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Sharp Stolarsky mean bounds for the complete elliptic integral of the second kind

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Abstract

In the article, we prove that the double inequality

$$25/16 < \mathcal{E}(r)/S_{5/2,2}(1, r') < \pi/2,$$

holds for all $r \in (0, 1)$ with the best possible constants $25/16$ and $\pi/2$, where $r' = (1-r^2)^{1/2}$, $\mathcal{E}(r) = \int_0^{\pi/2}\sqrt{1-r^2\sin^2(t)}dt$, is the complete elliptic integral of the second kind and $S_{p,q}(a, b) = [q(a^p - b^p)/(p(a^q - b^q))]^{1/(p-q)}$, is the Stolarsky mean of a and b.

Keywords: Gaussian hypergeometric function, complete elliptic integral, Stolarsky mean.
2010 MSC: 33E05, 26D15, 26E60.

1. Introduction

For $r \in (0, 1)$, the complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ [1] of the first and second kinds are respectively given by

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-r^2\sin^2(t)}} = \frac{\pi}{2}F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}r^{2n},$$

$$\mathcal{E}(r) = \int_0^{\pi/2}\sqrt{1-r^2\sin^2(t)}dt = \frac{\pi}{2}F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(n!)^2}r^{2n}, \tag{1.1}$$

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where

$$F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!},\quad (-1<x<1)$$

is the Gaussian hypergeometric function, $(a)_n=\Gamma(a+n)/\Gamma(a)$ and $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$ $(x>0)$ is the gamma function. We clearly see that $\mathcal{K}(r)$ and $\mathcal{E}(r)$ satisfy the identities

$$\mathcal{K}(0^+)=\mathcal{E}(0^+)=\frac{\pi}{2},\quad \mathcal{K}(1^-)=\infty,\quad \mathcal{E}(1^-)=1.$$

It is well-known that the double inequality

$$\frac{\pi}{2}A_{3/2}(1,r')<\mathcal{E}(r)<\frac{\pi}{2}A_2(1,r'), \tag{1.2}$$

holds for all $r\in(0,1)$ (see [12, 19.9.4]). Here and in what follows $r'=(1-r^2)^{1/2}$, $A_p(a,b)=[(a^p+b^p)/2]^{1/p}$. The first inequality of (1.2) is due to Qiu and Shen [13] and the second inequality of (1.2) is due to Barnard et al. [2].

In [4, 5], the authors proved that the inequalities

$$\mathcal{E}(r)<\frac{\pi}{2}L_{1/4}(1,r'), \tag{1.3}$$

$$\mathcal{E}(r)<\frac{\pi}{32}[18A(1,r')-5G(1,r')+3Q(1,r')], \tag{1.4}$$

hold for all $r\in(0,1)$, where $A(a,b)=(a+b)/2$, $G(a,b)=\sqrt{ab}$, $Q(a,b)=\sqrt{(a^2+b^2)/2}$ and $L_p(a,b)=(a^{p+1}+b^{p+1})/(a^p+b^p)$ are the arithmetic, geometric, quadratic and p-th Lehmer means of a and b, respectively.

Let $p,q\in\mathbb{R}$ with $p\neq q$ and $pq\neq 0$, and $a,b>0$. Then the Stolarsky mean $S_{p,q}(a,b)$ [14] is defined by

$$S_{p,q}(a,b)=\left[\frac{q(a^p-b^p)}{p(a^q-b^q)}\right]^{1/(p-q)},\quad (a\neq b),\quad S_{p,q}(a,a)=a. \tag{1.5}$$

Recently, the Stolarsky mean $S_{p,q}(a,b)$ has attracted the attention of many researchers. In particular, many remarkable inequalities involving the Stolarsky mean $S_{p,q}(a,b)$ can be found in the literature [6–11].

The main purpose of this paper is to present the best possible constants λ and μ such that the double inequality

$$\lambda<\mathcal{E}(r)/S_{5/2,2}(1,r')<\mu,$$

holds for all $r\in(0,1)$. Some complicated computations are carried out using Mathematica computer algebra system.

2. Lemmas

Lemma 2.1 ([3]). *Let $A(t)=\sum_{k=0}^{\infty}a_kt^k$ and $B(t)=\sum_{k=0}^{\infty}b_kt^k$ be two real power series converging on $(-r,r)$ $(r>0)$ with $b_k>0$ for all k. If the non-constant sequence $\{a_k/b_k\}_{k=0}^{\infty}$ is increasing (decreasing) for all k, then the function $t\mapsto A(t)/B(t)$ is strictly increasing (decreasing) on $(0,r)$.*

Lemma 2.2 ([5]). *The inequality*

$$L_{1/4}(a,b)>\frac{18A(a,b)-5G(a,b)+3Q(a,b)}{16},$$

holds for all $a,b>0$ with $a\neq b$.

Lemma 2.3. *Let* $n = 1, 2, 3, \cdots$, *and* u_n *and* v_n *be defined by*

$$u_n = \frac{6\left(\frac{1}{2}\right)_{n-1} + 2\left(\frac{3}{4}\right)_n}{5(n+2)!}, \qquad v_n = \frac{\left(\frac{1}{2}\right)_{n-1}\left(\frac{1}{2}\right)_n}{2(n!)^2}. \tag{2.1}$$

Then the non-constant sequence $\{v_n/u_n\}_{n=1}^{\infty}$ *is increasing for all* $n \geqslant 1$.

Proof. Let

$$w_n = \frac{v_{n+1}u_n}{v_n} - u_{n+1}. \tag{2.2}$$

Then it follows from (2.1) that

$$w_n = \frac{6(3n+1)\left(\frac{1}{2}\right)_n + (n^2 - 11n - 6)\left(\frac{3}{4}\right)_n}{10(n+1)^2(n+3)!}. \tag{2.3}$$

Elaborated computations lead to

$$w_1 = w_2 = 0, \quad w_3 = \frac{3}{81920}, \quad w_4 = \frac{21}{512000}, \quad w_5 = \frac{47}{1310720}, \tag{2.4}$$

$$w_6 = \frac{1881}{64225280}, \quad w_7 = \frac{157531}{6710886400}, \quad w_8 = \frac{42559}{2264924160}, \tag{2.5}$$

$$w_9 = \frac{507577}{33554432000}, \quad w_{10} = \frac{997177}{81201725440}, \quad w_{11} = \frac{20743573}{2061584302080}. \tag{2.6}$$

Note that

$$n^2 - 11n - 6 > 0, \tag{2.7}$$

for all $n \geqslant 12$.

Therefore, Lemma 2.3 follows easily from (2.2), (2.3), (2.4), (2.5), (2.6), (2.7). □

Lemma 2.4. *The inequality*

$$L_{1/4}(1, x) > S_{5/2,2}(1, x),$$

holds for all $x \in (0, 1)$.

Proof. It follows from (1.5) that

$$\begin{aligned} L_{1/4}(1,x) - S_{5/2,2}(1,x) &= \frac{1 + x^{5/4}}{1 + x^{1/4}} - \frac{16\left(1 - x^{5/2}\right)^2}{25\left(1 - x^2\right)^2} \\ &= \frac{\left(1 - x^{1/4}\right)^6 \left(1 + x^{1/4}\right)\left(1 + x^{5/4}\right)}{25\left(1 - x^2\right)^2} \\ &\quad \times \left(9x^2 + 20x^{7/4} + 44x^{3/2} + 60x^{5/4} + 74x + 60x^{3/4} + 44x^{1/2} + 20x^{1/4} + 9\right) > 0. \end{aligned}$$

□

Lemma 2.5. *Let* $f(x)$ *and* $g(x)$ *be defined by*

$$\begin{aligned} f(x) &= S_{5/2,2}\left(1, x^2\right), \\ g(x) &= \frac{9}{8}A\left(1, x^2\right) - \frac{5}{16}G\left(1, x^2\right) + \frac{3}{16}Q\left(1, x^2\right). \end{aligned}$$

Then there exists $x_0 \in (0, 1)$ *such that* $f(x) > g(x)$, *for* $x \in (0, x_0)$ *and* $f(x) < g(x)$, *for* $x \in (x_0, 1)$.

Proof. Let $x \in (0,1)$, $u = x + 1/x \in (2,\infty)$, $v = u - 2 = x + 1/x - 2 \in (0,\infty)$, $h_1(x)$, $h_2(x)$ and $h(x)$ be defined by

$$h_1(x) = \frac{31x^8 + 187x^7 + 118x^6 + 49x^5 + 430x^4 + 49x^3 + 118x^2 + 187x + 31}{(x^2+1)^2(x+1)^2},$$
$$h_2(x) = \frac{75\sqrt{2}\sqrt{1+x^4}}{2},$$
$$h(x) = 3703x^6 + 14124x^5 - 16260x^4 - 98560x^3 - 23040x^2 + 98304x - 32768.$$

Then elaborated computations give

$$f(x) - g(x) = \frac{16(1-x^5)^2}{25(1-x^4)^2} - \left[\frac{9(1+x^2)}{16} - \frac{5x}{16} + \frac{3}{16}\sqrt{\frac{1+x^4}{2}}\right], \tag{2.8}$$
$$= \frac{h_1(x) - h_2(x)}{400} = \frac{h_1^2(x) - h_2^2(x)}{400[h_1(x) + h_2(x)]},$$

$$h_1(x) = \frac{31u^4 + 187u^3 - 6u^2 - 512u + 256}{u^2(u+2)}x, \quad h_2(x) = 75x\sqrt{\frac{u^2-2}{2}},$$

$$h_1^2(x) - h_2^2(x) = -\frac{x^2(u-2)^2}{2u^4(u+2)^2}h(u), \tag{2.9}$$

$$h(u) = h(v+2) = 3703v^6 + 58560v^5 + 347160v^4 + 928800v^3 + 1014000v^2 + 144000v - 288000. \tag{2.10}$$

From (2.10) we clearly see that there exists $v_0 \in (0,\infty)$ such that $h(v+2) < 0$ for $v \in (0, v_0)$ and $h(v+2) > 0$ for $v \in (v_0, \infty)$. Then (2.8), (2.9), (2.10) lead to Lemma 2.5. Numerical computations show that $v_0 = 0.3994\cdots$ and $x_0 = [v_0 + 2 - \sqrt{v_0(v_0+4)}]/2 = 0.5368\cdots$. □

3. Main results

Theorem 3.1. *The double inequality*

$$\lambda S_{5/2,2}(1,r') < \mathcal{E}(r) < \mu S_{5/2,2}(1,r'),$$

holds for all $r \in (0,1)$, *if and only if* $\lambda \leqslant 25/16 = 1.5625$ *and* $\mu \geqslant \pi/2 = 1.5707\cdots$.

Proof. Let $r \in (0,1)$, u_n and v_n be defined by (2.1) and

$$F(r) = \frac{1 - \frac{2}{\pi}\mathcal{E}(r)}{1 - S_{5/2,2}(1,r')}. \tag{3.1}$$

Then it follows from (1.1), (1.5), (2.1), (3.1) and $(a)_n = a(a+1)_{n-1}$ that

$$u_1 = v_1 = \frac{1}{4}, \tag{3.2}$$

$$F(1^-) = \frac{25(\pi-2)}{9\pi}, \tag{3.3}$$

$$1 - \frac{2}{\pi}\mathcal{E}(r) = -\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(n!)^2}r^{2n} = \sum_{n=1}^{\infty} v_n r^{2n}, \tag{3.4}$$

$$
\begin{aligned}
1-S_{5/2,2}(1,r') =&1-\frac{16\left[(1-r^2)^{5/2}-2(1-r^2)^{5/4}+1\right]}{25r^4}\\
=&1-\frac{16}{25r^4}\left[\sum_{n=0}^{\infty}\frac{\left(-\frac{5}{2}\right)_n}{n!}r^{2n}-2\sum_{n=0}^{\infty}\frac{\left(-\frac{5}{4}\right)_n}{n!}r^{2n}+1\right]\\
=&\frac{16}{25}\sum_{n=1}^{\infty}\frac{2\left(-\frac{5}{4}\right)_{n+2}-\left(-\frac{5}{2}\right)_{n+2}}{(n+2)!}r^{2n}=\sum_{n=1}^{\infty}u_nr^{2n}.
\end{aligned}
\tag{3.5}
$$

From Lemmas 2.1 and 2.3 , (3.1), (3.2), (3.4) and (3.5) we know that $F(r)$ is strictly increasing on $(0,1)$ and

$$
F(0^+)=\frac{v_1}{u_1}=1. \tag{3.6}
$$

Equations (3.1), (3.3) and (3.6) together with the monotonicity of $F(r)$ on the interval $(0,1)$ lead to the conclusion that

$$
-\frac{8\pi-25}{9}+\frac{25(\pi-2)}{18}S_{5/2,2}(1,r')<\mathcal{E}(r)<\frac{\pi}{2}S_{5/2,2}(1,r'). \tag{3.7}
$$

It follows from (1.5) that

$$
\begin{aligned}
-\frac{8\pi-25}{9}+\frac{25(\pi-2)}{18}S_{5/2,2}(1,r')-\frac{25}{16}S_{5/2,2}(1,r') =&\frac{25(8\pi-25)}{144}\left(S_{5/2,2}(1,r')-\frac{16}{25}\right)\\
>&\frac{25(8\pi-25)}{144}\left(S_{5/2,2}(1,0^+)-\frac{16}{25}\right)=0,
\end{aligned}
\tag{3.8}
$$

for $r\in(0,1)$ and

$$
\lim_{r\to 0^+}\frac{\mathcal{E}(r)}{S_{5/2,2}(1,r')}=\frac{\pi}{2},\quad \lim_{r\to 1^-}\frac{\mathcal{E}(r)}{S_{5/2,2}(1,r')}=\frac{25}{16}. \tag{3.9}
$$

Therefore, Theorem 3.1 follows from (3.7), (3.8), (3.9). □

From Theorem 3.1 we get Corollary 3.2 immediately.

Corollary 3.2. *The double inequality*

$$
3.125<\frac{2\mathcal{E}(r)}{S_{5/2,2}(1,r')}<\pi,
$$

holds for all $r\in(0,1)$.

Corollary 3.3. *The inequality*

$$
\mathcal{E}(r)>\frac{\pi}{2}S_{5/2,2}(1,r')-\left(\frac{8\pi}{25}-1\right),
$$

holds for all $r\in(0,1)$.

Proof. Let $r\in(0,1)$, $F(r)$ be defined by (3.1) and $H(r)$ be defined by

$$
H(r)=S_{5/2,2}(1,r')-\frac{2}{\pi}\mathcal{E}(r). \tag{3.10}
$$

Then

$$
H(r)=[F(r)-1]\left[1-S_{5/2,2}(1,r')\right]. \tag{3.11}
$$

We clearly see that the function $r\mapsto 1-S_{5/2,2}(1,r')$ is strictly increasing from $(0,1)$ onto $(0,9/25)$. From the proof of Theorem 3.1 we know that $F(r)-1$ is strictly increasing from $(0,1)$ onto $(0,2(8\pi-25)/(9\pi))$. Then (3.11) leads to the conclusion that $H(r)$ is strictly increasing from $(0,1)$ onto $(0,16/25-2/\pi)$ and

$$
H(r)<\frac{16}{25}-\frac{2}{\pi}, \tag{3.12}
$$

for all $r\in(0,1)$. Therefore, Corollary 3.3 follows from (3.10) and (3.12). □

Remark 3.4. Extensive numerical computations show that in all cases the upper bound for $\mathcal{E}(r)$ in Theorem 3.1 is sharper than the corresponding upper bound derived from (1.2), but the lower bound for $\mathcal{E}(r)$ derived from (1.2) is better than that obtained with the use of Theorem 3.1.

Remark 3.5. For all $r \in (0,1)$, from Lemmas 2.2, 2.4 and 2.5, we clearly see that the upper bounds $\pi S_{5/2,2}(1,r')/2$ and $\pi[18A(1,r') - 5G(1,r') + 3Q(1,r')]/32$ for $\mathcal{E}(r)$ given in Theorem 3.1 and (1.4) are better than the upper bound $\pi L_{1/4}(1,r')/2$ given in (1.3) and there exists $r_0 \in (0,1)$ such that the upper bound $\pi S_{5/2,2}(1,r')/2$ for $\mathcal{E}(r)$ given in Theorem 3.1 is better than the upper bound $\pi[18A(1,r') - 5G(1,r') + 3Q(1,r')]/32$ given in (1.4) for $r \in (0, r_0)$.

Remark 3.6. Let $r \in (0,1)$, $\triangle_1(r) = L_{1/4}(1,r') - 2\mathcal{E}(r)/\pi$, $\triangle_2(r) = [18A(1,r') - 5G(1,r') + 3Q(1,r')]/16 - 2\mathcal{E}(r)/\pi$ and $\triangle_3(r) = S_{5/2,2}(1,r') - 2\mathcal{E}(r)/\pi$. Then making use of power series formulas and Corollary 3.3 we get

$$\triangle_1(r) = \frac{1}{2^{12}}r^8 + o(r^8), \quad \triangle_2(r) = \frac{7}{2^{20}}r^{12} + o\left(r^{12}\right), \quad \triangle_3(r) = \frac{3}{5\times 2^{14}}r^8 + o(r^8),$$

$$\sup_{r\in(0,1)} \triangle_1(r) \geqslant \triangle_1(1^-) = 1 - \frac{2}{\pi} = 0.3633802276\cdots,$$

$$\sup_{r\in(0,1)} \triangle_2(r) \geqslant \triangle_2(1^-) = \frac{18+3\sqrt{2}}{32} - \frac{2}{\pi} = 0.0584627491\cdots,$$

$$\sup_{r\in(0,1)} \triangle_3(r) \leqslant \frac{16}{25} - \frac{2}{\pi} = 0.0033802276\cdots.$$

Corollary 3.7. *The double inequality*

$$\frac{\pi}{2}S_{5/2,2}(1,r') - \left(\frac{8\pi}{25} - 1\right)r^8 < \mathcal{E}(r) < \frac{\pi}{2}S_{5/2,2}(1,r') - \frac{3\pi}{5}\times 2^{-15}r^8,$$

holds for all $r \in (0,1)$.

Proof. Let u_n, v_n and w_n be respectively defined by (2.1) and (2.2) and σ_n be defined by

$$\sigma_n = v_n - u_n. \tag{3.13}$$

Then it follows from (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7) and (3.13) that

$$\sigma_1 = \sigma_2 = \sigma_3 = 0, \quad \sigma_4 = \frac{3}{5}\times 2^{-14}, \tag{3.14}$$

$$\sigma_{n+1} - \frac{v_{n+1}}{v_n}\sigma_n = w_n > 0, \tag{3.15}$$

for all $n \geqslant 3$.

Inequality (3.15) leads to the conclusion that

$$\sigma_n > 0, \tag{3.16}$$

for all $n \geqslant 4$.

From (3.4), (3.5), (3.13), (3.14) and (3.16) we get

$$\frac{S_{5/2,2}(1,r') - \frac{2}{\pi}\mathcal{E}(r)}{r^8} = \frac{3}{5}\times 2^{-14} + \sum_{n=5}^{\infty}\sigma_n r^{2n-8}. \tag{3.17}$$

Equation (3.17) leads to

$$\frac{3}{5}\times 2^{-14} < \frac{S_{5/2,2}(1,r') - \frac{2}{\pi}\mathcal{E}(r)}{r^8} < \lim_{r\to 1^-}\frac{S_{5/2,2}(1,r') - \frac{2}{\pi}\mathcal{E}(r)}{r^8} = \frac{16}{25} - \frac{2}{\pi}, \tag{3.18}$$

for $r \in (0,1)$.

Therefore, Corollary 3.7 follows easily from (3.18) . □

Corollary 3.8. *The inequality*

$$\mathcal{E}(r) < \frac{875\pi}{1744}S_{5/2,2}(1,r') - \frac{3\pi}{1744}\left(1 - \frac{1}{4}r^2 - \frac{3}{64}r^4 - \frac{5}{256}r^6\right),$$

holds for all $r \in (0,1)$.

Proof. Let u_n, v_n and σ_n be respectively defined by (2.1) and (3.13). Then it follows from Lemma 2.3, (2.1), (3.3), (3.4) and (3.13), (3.14), (3.15) that

$$\sigma_n > \frac{\sigma_{n-1}}{v_{n-1}}v_n > \frac{\sigma_4}{v_4}v_n = \frac{\frac{3}{5}\times 2^{-14}}{175\times 2^{-14}}v_n = \frac{3}{875}v_n,$$

for $n \geqslant 5$ and

$$\begin{aligned} S_{5/2,2}(1,r') - \frac{2}{\pi}\mathcal{E}(r) &= \sum_{n=4}^{\infty}\sigma_n r^{2n} > \frac{3}{875}\sum_{n=4}^{\infty}v_n r^{2n} = \frac{3}{875}\left(\sum_{n=1}^{\infty}v_n r^{2n} - \sum_{n=1}^{3}v_n r^{2n}\right) \\ &= \frac{3}{875}\left[1 - \frac{2}{\pi}\mathcal{E}(r) - \left(\frac{1}{4}r^2 + \frac{3}{64}r^4 + \frac{5}{256}r^6\right)\right], \end{aligned} \tag{3.19}$$

for $r \in (0,1)$.

Therefore, Corollary 3.8 follows easily from (3.19). □

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