Convergence analysis of a novel iteration algorithm for solving split feasibility problems

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Abstract
In this paper, our aim is to construct a convergence theorem in Banach spaces via the following Ishikawa recursive algorithm
\[
\begin{align*}
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T_n y_n, \\
y_n &= (1 - \beta_n)x_n + \beta_n T_n x_n,
\end{align*}
\]
where \(\{\alpha_n\}, \{\beta_n\}\) are sequences in \([0, 1]\) and \(\{T_n\}\) is a sequence of nonexpansive mappings. Moreover, we also apply these results to solve a split feasibility problem. ©2017 All rights reserved.

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1. Introduction and preliminaries
Throughout this paper, we always assume that \(E\) is a real Banach space and \(H\) is a real Hilbert space, respectively. Let \(C\) and \(Q\) (\(C_n\) and \(Q_n\), \(n = 0, 1, 2, \ldots\)) denote the nonempty closed convex subsets of the Hilbert spaces \(H_1\) and \(H_2\). Let \(T\) be a self-mapping of \(C\). Recall that \(T\) is said to be a nonexpansive mapping, if
\[\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.\]
Here \(F(T)\) denotes the set of fixed points of \(T\), i.e., \(F(T) = \{x \in C : x = Tx\}\). We use \(\rightharpoonup (\rightarrow)\) to denote weak (strong) convergence, \(\omega_w(\{x_n\}) = \{x : \exists x_n \rightharpoonup x\}\) to denote the w-limit set of \(\{x_n\}\).

On the fixed point problems of the nonexpansive mappings which is an important class of nonlinear mappings, there are many interesting convergence results during the past decades, see [7, 21, 31] and the references therein.

Krasnosel’skii [15] and Mann [17] used the following algorithm which is now called the K-M algorithm
\[x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,\]
where \(\alpha_n \subset [0, 1]\) and the initial point \(x_0 \in C\) have no restrictions.

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In 1979, Reich [24] proved that the sequence defined by (1.1) converges weakly to \( q \in F(T) \), if \( E \) is a uniformly convex Banach space with a Fréchet differentiable norm, \( T : C \to C \) is a nonexpansive self-mapping with \( F(T) \neq \emptyset \) and \( \sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty \).

In 2011, Zhang et al. [29] proposed modified Halpern and Ishikawa iteration algorithms for solving the fixed points of nonexpansive mappings in Banach spaces. For the convergence of modified Halpern and Ishikawa iterative algorithms, we refer authors to [8, 9, 20, 28] for more details. In 2016, Hieu et al. [11] introduced three parallel hybrid extragradient methods and obtained the set of fixed points of nonexpansive mappings in a real Hilbert space.

The split feasibility problem (SFP) is to find a point

\[ x \in C, \quad \text{such that} \quad Ax \in Q, \]

where \( A : H_1 \to H_2 \) is a bounded linear operator. Censor and Elfving [5] first introduced the SFP in a Hilbert space. Recently, SFP which attracts attentions of many researchers, has been widely used in many applications such as signal processing and other fields, see [3, 4, 10, 14] and the references therein.

It has been proved that if the SFP (1.2) has a solution, it is not hard to find a solution \( x^* \) to (1.2) is equivalent to a fixed point equation

\[ P_C(I - \gamma A^*(I - P_Q)A)x^* = x^*. \]

In order to solve the SFP (1.2), Byren [3] proposed the popular CQ algorithm which generates a sequence \( \{x_n\} \) by

\[ x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), \quad n \geq 0, \]

where \( \gamma \in (0, 2/\lambda) \) with \( \lambda \) being the spectral radius of the operator \( A^*A \).

As we know, the CQ algorithm (1.4) is a special case of the \( K - M \) algorithm (1.1) (see [27]). Due to the fixed point formulation (1.3) of the SFP (1.2), we can apply the \( K - M \) algorithm (1.1) to the operator

\[ P_C(I - \gamma A^*(I - P_Q)A), \]

to produce a sequence \( \{x_n\} \) given by

\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_C(x_n - \gamma A^*(I - P_Q)Ax_n), \quad n \geq 0, \]

where \( \gamma \in (0, 2/\lambda) \) and again \( \lambda \) being the special radium of the operator \( A^*A \). Then we can see that as long as \( \{\alpha_n\} \) satisfies condition \( \sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty \), we have weak convergence of the sequence \( \{x_n\} \) generated by the algorithm (1.5). If possible errors are taken into consideration, then we should study perturbations of the closed convex sets \( C \) and \( Q \). For example, Zhao and Yang [30] considered the following perturbed algorithm:

\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_{C_n}(x_n - \gamma A^*(I - P_{Q_n})Ax_n), \]

where \( \{C_n\} \) and \( \{Q_n\} \) are sequences of closed convex subsets of \( H_1 \) and \( H_2 \), respectively, which converges to \( C \) and \( Q \), respectively, in the sense of Mosco [1]. This is a motivation for the authors to study the following more general algorithm which generates a sequence \( \{x_n\} \) according to the recursive formula

\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n x_n, \]

where \( \{T_n\} \) is a sequence of nonexpansive mapping in Hilbert space \( H \).

In 2005, under certain conditions, Zhao and Yang [30] studied the convergence of (1.6) in a finite-dimensional Hilbert space.

**Theorem 1.1.** Let \( T \) and \( T_n \) be nonexpansive operators in Hilbert space \( H \) for \( k = 0, 1, 2, \cdots, T_n \to T \) and \( \{\alpha_n\} \subset (0, 1) \) satisfying
Remark 1.2. In [30, page 1794], the lim inf \( n \to \infty \) of \( \{x_n\} \) weakly converges to a fixed point \( z \) of \( T \) do not imply that lim inf \( j \to \infty \) of \( \{x_{n_j}\} \) weakly converges to a fixed point \( z \) of \( T \) unless the space is finite dimensional.

In 2006, Xu [26] extended Zhao and Yang [30] from finite dimensional Hilbert spaces to infinite dimensional Banach spaces and they obtained the following result.

**Theorem 1.3.** Assume that \( X \) is a uniformly convex Banach space which has a Fréchet differentiable norm. Let \( T \) be a nonexpansive operator in the Banach space \( X \), \( F(T) \) is the set of fixed points and \( F(T) \) is nonempty. Let \( \{T_n\} \) be a sequence of nonexpansive mappings on \( C \). If assumptions (i) and (ii) of Theorem 1.1 are satisfied, then the sequence \( \{x_n\} \) generated by the algorithm (1.6) converges weakly to a fixed point of \( T \).

Recently, Qu et al. [22] and Moudafi [18] studied the split feasibility problem by the relaxed alternating CQ-algorithm and CQ-like algorithms. In 2014, [6] present weak and strong convergence theorems of solutions to a split feasibility problem for a family of nonspreading-type mapping in Hilbert spaces.

For each \( x_0 \in C \), the iteration sequence \( \{x_n\} \) is called an Ishikawa iteration scheme, if

\[
\begin{align*}
\alpha_n &\in [0, 1], \\
(1 - \alpha_n)x_n &+ \alpha_n T_n y_n, \\
y_n &\in (1 - \beta_n)x_n + \beta_n T_n x_n.
\end{align*}
\]

The Ishikawa iteration scheme was introduced by Ishikawa [12] and he proved that the sequence generated by this algorithm must converge to a fixed point of a Lipschitzian pseudo-contractive mapping in a convex compact subset of Hilbert spaces. After that, lots of authors studied the Ishikawa (two-step) iteration algorithm for solving the zero points of nonlinear operators, the equilibrium problems, the variational inequalities problems in Hilbert spaces and Banach spaces, see [13, 16, 19, 23] and the references therein.

In this paper, motivated by Zhao and Yang [30], Xu [26] and the above works, we proposed the following Ishikawa iteration algorithm, given \( x_0 \in C \)

\[
\begin{align*}
\alpha_n &\in [0, 1], \\
(1 - \alpha_n)x_n &+ \alpha_n T_n y_n, \\
y_n &\in (1 - \beta_n)x_n + \beta_n T_n x_n,
\end{align*}
\]

where \( 0 \leq \alpha_n, \beta_n \leq 1 \) and \( \{T_n\} \) is a sequence of nonexpansive mappings. We show that the sequence \( \{x_n\} \) weakly converges to a fixed point of \( T \). We also apply this result to solve the SFP (1.2) via the following iteration algorithm

\[
\begin{align*}
\alpha_n &\in [0, 1], \\
(1 - \alpha_n)x_n &+ \alpha_n P_{C_n}(y_n - \gamma A^*(I - P_{Q_n})Ay_n), \\
y_n &\in (1 - \beta_n)x_n + \beta_n P_{C_n}(x_n - \gamma A^*(I - P_{Q_n})Ax_n),
\end{align*}
\]

We show that \( \{x_n\} \) weakly converges to a solution of the SFP (1.2).

The aim of this paper is to present the above Ishikawa algorithm to solve the SFP, these results mainly improve the exited results in Zhao et al. [30], Xu [26] and Qu et al. [22]. Specifically, we list the following highlights:

- The results extend and improve the corresponding results from finite dimensional Hilbert spaces to infinite dimensional Banach spaces.
- The conditions in this paper are much mild. Indeed, we remove the assumptions in [30, Theorem 2.2] that the sequence \( \{C_n\} \) and \( \{Q_n\} \) Mosco converge to \( C \) and \( Q \), respectively.
• Our results extend the K-M algorithm to the Ishikawa algorithm.

• Our algorithm is efficient for solving the SFP.

In order to get our main results, we need the following preliminaries.

Definition 1.4. An operator $S : H \to H$ is called an averaged operator, if it can be shown as the following combining form:

$$S = (1 - \alpha)I + \alpha T,$$

where $I$ is the identity operator and $T : H \to H$ is a nonexpansive operator and $\alpha \in (0, 1)$.

As an special case, if $\alpha = 1/2$, the projections are averaged operators.

Definition 1.5. If $T$ is an operator with domain $D(T)$ and range $R(T)$ in $H$.

(i) $T$ is called monotone, if

$$\langle x - y, Tx - Ty \rangle \geq 0, \quad \forall x, y \in D(T).$$

(ii) For a real number $\nu > 0$, $T$ is called to be $\nu$-inverse strongly monotone ($\nu$-ism) (or co-coercive), if it satisfies the following inequality

$$\langle x - y, Tx - Ty \rangle \geq \nu \| Tx - Ty \|^2.$$

So, we can easily get the following conclusions.

(i) If $T$ is nonexpansive, then $I - T$ is monotone and a projection $P_K$ is $1$-ism.

(ii) $T$ is averaged $\iff$ the complement $I - T$ is $\nu$-ism for some $\nu > \frac{1}{2}$.

The following lemma is trivial.

Lemma 1.6. Let $\{\mu_n\}$ and $\{\nu_n\}$ be nonnegative sequences satisfying $\sum_{n=0}^{\infty} \mu_n < \infty$ and $\nu_{n+1} \leq \nu_n + \mu_n$, $n = 0, 1, \ldots$. Then $\{\nu_n\}$ is a convergent sequence.

2. Main Results

Theorem 2.1. Let $E$ be a real uniformly convex Banach space, $C$ be a nonempty closed convex subset of $E$ and $T : C \to C$ be nonexpansive mapping and $(T_n)$ be a sequence of nonexpansive mappings on $C$. Let $(x_n)$ be defined in (1.7), where $0 \leq \alpha_n, \beta_n \leq 1$ satisfy the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \alpha$, $\sum_{n=0}^{\infty} \alpha_n \beta_n < \infty$, $\lim_{n \to \infty} \beta_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n D_\rho(T_n, T) < \infty$, for every $\rho > 0$, where

$$D_\rho(T_n, T) = \sup\{\|Tx - Ty\| : \|x\| \leq \rho\}.$$

Then $\{x_n\}$ converges weakly to a fixed point $P$ of $T$.

Proof. First we show that $\{x_n\}$ is bounded.

Take $z \in F(T)$, it follows that

$$\|x_{n+1} - z\| \leq (1 - \alpha_n)\|x_n - z\| + \alpha_n\|T_n y_n - T_n z\| + \alpha_n\|T_n z - Tz\| \leq \|x_n - z\| + \alpha_n\|T_n y_n - T_n z\| + \alpha_n\|T_n z - Tz\|.$$

(2.1)

Similarly, we have

$$\|y_n - z\| \leq (1 - \beta_n)\|x_n - z\| + \beta_n\|T_n x_n - T_n z\| + \beta_n\|T_n z - Tz\| \leq \|x_n - z\| + \beta_n\|T_n z - Tz\|.\quad (2.2)$$

Take $z \in \text{Fix}(T)$, it follows that

$$\|x_{n+1} - z\| = \lim_{n \to \infty} \alpha_n \|T_n y_n - T_n z\| + \alpha_n\|T_n z - Tz\|.$$

(2.3)
It follows from (2.1) and (2.2) that
\[
\|x_{n+1} - z\| \leq \|x_n - z\| + \alpha_n (1 + \beta_n) \|T_n z - Tz\|
\leq \|x_n - z\| + 2\alpha_n D \|z\| (T_n, T).
\]
By condition (ii), we see that \(\lim_{n \to \infty} \|x_n - z\|\) exists. Hence, \(\{x_n\}\) is bounded, so \(\{T_n x_n\}\) and \(\{T x_n\}\) are bounded, too.

Next, we claim that \(\|x_n - T x_n\| \to 0\) as \(n \to \infty\).

Let \(\rho = \sup\{\|x_n\|, \|T_n x_n\| : n \geq 0\} < \infty\) and let \(r = \rho + \|z\| + 2\sup(\alpha_n D \rho(T_n, T))\).

Now since \(E\) is uniformly convex, there exists a continuous strictly convex function \(\varphi\) with \(\varphi(0) = 0\) by [25]. Hence, we have
\[
\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda) \varphi(\|x - y\|),
\]
for all \(x, y \in E\) such that \(\|x\| \leq r\) and \(\|y\| \leq r\) and for all \(\lambda \in [0, 1]\). In particular, setting \(e_n = T_n y_n - T y_n\) (note that \(\|e_n\| \leq D \rho(T_n, T)\)) and a constant \(M_1\) so that, \(M_1 \geq \sup\{\|x_n - z\| + \alpha_n \|e_n\| : n \geq 0\}\). By (2.3) and condition (i), we have
\[
\|x_{n+1} - z\|^2 = \|(1 - \alpha_n)(x_n - z + \alpha_n e_n) + \alpha_n (T y_n - z + \alpha_n e_n)\|^2
\leq (1 - \alpha_n) \|x_n - z + \alpha_n e_n\|^2 + \alpha_n \|T y_n - z + \alpha_n e_n\|^2
- \alpha_n (1 - \alpha_n) \varphi(\|x_n - T y_n\|)
\leq (1 - \alpha_n) (\|x_n - z\|^2 + 2\alpha_n \|x_n - z\| \|e_n\| + \alpha_n^2 \|e_n\|^2)
+ \alpha_n (\|T y_n - z\|^2 + 2\alpha_n \|e_n\| \|T y_n - z\| + \alpha_n^2 \|e_n\|^2)
- \alpha_n (1 - \alpha_n) \varphi(\|x_n - T y_n\|)
\leq \|x_n - z\|^2 + M_1 \alpha_n D \rho(T_n, T) - \alpha_n (1 - \alpha_n) \varphi(\|x_n - T y_n\|).
\]
It follows that
\[
\alpha_n (1 - \alpha_n) \varphi(\|x_n - T y_n\|) \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + M_1 \alpha_n D \rho(T_n, T).
\]
Since \(\lim_{n \to \infty} \|x_n - z\|\) exists, condition (i) and (2.4) imply that
\[
\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) \varphi(\|x_n - T y_n\|) < \infty,
\]
which further implies that \(\liminf_{n \to \infty} \varphi(\|x_n - T y_n\|) = 0\). Hence,
\[
\liminf_{n \to \infty} \|x_n - T y_n\| = 0.
\]
It follows from (1.7) that
\[
\|x_n - y_n\| = \beta_n \|x_n - T_n x_n\| \to 0.
\]
Then
\[
\liminf_{n \to \infty} \|x_n - T x_n\| = 0.
\]
(2.5)
Since \(\{x_n\}\) and \(\{T_n x_n\}\) are bounded, there exists a constant \(M_2\) satisfying \(2\|T_n x_n - x_n\| \leq M_2\). Hence, we have
\[
\|x_{n+1} - T x_{n+1}\| = \|(1 - \alpha_n)x_n + \alpha_n T_n y_n - T x_{n+1}\|
\leq \|(1 - \alpha_n)x_n + \alpha_n T_n y_n - T x_n + T x_n - T x_{n+1}\|
\leq (1 - \alpha_n) \|x_n - T x_n\| + \alpha_n \|T_n y_n - T x_n\|
\leq (1 - \alpha_n) \|x_n - T x_n\| + \alpha_n M_2.
\]
\[
\begin{align*}
&+ \|x_{n+1} - x_n\| \\
&= (1 - \alpha_n)\|x_n - Tx_n\| + \alpha_n\|T_n y_n - Tx_n\| \\
&\quad + \alpha_n\|x_n - T_n y_n\| \\
&\leq (1 - \alpha_n)\|x_n - Tx_n\| + \alpha_n\|T_n y_n - Tx_n\| \\
&\quad + \alpha_n\|x_n - T_n y_n\| \\
&= \|x_n - Tx_n\| + 2\alpha_n\|T_n y_n - Tx_n\| \\
&\leq \|x_n - Tx_n\| + 2\alpha_n\|T_n y_n - Ty_n\| + 2\alpha_n\|y_n - x_n\| \\
&\leq \|x_n - Tx_n\| + 2\alpha_n D_\rho(T_n, T) + 2\alpha_n\|T_n x_n - x_n\| \\
&\leq \|x_n - Tx_n\| + 2\alpha_n D_\rho(T_n, T) + \alpha_n\beta_n M_2.
\end{align*}
\]

Since \(\sum_{n=1}^{\infty} \alpha_n D_\rho(T_n, T) < \infty\), \(\sum_{n=1}^{\infty} \alpha_n\beta_n M_2 < \infty\), by Lemma 1.6, we obtain
\[
\lim_{n \to \infty} \|x_n - Tx_n\| \text{ exists.}
\]
This together with (2.5) implies that
\[
\lim_{n \to \infty} \|x_n - Tx_n\| = 0.
\]
The demiclosedness principle for nonexpansive mappings (see [2]) implies that
\[
\omega_w(x_n) \subseteq F(T).
\]
To prove that \(\{x_n\}\) is weakly convergent to a fixed point of \(T\), it now suffices to prove that \(\omega_w(x_n)\) consists of exactly one point.

Indeed, there are \(\bar{x}, \bar{x} \in \omega_w(x_n)\) \((x_{n_i} \rightharpoonup \bar{x}, \ x_{m_j} \rightharpoonup \bar{x})\). Note that \(\lim_{n \to \infty} \|x_n - \bar{x}\|\) and \(\lim_{n \to \infty} \|x_n - \bar{x}\|\) exist. If \(\bar{x} \neq \bar{x}\), then
\[
\lim_{n \to \infty} \|x_n - \bar{x}\|^2 = \lim_{j \to \infty} \|(x_{m_j} - \bar{x}) + (\bar{x} - \bar{x})\|^2 \\
= \lim_{j \to \infty} \|x_{m_j} - \bar{x}\|^2 + \|\bar{x} - \bar{x}\|^2 \\
> \lim_{l \to \infty} \|x_{n_l} - \bar{x}\|^2 \\
= \lim_{l \to \infty} \|(x_{n_l} - \bar{x}) + (\bar{x} - \bar{x})\|^2 \\
= \lim_{l \to \infty} \|x_{n_l} - \bar{x}\|^2 + \|\bar{x} - \bar{x}\|^2 \\
> \lim_{l \to \infty} \|x_{n_l} - \bar{x}\|^2 \\
= \lim_{n \to \infty} \|x_n - \bar{x}\|^2.
\]
This is a contradiction. The proof is completed. \(\Box\)

Corollary 2.2. Let \(C\) be a closed convex subset of a Hilbert space \(H\). Assume that \(T : C \to C\) is a nonexpansive mapping such that \(F(T) \neq \emptyset\). Assume also that \(\{T_n\}\) is a sequence of nonexpansive mappings on \(C\). Let the sequence \(\{x_n\}\) be defined by (1.7), where \(0 \leq \alpha_n, \beta_n \leq 1\) satisfying the following conditions:

(i) \(\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty\), \(\sum_{n=0}^{\infty} \alpha_n\beta_n < \infty\), \(\lim_{n \to \infty} \beta_n = 0\);

(ii) \(\sum_{n=0}^{\infty} \alpha_n D_\rho(T_n, T) < \infty\), for every \(\rho > 0\), where \(D_\rho(T_n, T) = \sup\{\|T_n x - Tx\| : \|x\| \leq \rho\}\).

Then \(\{x_n\}\) converges weakly to a fixed point of \(T\).

Below we show the applications of algorithm (1.7) to the split feasibility problem.

We now apply Corollary 2.2 to the SFP (1.2).

Recall that \( p \)-distance between two closed convex subsets \( E_1 \) and \( E_2 \) of a Hilbert space \( H \) is defined by

\[
d_p(E_1, E_2) = \sup_{\|x\| \leq \rho} \|P_{E_1}x - P_{E_2}x\|.
\]

**Theorem 2.4.** Assume that the sequence \( \{x_n\} \) is generated by the perturbed averaging CQ algorithm (1.8), the sequences \( \{\alpha_n\}, \{\beta_n\} \in [0, 1] \) satisfy the conditions:

(i) \( \sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty, \sum_{n=0}^{\infty} \alpha_n \beta_n < \infty, \lim_{n \to \infty} \beta_n = 0; \)

(ii) \( \sum_{n=0}^{\infty} \alpha_n d_p(C_n, C) < \infty \) and \( \sum_{n=0}^{\infty} \alpha_n d_p(Q_n, Q) < \infty, \forall \rho > 0. \)

Then \( \{x_n\} \) converges weakly to a solution of the SFP (1.2).

**Proof.** Set \( T = P_{C}(I - \gamma A^* (I - P_Q)A) \) and \( T_n = P_{C_n}(I - \gamma A^* (I - P_{Q_n})A) \). Then \( T \) and \( T_n \) are nonexpansive with \( \gamma < \frac{2}{\|A\|^2} \). Indeed, write

\[
U = A^* (I - P_Q)A,
\]

and

\[
S = P_C(I - \gamma U).
\]

Since \( P_Q \) and \( I - P_Q \) are 1-isometry, we calculate

\[
\langle x - y, Ux - Ut_y \rangle = \langle x - y, A^* (I - P_Q)Ax - A^* (I - P_Q)Ay \rangle = \langle Ax - Ay, (I - P_Q)Ax - (I - P_Q)Ay \rangle \geq \| (I - P_Q)Ax - (I - P_Q)Ay \|^2 \geq \frac{1}{\|A\|^2} \| Ux - Ut_y \|^2.
\]

Hence, \( U \) is \( \frac{1}{\|A\|^2} \)-inverse strongly monotone, which implies that \( \gamma U \) is \( \frac{1}{\gamma \|A\|^2} \)-isometry, which in turn implies that \( I - \gamma U \) is averaged for \( \|A\|^2 \gamma < 2 \), i.e. \( \gamma < \frac{2}{\|A\|^2} \).

Hence, we get that \( S = P_C(I - \gamma U) \) is averaged. Then \( S = P_C(I - \gamma U) \) is nonexpansive, so are \( S_n = P_{C_n}(I - \gamma U_n) \). Since the SFP (1.2) is consistent, \( F(T) \) is nonempty. Note that \( F(T) \) is the solution set of the SFP (1.2). Also the perturbed averaging CQ algorithm (1.8) can be written as

\[
\begin{cases}
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n y_n, \\
y_n = (1 - \beta_n)x_n + \beta_n T_n x_n.
\end{cases}
\]

Given \( \rho > 0 \). Letting

\[
\hat{\rho} = \sup \{ \max \{ \|Ax\|, \|x - \gamma A^* (I - P_Q)Ax\| : \|x\| \leq \rho \} : \|x\| \leq \rho \} < \infty,
\]

we compute for \( x \in H \), such that \( \|x\| \leq \rho \),

\[
\|T_n x - Tx\| \leq \|P_{C_n}(x - \gamma A^* (I - P_{Q_n})Ax) - P_{C_n}(x - \gamma A^* (I - P_Q)Ax)\|
+ \|P_{C_n}(x - \gamma A^* (I - P_{Q_n})Ax) - P_C(x - \gamma A^* (I - P_Q)Ax)\|
\leq \|P_{C_n}(x - \gamma A^* (I - P_{Q_n})Ax) - P_C(x - \gamma A^* (I - P_Q)Ax)\|
+ \gamma \|A^* (P_Q Ax - P_{Q_n} Ax)\|
\leq d_{\hat{\rho}}(C_n, C) + \gamma \|A\| d_{\hat{\rho}}(Q_n, Q).
\]
This shows that \( D_\rho(T_n, T) \leq d_\rho(C_n, C) + \gamma \|A\| d_\rho(Q_n, Q). \)

It follows from condition (ii) that
\[
\sum_{n=0}^{\infty} \alpha_n D_\rho(T_n, T) \leq \sum_{n=0}^{\infty} \alpha_n d_\rho(C_n, C) + \gamma \|A\| \sum_{n=0}^{\infty} \alpha_n d_\rho(Q_n, Q) < \infty.
\]

Now we can apply Corollary 2.2 to conclude that the sequence \( \{x_n\} \) defined by the perturbed averaging CQ algorithm (1.8) converges weakly to a solution of SFP (1.2).

**Remark 2.5.** Theorem 2.4 extends [30, Theorem 2.2] from the K-M algorithm to the Ishikawa iteration algorithm and removes the assumption in [30, Theorem 2.2] that the sequences \( \{C_n\} \) and \( \{Q_n\} \) Mosco converge to \( C \) and \( Q \), respectively.

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