u \).
3. Fixed point theorems

**Theorem 3.1.** Let $D$ be a nonempty bounded closed convex subset of a complete CAT(0) space $X$. Suppose $T : D \to D$ is a mapping satisfying condition (L). Then $T$ has a fixed point in $D$.

**Proof.** Since $T$ satisfies condition (L), there exists an a.f.p.s. for $T$ in $D$, say $\{x_n\}$. By Proposition 7 of [10] we let $A(\{x_n\}) = \{z\}$. It follows from Lemma 2.10 that $z \in D$. By condition (L), we get

$$
\limsup_{n \to \infty} d(x_n, Tz) \leq \limsup_{n \to \infty} d(x_n, z),
$$

which means

$$
r(Tz, \{x_n\}) \leq r(z, \{x_n\}).
$$

By the uniqueness of asymptotic centers, we have $z = Tz$. □

Theorem 3.1 extends [23, Theorem 4.2]. By using this theorem along with Proposition 2.4 and [8, Theorem 1.3], we can obtain the following corollary.

**Corollary 3.2.** Let $D$ be a nonempty bounded closed convex subset of a complete CAT(0) space $X$. Suppose $T : D \to D$ is a mapping satisfying condition (L). Then $F(T)$ is nonempty closed, convex and hence contractible.

In 1968, Browder proved demiclosedness principle [5] for nonexpansive mappings which has been one of the fundamental and celebrated results in fixed point theory. Demiclosedness principle states that if $D$ is a nonempty closed convex subset of a uniformly convex Banach space $X$, and $T : D \to X$ is a nonexpansive mapping, then $I - T$ is demiclosed at 0, that is, for any sequence $\{x_n\}$ in $D$, if $\{x_n\}$ weakly converges to $x$ and $(I - T)x_n$ strongly converges to 0, then $x = Tx$ (here $I$ is the identity operator of $X$ into itself). The principle is also valid in a space satisfying Opial’s condition. It has been known that the demiclosedness principle plays a key role in studying the asymptotic and ergodic behavior of nonexpansive mapping, see for example [15, 22].

**Remark 3.3.** Let $D$ be a closed convex subset of a CAT(0) space $X$ and $\{x_n\}$ be a bounded sequence in $D$. We need the following notation:

$$
\{x_n\} \rightharpoonup \omega \quad \text{if and only if} \quad \Phi(\omega) = \inf_{x \in D} \Phi(x),
$$

where $\Phi(x) := \limsup_{n \to \infty} d(x_n, x)$.

Theorem 3.4 in the following takes use of the notion defined above to prove demiclosedness principle for (L)-type mappings which extend [23, Theorem 4.6] to CAT(0) spaces.

**Theorem 3.4 (Demiclosed principle).** Suppose $D$ is a bounded closed convex subset of a complete CAT(0) space $X$ and $T : D \to D$ is a mapping satisfying condition (L). If $\{x_n\} \subset D$ is an a.f.p.s. for $T$ such that $\{x_n\} \rightharpoonup p$, then $Tp = p$.

**Proof.** By the definition, $\{x_n\} \rightharpoonup p$ if and only if $A_D(\{x_n\}) = \{p\}$. We have $A(\{x_n\}) = \{p\}$ from Lemma 2.10 and Lemma 2.11. Since $\{x_n\}$ is an a.f.p.s. for $T$, we have

$$
\Phi(x) := \limsup_{n \to \infty} d(x_n, x) = \limsup_{n \to \infty} d(Tx_n, x). \quad (3.1)
$$

Taking $x = Tp$ in (3.1), we have

$$
\Phi(Tp) = \limsup_{n \to \infty} d(x_n, Tp) \leq \limsup_{n \to \infty} d(x_n, p) = \Phi(p).
$$

Furthermore, for any $n \geq 1$, it follows from (CN) inequality with $t = \frac{1}{2}$ that

$$
\frac{d^2(x_n, \frac{p \oplus Tp}{2})}{\leq \frac{1}{2} d^2(x_n, p) + \frac{1}{2} d^2(x_n, Tp) - \frac{1}{4} d^2(p, Tp)}.
$$
Letting \( n \to \infty \) and taking superior limit on the both sides of the above inequality, we get
\[
\Phi \left( \frac{p + Tp}{2} \right) \leq \frac{1}{2} \Phi (p) + \frac{1}{2} \Phi (Tp) - \frac{1}{4} d^2 (p, Tp).
\]
Since \( A(\{x_n\}) = \{p\} \), we have
\[
\Phi (p) \leq \Phi \left( \frac{p + Tp}{2} \right) \leq \frac{1}{2} \Phi (p) + \frac{1}{2} \Phi (Tp) - \frac{1}{4} d^2 (p, Tp),
\]
which implies that
\[
d(p, Tp) = 0,
\]
i.e., \( p = Tp \).

**Lemma 3.5** (cf. [20, 27]). Let \( X \) be a complete \( \text{CAT}(0) \) space, then the intersection of any decreasing sequence of nonempty bounded closed convex subsets of \( X \) is nonempty.

Together with Theorem 3.1 and Lemma 3.5, we have a common fixed point theorem of a countable family of mappings which satisfy condition (L).

**Theorem 3.6.** Let \( D \) be a nonempty bounded closed and convex subset of a complete \( \text{CAT}(0) \) space \( X \). Let \( \{T_i\}_{i=1}^\infty \) be a countable family of commuting mappings on \( D \) satisfying condition (L). Then \( \{T_i\}_{i=1}^\infty \) has a common fixed point.

**Proof.** Let \( C_n := \bigcap_{i=1}^n F(T_i) \) for each \( n \). From Corollary 3.2, \( C_1 = F(T_1) \) is nonempty bounded closed and convex subset of \( X \). Now we assume that \( C_{k-1} \) is nonempty bounded closed and convex for \( k \in \mathbb{N} \). We are going to show that \( C_k \) is also nonempty bounded closed and convex. Let \( p \in C_{k-1} \) and \( i \in \mathbb{N} \) with \( 1 \leq i < k \). Since \( T_k \) and \( T_i \) commute, we have
\[
T_k p = T_k \circ T_i p = T_i \circ T_k p.
\]
Thus \( T_k p \) is a fixed point of \( T_i \), which implies that \( T_k p \in C_{k-1} \). Hence we get \( T_k (C_{k-1}) \subseteq C_{k-1} \). By Theorem 3.1, \( T_k \) has a fixed point in \( C_{k-1} \), that is,
\[
C_k = C_{k-1} \bigcap F(T_k) \neq \emptyset.
\]
Also, it is closed and convex by Corollary 3.2. By induction, \( C_n \) is nonempty bounded closed and convex for all \( n \in \mathbb{N} \). Since \( C_n \subseteq C_{n-1} \) for all \( n \in \mathbb{N} \), by Lemma 3.5 we have
\[
\bigcap_{i=1}^\infty F(T_i) = F_{\infty} \neq \emptyset.
\]
This completes the proof.

**Theorem 3.7.** Let \( t : D \to D \) and \( T : D \to C(D) \) be a single-valued mapping and a set-valued mapping, respectively. If both \( t \) and \( T \) satisfy the condition (L) and in the meantime, they have common a.f.p.s., then they have a common fixed point, that is, there exists a point \( z \in D \) such that \( z = tz \in Tz \).

**Proof.** By Theorem 3.1 and Corollary 3.2, we know that the mapping \( t \) has a fixed point set \( F(t) \) which is a nonempty closed convex subset of \( X \). Let \( p \in F(t) \). Since \( Tp \) is a bounded closed convex subset of \( X \), we can obtain that \( t \) has a fixed point in \( Tp \) for \( p \in F(t) \). From the assumption, let \( \{u_n\} \) be the common a.f.p.s. and \( A(\{u_n\}) = \{z\} \). By the proof of Theorem 3.1, we have that \( z \in F(t) \). Since \( Tz \) is a compact set, there exists \( v_n \in Tz \) such that
\[
d(u_n, v_n) = \text{dist}(u_n, Tz).
\]
Again from the compactness of \( Tz \), we may assume that \( v_n \to z' \in Tz \). Since \( T \) satisfies condition (L),
\[
\lim_{n \to \infty} d(u_n, z') \leq \lim_{n \to \infty} d(u_n, v_n) + \lim_{n \to \infty} d(v_n, z') = \lim_{n \to \infty} \text{dist}(u_n, Tz) \leq \lim_{n \to \infty} \text{dist}(u_n, z).
\]
This implies that
\[
r(z', \{u_n\}) \leq r(z, \{u_n\}).
\]
By the uniqueness of asymptotic centers, we have \( z = z' \in Tz \). Hence \( z = tz \in Tz \).
4. Convergence theorems

In this section, we shall prove $\Delta$ and strong convergence theorems for (L)-type mappings of a threestep iteration scheme introduced by Thakur et al. in [29] which not only converges faster than the known iterations but also is stable. Give $x_1 \in D$, the sequence $\{x_n\}$ is generated by

$$\begin{cases}
x_1 \in D, \\
x_{n+1} = Ty_n, \\
y_n = T((1-\alpha_n)x_n \oplus \alpha_n z_n), \\
z_n = (1-\beta_n)x_n \oplus \beta_n T x_n,
\end{cases} \tag{4.1}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences with $0 < \alpha \leq \alpha_n, \beta_n \leq b < 1$.

We now establish the following useful lemma.

**Lemma 4.1.** Let $D$ be a nonempty bounded closed convex subset of a complete CAT(0) space $X$ and let $T : D \to D$ be a mapping satisfying condition (L). For arbitrary chosen $x_1 \in D$ and $\{x_n\}$ generated by (4.1), $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.

**Proof.** By Theorem 3.1, $F(T)$ is nonempty. Given $p \in F(T)$, by Lemma 2.1 (2) and Proposition 2.4 we have

$$d(z_n, p) = d((1-\beta_n)x_n \oplus \beta_n T x_n, p)$$

$$\leq (1-\beta_n)d(x_n, p) + \beta_n d(T x_n, p)$$

$$\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, T p)$$

$$\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p)$$

$$= d(x_n, p), \tag{4.2}$$

and from (4.2),

$$d(y_n, p) = d(T((1-\alpha_n)x_n \oplus \alpha_n z_n), p)$$

$$\leq d((1-\alpha_n)x_n \oplus \alpha_n z_n, p)$$

$$\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(z_n, p)$$

$$\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p)$$

$$= d(x_n, p). \tag{4.3}$$

By (4.3) we can obtain that

$$d(x_{n+1}, p) = d(T y_n, p) \leq d(y_n, p) \leq d(x_n, p). \tag{4.4}$$

Thus, $\{d(x_n, p)\}$ is bounded and decreasing for all $p \in F(T)$, i.e., $\lim_{n\to\infty} d(x_n, p)$ exists.

**Lemma 4.2 ([7, Lemma 3.2]).** Let $X$ be a CAT(0) space, $x \in X$ be a given point, and $\{t_n\}$ be a sequence in $[a, b]$ for some $a, b \in [0, 1]$. If $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that

$$\limsup_{n\to\infty} d(x_n, x) \leq r, \quad \limsup_{n\to\infty} d(y_n, x) \leq r,$$

and

$$\lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r,$$

for some $r \geq 0$. Then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

**Theorem 4.3.** Let $D$ be a nonempty bounded closed convex subset of a complete CAT(0) space $X$. Suppose $T : D \to D$ is a mapping satisfying condition (L). For arbitrary chosen $x_1 \in D$ and $\{x_n\}$ generated by (4.1), $\{x_n\}$ $\Delta$-converges to a fixed point of $T$. 


Hence, together with (4.7) and (4.8), we have
\[ \lim_{n \to \infty} d(x_n, Tx_n) = 0. \]
In fact, it follows from Lemma 4.1 that for each given \( p \in F(T) \), \( \lim_{n \to \infty} d(x_n, p) \) exists, without loss of generality, let
\[ \lim_{n \to \infty} d(x_n, p) = r \geq 0. \]
(4.5)
By Proposition 2.4, we have
\[ \limsup_{n \to \infty} d(Tx_n, p) \leq \limsup_{n \to \infty} d(x_n, p) = r. \]
(4.6)
Since \( \{\alpha_n\} \) is a sequence with \( 0 < a \leq \alpha_n \leq b < 1 \), we can assume that \( \lim_{n \to \infty} \alpha_n = \alpha \in [a, b] \). By using (4.3)-(4.5), we get
\[ r = \lim_{n \to \infty} d(x_{n+1}, p) = \lim_{n \to \infty} d(Ty_n, p) \leq \lim_{n \to \infty} d((1 - \alpha_n)x_n + \alpha_n z_n, p) \leq \lim_{n \to \infty} (1 - \alpha_n) d(x_n, p) + \lim_{n \to \infty} \alpha_n d(z_n, p) = (1 - \alpha) r + \alpha \lim_{n \to \infty} d(z_n, p), \]
which implies that
\[ \lim_{n \to \infty} d(z_n, p) \geq r. \]
(4.7)
On the other hand, it follows from (4.2) and (4.5) that
\[ \lim_{n \to \infty} d(z_n, p) \leq \lim_{n \to \infty} d(x_n, p) = r. \]
(4.8)
Hence, together with (4.7) and (4.8), we have
\[ r \leq \lim_{n \to \infty} d(z_n, p) = \lim_{n \to \infty} d((1 - \beta_n)x_n + \beta_n Tx_n, p) \leq r, \]
which implies that
\[ \lim_{n \to \infty} d((1 - \beta_n)x_n + \beta_n Tx_n, p) = r, \]
(4.9)
where \( 0 < a \leq \beta_n \leq b < 1 \). By (4.5), (4.6), (4.9), as well as Lemma 4.2, it gets that
\[ \lim_{n \to \infty} d(x_n, Tx_n) = 0, \]
(4.10)
i.e., \( \{x_n\} \) is an a.f.p.s. of \( T \) in \( D \).

Now we prove that
\[ \omega_w(x_n) := \bigcup_{\{u_n\} \subset \{x_n\}} A([u_n]) \subset F(T), \]
(4.11)
and \( \omega_w(x_n) \) consists of exactly one point.

In fact, \( u \in \omega_w(x_n) \), then, there exists a subsequence \( \{u_n\} \) of \( \{x_n\} \) such that \( A([u_n]) = \{u\} \). By Lemma 2.10 and Lemma 2.11, there exists a subsequence \( \{v_n\} \) of \( \{u_n\} \) such that \( \Delta - \lim_{n \to \infty} v_n = v \in D \). In view of (4.10) and Theorem 3.4, we have \( v \in F(T) \). Furthermore, \( u = v \) by Lemma 2.12. This implies that \( \omega_w(x_n) \subset F(T) \). Next we claim that \( \omega_w(x_n) \) consists of exactly one point. Let \( \{u_n\} \) be a subsequence of \( \{x_n\} \) with \( A([u_n]) = \{u\} \) and let \( A([x_n]) = \{x\} \). Since \( u \in \omega_w(x_n) \subset F(T) \), from Lemma 4.1 we know that \( d(x_n, u) \) is convergent. In view of Lemma 2.12, we have \( x = u \).

Finally we prove that \( \{x_n\} \Delta \)-converges to a fixed point of \( T \).

In fact, by Lemma 4.1 we know that \( d(x_n, p) \) is convergent for each \( p \in F(T) \). By (4.10), \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \). By (4.11), \( \omega_w(x_n) \subset F(T) \) and \( \omega_w(x_n) \) consists of exactly one point. This shows that \( \{x_n\} \Delta \)-converges to a point of \( F(T) \). This completes the proof. \( \square \)
Thus we have

\[ \exists z \in D. \]

This implies \( z = Tz \), i.e., \( z \in F(T) \). By Lemma 4.1, we have

\[ \lim_{n \to \infty} d(x_n, z) \]

exists, thus \( \{x_n\} \) converges to \( Tz \). This implies \( z = Tz \), i.e., \( z \in F(T) \). By Lemma 4.1, we have

\[ \lim_{n \to \infty} d(x_n, z) \]

exists, thus \( z \) is the strong limit of the sequence \( \{x_n\} \) itself.

\[ \square \]

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