Vector valued Orlicz-Lorentz sequence spaces and their operator ideals

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Communicated by M. De la Sen

Abstract

In the present paper we introduce and study vector valued Orlicz-Lorentz sequence spaces \(l^{p,q,M,u,\Delta,A}(X)\) on Banach space \(X\) with the help of a Musilak-Orlicz function \(M\) and for different positive indices \(p\) and \(q\). We also study their cross and topological duals. Finally, we introduce the operator ideals with the help of the corresponding scalar sequence spaces and \(s\)-numbers. ©2017 All rights reserved.

Keywords: Lorentz sequence spaces, \(s\)-numbers of operators, Musielak-Orlicz function, difference sequence spaces, operator ideals.

2010 MSC: 46A45, 47B06, 47L20.

1. Introduction and preliminaries

Let \(X\) and \(Y\) be two sequence spaces and \(A = (a_{nk})\) be an infinite matrix of real or complex numbers. Then we say that \(A\) defines a matrix mapping from \(X\) into \(Y\), if for every sequence \(x = (x_k)_{k=0}^{\infty} \in X\), the sequence \(Ax = \{A_n(x)\}_{n=0}^{\infty}\), the \(A\)-transform of \(x\), is in \(Y\), where

\[
A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k, \quad (n \in \mathbb{N}).
\]  

(1.1)

By \((X,Y)\), we denote the class of all matrices \(A\) such that \(A : X \to Y\). Thus, \(A \in (X,Y)\), if and only if the series on the right-hand side of (1.1) converges for each \(n \in \mathbb{N}\) and every \(x \in X\).

The matrix domain \(X_A\) of an infinite matrix \(A\) in a sequence space \(X\) is defined by

\[
X_A = \{x = (x_k) : Ax \in X\}.
\]

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by several authors (see [20]).

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doi:10.22436/jnsa.010.02.01

Received 2016-08-05
The study of vector-valued sequence spaces (VVSS) was provoked by the work of Grothendieck in [6]. Since then this theory has developed considerably in different directions, (see [3, 14] and references given therein).

An Orlicz function $M: [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing and convex function such that $M(0) = 0, M(x) > 0$, for $x > 0$. Such function $M$ always has the integral representation

$$M(x) = \int_0^x p(t)dt,$$

where $p(t)$, known as the kernel of $M$, is right continuous, non-decreasing function for $t > 0$. It is clear that an Orlicz function $M$ is always increasing as $M(x) \to \infty$ as $x \to \infty$. Also $tp(t) \to \infty$ as $t \to \infty$ and $tp(t) = 0$ for $t = 0$, [11]. However $p(t) > 0$ for $t = 0$ is equivalent to the fact that the Orlicz sequence space $l_M$ is isomorphic to $l_1$, [8]. Therefore, we presume here that the kernel $p(t)$ has value $0$ for $t = 0$ and obviously $p(t) \to \infty$ as $t \to \infty$.

For Orlicz function $M$ and kernel $p$, we define $q(s) = \sup\{t : p(t) \leq s\}, s \geq 0$. Clearly $q$ possesses the same properties as $p$ and the function $N$ defined as $N(x) = \int_0^x q(t)dt$, is an Orlicz function. The functions $M$ and $N$ are called mutually complementary functions. These functions $M$ and $N$ satisfy Young’s inequality: $xy \leq M(x) + N(y)$, for $xy \geq 0$ and also $M(\alpha x) \leq \alpha M(x)$ for $0 < \alpha < 1$.

An Orlicz function $M$ is said to satisfy the $\Delta_2$-condition for small $x$ or at $0$, if for each $k > 1$, there exist $R_k > 0$ and $x_k > 0$ such that

$$M(kx) \leq R_k M(x), \text{ for all } x \in (0, x_k].$$

Suppose $X$ and $Y$ are vector spaces over the same field $\mathbb{K}$ of real or complex numbers, generates a dual system $(X, Y)$ with respect to the bilinear functional $\langle x, y \rangle$. We shall denote the vector space of all sequences formed by the elements of $\Lambda$ with respect to the operations of pointwise addition and scalar multiplication by $\Omega(X)$ and the space of all finitely non-zero sequences from $\Omega(X)$ by $\phi(X)$. A vector-valued sequence space $\Lambda(X)$ is a subspace of $\Omega(X)$ containing $\phi(X)$. The symbol $\delta_x^n$ exists for the sequence $\{0, 0, \ldots, 0, x, 0, 0, \ldots\}$, where $x$ is placed at the $i$th coordinate. The notation $\vec{x}^{(n)}$ denotes the $n$-th section of $x$ given by $\{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}$.

A subset $M$ of $\Lambda(X)$ is said to be normal, if for $\{x_i\} \in M$ and $\{\alpha_i\} \in \mathbb{K}$, with $|\alpha_i| \leq 1, i \geq 1$, the sequence $\{\alpha_i x_i\} \in M$. The generalized Köthe dual of $\Lambda(X)$ is the space

$$\Lambda^\times(Y) = \left\{ \vec{y} = \{y_i\} \in Y : \sum_{i \geq 1} |\langle x_i, y_i \rangle| < \infty \text{ for all } \{x_i\} \in \Lambda(X) \right\}.$$

The generalized Köthe dual of $\Lambda^\times(Y)$ is denoted by $\Lambda^{\times\times}(X)$. The space $\Lambda(X)$ is said to be perfect, if $\Lambda(X) = \Lambda^{\times\times}(X)$.

A vector-valued sequence space $\Lambda(X)$ equipped with a Hausdorff locally convex topology $T$ is called

(i) a GK-space, if the maps $P_{n, \Lambda(X)} : \Lambda(X) \to X, P_{n, \Lambda(X)}(\vec{x}) = x_n$, for each $n \geq 1$, are continuous;

(ii) a GAK-space, if $\Lambda(X)$ is a GK-space and for each $\{x_i\} \in \Lambda(X), \vec{x}^{(n)} \to \vec{x}$ as $n \to \infty$, in $T$;

(iii) a GAD-space, if $\vec{x} \in \overline{\phi(X)}$, for every $\vec{x} \in \Lambda(X)$, i.e., $\overline{\phi(X)} = \Lambda(X)$.

Remark 1.1. Every perfect sequence space $\Lambda(X)$ is normal [14].

Let us state here that if the dual system is $\langle X, X^\ast \rangle$ where $X$ is a Banach space and $X^\ast$ is its topological dual, then we may interchangeably use the notations $\langle x, f \rangle$ or $f(x)$ for $x \in X$ and $f \in X^\ast$ in the sequel.

We write $w$ for $\Omega(X)$, $\phi$ for $\phi(X)$ and $\lambda$ for $\Lambda(X)$, if we take $X = \mathbb{K}$, the field of scalars. If $e_n$’s are the $n$-th unit vectors in $w$, i.e., $e^n = [\delta_nj]_{j=1}^\infty$, where $\delta_nj$ is the Kronecker delta, $\phi$ is clearly the subspace of $w$ spanned by $e_n$’s, $n \geq 1$.

A sequence space $\lambda$ is said to be symmetric, if $\vec{x}_\sigma = \{\alpha_{\sigma(j)}\} \in \lambda$ whenever $\vec{x} \in \lambda$ and $\sigma \in \Pi$, where $\Pi$ is the collection of all permutations of $\mathbb{N}$. The Köthe dual $\lambda^\times$ of a symmetric sequence space $\lambda$ is symmetric [8].
The $\delta$-dual for scalar-valued sequence space $\lambda$ is defined as
\[
\lambda^\delta = \left\{ \overline{x} \in w : \sum_{i \geq 1} |\alpha_i \beta_{\rho(i)}| < \infty \text{ for all } \overline{\beta} \in \lambda \text{ and } \rho \in \Pi \right\}.
\]

$\lambda^\times$ coincides with $\lambda^\delta$, if $\lambda$ is symmetric.
We define
\[
\lambda(X) = \left\{ \{x_n\} : x_n \in X, n \geq 1 \text{ and } \|\|x_n\|\| \in \lambda \right\},
\]
for a scalar-valued sequence space $\lambda$ and a Banach space $X$. In case, $\lambda$ equipped with the norm $\|\cdot\|_{\lambda}$, is a Banach space. Therefore, $\lambda(X)$ is also a Banach space with respect to the norm
\[
\|\overline{x}\|_{\lambda(X)} = \|\|x_n\|\|_{\lambda}, \quad \text{(see [1, 3])}.
\]
As particular cases, we have $l^\infty_{\lambda}(X)$ for $\lambda = l^\infty$ and $c_0(X)$ corresponding to $\lambda = c_0$.
We define the set $\tilde{l}_M(X)$ as
\[
\tilde{l}_M(X) = \{ \overline{x} \in \Omega(X) : \sum_{i \geq 1} M(\|x_i\|) < \infty \},
\]
for a Banach space $X$ corresponding to an Orlicz function $M$.
The vector-valued Orlicz sequence space is defined as
\[
l_M(X) = \{ \overline{x} \in \Omega(X) : \sum_{i \geq 1} f_i(x_i) \text{ converges for all } \{f_i\} \in \tilde{l}_N(X^*) \},
\]
for mutually complementary functions $M$ and $N$.
A corresponding way of defining $l_M(X)$ is
\[
l_M(X) = \{ \overline{x} \in \Omega(X) : \sum_{i \geq 1} M\left(\frac{\|x_i\|}{\rho}\right) < \infty \text{ for some } \rho > 0 \}.
\]
Two norms
\[
\|\overline{x}\|_{(M)} = \sup \left\{ \left\| \sum_{i \geq 1} f_i(x_i) \right\| : \sum_{i \geq 1} N(\|f_i\|) \leq 1 \right\},
\]
and
\[
\|\overline{x}\|_{M} = \inf \left\{ \rho > 0 : \sum_{i \geq 1} M\left(\frac{\|x_i\|}{\rho}\right) \leq 1 \right\},
\]
are equivalent on $l_M$ and hence we have
\[
\|\overline{x}\|_M \leq \|\overline{x}\|_{(M)} \leq 2\|\overline{x}\|_M, \quad \text{for } \overline{x} \in l_M(X), \quad \text{(see [21]).}
\]
We shall write $l_M(X)$ as $l_M$ for $X = K$. If $M$ satisfies $\Delta_2$-condition at 0 and $M$, $N$ are mutually complementary Orlicz functions, then $(l_M)^\times = l_N$ [8].
A Musielak-Orlicz function $M = \{M_n\}$ is a sequence of Orlicz functions (see [5, 13]). A Musielak-Orlicz function $M$ is said to satisfy $L_1$ condition, if $p_n(x) \geq p_{n+1}(x)$ for all $x \in [0, \infty)$, where $p_n$ be the kernel of $M_n$, for all $n \in N$. A convex modular $\rho_M$ on $w$ for a Musielak-Orlicz function $M$ is defined as
\[
\rho_M(\{\alpha_n\}) = \sup_{\sigma \in \Pi} \sum_{n=1}^{\infty} M_n(\alpha_{\sigma(n)}).
\]
Analogous to a convex modular $\rho_M$, we define modular space as
\[
\lambda_M = \{ \overline{x} = \{\alpha_n\} \in w : \rho_M(\beta \overline{x}) < \infty, \text{ for some } \beta > 0 \}.
\]
This space becomes a normed space under the Luxemburg norm
\[ \|\varpi\| = \inf\{\beta > 0 : \rho_M(\varpi/\beta) \leq 1\}. \]

A modular sequence space \( \lambda_M \) is always a symmetric sequence space.

The decreasing rearrangement of the absolute values of a sequence \( \varpi = \{\alpha_n\} \) in \( l_\infty \) is given by \( (t_n(\varpi)) \), where
\[ t_n(\varpi) = \inf\{\rho > 0 : \text{card}\{k : |\alpha_k| > \rho\} < n\}. \]

Here \( \text{card} A \) denotes the cardinality of the set \( A \). The sequence \( \{t_n(\varpi)\} \) satisfies the following properties [16]:

(i) \( \|\varpi\|_\infty = t_1(\varpi) \geq t_2(\varpi) \geq ... \geq 0 \) for \( \varpi \in l_\infty \).

(ii) \( t_{m+n-1}(\varpi + \overline{\beta}) \leq t_m(\varpi) + t_m(\overline{\beta}) \) for \( \varpi, \overline{\beta} \in l_\infty \).

(iii) \( t_{m+n-1}(\varpi \overline{\beta}) \leq t_m(\varpi) t_m(\overline{\beta}) \) for \( \varpi, \overline{\beta} \in l_\infty \).

Here \( \varpi \overline{\beta} = \{\alpha_n \beta_n\} \).

For \( \varpi = \{x_n\} \in l_\infty(X) \), we denote by
\[ t_n(\varpi) = t_n(\{x_n\}) = t_n(\{\|x_n\|\}), \quad n \in \mathbb{N}. \]

The Lorentz sequence space \( l_{p,q} (0 < p, q \leq \infty) \) is given by
\[ l_{p,q} = \{\varpi = \{\alpha_n\} \in l_\infty : \{n^{\frac{1}{p} - \frac{1}{q}} t_n(\varpi)\} \in l_q\}. \]

For \( \varpi \in l_{p,q} \), let us consider the real-valued function \( \| . \|_{p,q} \) as follows
\[ \|\varpi\|_{p,q} = \begin{cases} \left\{ \frac{\sum_{n \geq 1} (n^{\frac{1}{p} - \frac{1}{q}} t_n(\varpi))^q}{n^{\frac{1}{p} t_n(\varpi)}} \right\}^{\frac{1}{q}} & \text{for } 0 < q < \infty, \\ \sup_{n \geq 1} n^{\frac{1}{p} t_n(\varpi)} & \text{for } q = \infty. \end{cases} \]

For a convex modular \( \rho_M \) defined on \( w \), it has been proved in [5], that
\[ \sum_{n \geq 1} M_n(t_n(\varpi)) = \rho_M(\varpi), \quad \text{(1.2)} \]

for \( \varpi \in w \), if and only if \( M \) satisfies L1 condition. We see that \( (l_{p,q}, \| . \|_{p,q}) \) is Banach spaces for \( p \geq q \) by (1.2). But for \( p < q \), it is a quasi-Banach space. Further, they are symmetric sequence spaces [15].

Throughout the paper, we shall denote the Banach spaces over the complex field \( \mathbb{C} \) by \( X \) and \( Y \) and the class of all bounded linear maps from \( X \) to \( Y \) by \( L(X,Y) \).

Let \( \mathcal{L} \) be the class of all bounded linear operators between any pair of Banach spaces and \( w^+ \) be the class of sequences of non-negative real numbers. A mapping \( s : \mathcal{L} \to w^+ \) is called an s-number function, if it satisfies the following conditions:

(i) \( \|S\| = s_1(S) \geq s_2(S) \geq ... \geq 0 \), \( s(S) = \{s_n(S)\} \), \( S \in \mathcal{L} \);

(ii) \( s_n(S + T) \leq s_n(S) + \|T\| \) for \( S, T \in L(X,Y) \) and \( n \in \mathbb{N} \);

(iii) \( s_n(RST) \leq \|R\| s_n(S) \|T\| \) for \( T \in L(X_0, X) \), \( S \in L(X,Y) \), \( R \in L(Y, Y_0) \) and \( n \in \mathbb{N} \);

(iv) if rank \( S < n \), then \( s_n(S) = 0 \), (v), if dim \( X \geq n \), then \( s_n(I_X) = 1 \), where \( I_X \) denotes the identity map of \( X \).
If the condition (ii) is replaced by

\[(ii)\quad s_{m+n-1}(S+T) \leq s_m(S) + s_n(T) \quad \text{for} \quad S, T \in \text{L}(X, Y) \quad \text{and} \quad m, n = 1, 2, \ldots,
\]
then the \(s\)-number function is called additive.

An \(s\)-number function is called multiplicative, if the condition (iii) is replaced by

\[(iii)\quad s_{m+n-1}(RT) \leq s_m(R)s_n(T) \quad \text{for} \quad R \in \text{L}(Y_0, Y) \quad \text{and} \quad T \in \text{L}(X, Y_0), \quad m, n = 1, 2, \ldots.
\]

We write \(A(X, Y) = A \cap \text{L}(X, Y)\) for a subset \(A\) of \(L\). An operator ideal is a collection of \(A\), if it satisfies the following:

(i) \(A\) contains all finite rank operators;
(ii) \(T + S \in A(X, Y)\) for \(S, T \in A(X, Y)\);
(iii) if \(T \in A(X, Y)\) and \(S \in \text{L}(Y, Z)\), then \(ST \in A(X, Z)\) and also if \(T \in \text{L}(X, Y)\) and \(S \in A(Y, Z)\), then \(ST \in A(X, Z)\).

For the Banach spaces \(X\) and \(Y\) the collection \(A(X, Y)\) is called a component of \(A\).

A real-valued function \(f\) is said to be an ideal quasi-norm, if \(f\) is defined on an operator ideal \(A\) and satisfies the following properties:

(i) \(0 \leq f(T) < \infty\) for each \(T \in A\) and \(f(T) = 0\), if and only if \(T = 0\);
(ii) there exists a constant \(\sigma \geq 1\) such that \(f(S + T) \leq \sigma[f(S) + f(T)]\) for \(S, T \in A(X, Y)\), where \(A(X, Y)\) is any component of \(A\);
(iii) \(a)\quad f(RS) \leq \|R\|f(S)\) for \(S \in A(X, Z)\), \(R \in \text{L}(Z, Y)\), and
(b) \(f(RS) \leq \|S\|f(R)\) for \(S \in \text{L}(X, Z)\), \(R \in A(Z, Y)\).

An operator ideal is said to be quasi-normed operator ideal, if it is equipped with an ideal quasi-norm and a quasi-Banach operator ideal is a quasi-normed operator ideal of which each component is complete with respect to the ideal quasi-norm.

The notion of difference sequence spaces was introduced by Kizmaz [10] who studied the difference sequence spaces \(l_\infty(\Delta)\), \(c(\Delta)\) and \(c_0(\Delta)\). The notion was further generalized by Et and Çölak [4] by introducing the spaces \(l_\infty(\Delta^m)\), \(c(\Delta^m)\) and \(c_0(\Delta^m)\). Let \(m\) be a non-negative integer, then for \(Z = c, c_0\) and \(l_\infty\), we have sequence spaces

\[Z(\Delta^m) = \{x = (x_k) \in w : (\Delta^m x_k) \in Z\},\]

where \(\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})\) and \(\Delta^0 x_k = x_k\) for all \(k \in \mathbb{N}\), which is equivalent to the following binomial representation

\[\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}.
\]

Taking \(m = 1\), we get the spaces studied by Et and Çölak [4]. For more details about this work one can refer to [1, 2, 9, 12, 16–18].

2. The vector-valued sequence spaces \(l_{p,q,M,u,\Delta,A}(X)\) and \(h_{p,q,M,u,\Delta,A}(X)\)

Let \(X\) be a Banach space. Let \(M = (M_k)\) be an Musielak-Orlicz function, that is, \(M\) is a sequence of Orlicz functions, \(u = (u_k)\) be a sequence of strictly positive real numbers and \(A = (a_{nk})\) be a nonnegative
two-dimensional bounded-regular matrix. In this paper we define the following classes of sequences:

\[ l_{p,q,M,u\Delta,A}(X) = \left\{ x = (x_k) \in l_\infty(X) : \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\| A k^{1 - \frac{1}{q}} t_k(\Delta^m x_k) \|}{\rho} \right) \right] < \infty, \text{ for some } \rho > 0 \right\}, \]

\[ h_{p,q,M,u\Delta,A}(X) = \left\{ x = (x_k) \in l_\infty(X) : \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\| A k^{1 - \frac{1}{q}} t_k(\Delta^m x_k) \|}{\delta} \right) \right] < \infty, \text{ for all } \delta > 0 \right\}. \]

For \( x \in l_{p,q,M,u\Delta,A}(X) \), we define

\[ \| x \|_{p,q,M,u\Delta,A}(X) = \inf \left\{ \rho > 0 : \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\| A k^{1 - \frac{1}{q}} t_k(\Delta^m x_k) \|}{\rho} \right) \right] \leq 1 \}. \]

If we take \( M(x) = x \) in \( l_{p,q,M,u\Delta,A}(X) \) and \( h_{p,q,M,u\Delta,A}(X) \), then we have the following spaces:

\[ l_{p,q,u\Delta,A}(X) = \left\{ x = (x_k) \in l_\infty(X) : \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\| A k^{1 - \frac{1}{q}} t_k(\Delta^m x_k) \|}{\rho} \right) \right] < \infty, \text{ for some } \rho > 0 \right\}, \]

\[ h_{p,q,u\Delta,A}(X) = \left\{ x = (x_k) \in l_\infty(X) : \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\| A k^{1 - \frac{1}{q}} t_k(\Delta^m x_k) \|}{\delta} \right) \right] < \infty, \text{ for all } \delta > 0 \right\}. \]

Let \( u = (u_k) = 1 \), for all \( k \in \mathbb{N} \). Then the spaces \( l_{p,q,M,u\Delta,A}(X) \) and \( h_{p,q,M,u\Delta,A}(X) \) are reduced to \( l_{p,q,u\Delta,A}(X) \) and \( h_{p,q,u\Delta,A}(X) \), respectively, as follow:

\[ l_{p,q,u\Delta,A}(X) = \left\{ x = (x_k) \in l_\infty(X) : \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\| A k^{1 - \frac{1}{q}} t_k(\Delta^m x_k) \|}{\rho} \right) \right] < \infty, \text{ for some } \rho > 0 \right\}, \]

\[ h_{p,q,u\Delta,A}(X) = \left\{ x = (x_k) \in l_\infty(X) : \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\| A k^{1 - \frac{1}{q}} t_k(\Delta^m x_k) \|}{\delta} \right) \right] < \infty, \text{ for all } \delta > 0 \right\}. \]

If we take \( A = (C,1) \) in \( l_{p,q,u\Delta,A}(X) \) and \( h_{p,q,u\Delta,A}(X) \), then we have the following spaces:

\[ l_{p,q,M,u\Delta}(X) = \left\{ x = (x_k) \in l_\infty(X) : \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\| A k^{1 - \frac{1}{q}} t_k(\Delta^m x_k) \|}{\rho} \right) \right] < \infty, \text{ for some } \rho > 0 \right\}, \]

\[ h_{p,q,M,u\Delta}(X) = \left\{ x = (x_k) \in l_\infty(X) : \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\| A k^{1 - \frac{1}{q}} t_k(\Delta^m x_k) \|}{\delta} \right) \right] < \infty, \text{ for all } \delta > 0 \right\}. \]

If we take \( A = (C,1) \) and \( M(x) = x \) in \( l_{p,q,M,u\Delta,A}(X) \) and \( h_{p,q,M,u\Delta,A}(X) \), then we have the following spaces:

\[ l_{p,q,u\Delta}(X) = \left\{ x = (x_k) \in l_\infty(X) : \sum_{k \geq 1} u_k \left[ \left\| A k^{1 - \frac{1}{q}} t_k(\Delta^m x_k) \right\| \right] < \infty, \text{ for some } \rho > 0 \right\}, \]

\[ h_{p,q,u\Delta}(X) = \left\{ x = (x_k) \in l_\infty(X) : \sum_{k \geq 1} u_k \left[ \left\| A k^{1 - \frac{1}{q}} t_k(\Delta^m x_k) \right\| \right] < \infty, \text{ for all } \delta > 0 \right\}. \]

If we take \( (M_k) = M, A = I, (u_k) = 1 \) for all \( k \in \mathbb{N} \) and \( m = 0 \), then we get the analogous of the spaces defined by Gupta and Bhar [7]. The aim of this paper is to study the vector-valued Orlicz-Lorentz sequence spaces. We also study their structural properties and investigate cross and topological duals of these spaces. Finally we prove that the operator ideals defined with the help of scalar-valued sequence spaces \( l_{p,q,M,u\Delta,A} \) and additive s-numbers are quasi-Banach operator ideals for \( p < q \) and Banach operator ideals for \( p \geq q \).
Theorem 2.1. Let $\mathcal{M} = (\mathcal{M}_k)$ be a Musielak-Orlicz function, $u = \{u_k\}$ be a sequence of strictly positive real numbers and $A = (a_{nk})$ be a nonnegative two-dimensional bounded-regular matrix. Then the space $l_{p,q,M,u,A,A}(X)$ equipped with $\|\cdot\|_{p,q,M,u,A,A}$ is a quasi-Banach space for $p < q$ and Banach space for $p \geq q$. Further for $x \in l_{p,q,M,u,A,A}(X)$, we have

$$\sum_{k \geq 1} u_k \left[ M_k \left( \frac{\|A k^{\frac{1}{p} - \frac{1}{q}} t_k(D^m x_k)\|}{\|x\|_{p,q,M,u,A,A}} \right) \right] \leq 1. \quad (2.1)$$

Proof. We can easily show that $l_{p,q,M,u,A,A}(X)$ is a vector space with usual coordinate wise addition and scalar multiplication. To show that $\|\cdot\|_{p,q,M,u,A,A}$ is a quasi-norm, let $\|x\|_{p,q,M,u,A,A} \geq 0$, for each $x \in l_{p,q,M,u,A,A}(X)$ and $\|x\|_{p,q,M,u,A,A} = 0$, for $x = 0$. Suppose that $\|x\|_{p,q,M,u,A,A} = 0$, for some $x = \{x_k\} \in l_{p,q,M,u,A,A}(X)$ and for given $\varepsilon > 0$, we can find $\rho > 0$ such that $\rho < \varepsilon$ and

$$\sum_{k \geq 1} u_k \left[ M_k \left( \frac{\|A k^{\frac{1}{p} - \frac{1}{q}} t_k(D^m x_k)\|}{\rho} \right) \right] \leq 1.$$

When $x \neq 0$, we get $\|x_k\| \neq 0$ for some $k \in \mathbb{N}$ and so $t_k(D^m x_k) = \|A k^{\frac{1}{p} - \frac{1}{q}} t_k(D^m x_k)\|$, for some $k \in \mathbb{N}$ implies

$$u_k \left[ M_k \left( \frac{\|A k^{\frac{1}{p} - \frac{1}{q}} t_k(D^m x_k)\|}{\rho} \right) \right] \leq u_k \left[ M_k \left( \frac{\|A k^{\frac{1}{p} - \frac{1}{q}} t_k(D^m x_k)\|}{\rho} \right) \right] \leq 1,$$

for any $\varepsilon > 0$. We get a contradiction to the fact, so $x = 0$.

To prove triangular-type inequality, let us consider $x = \{x_k\}$ and $y = \{y_k\} \in l_{p,q,M,u,A,A}(X)$. Thus for any $\varepsilon > 0$, there exist $\rho_1, \rho_2 > 0$ such that

$$\rho_1 < \|x\|_{p,q,M,u,A,A} + \frac{\varepsilon}{2} \quad \text{with} \quad \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\|A k^{\frac{1}{p} - \frac{1}{q}} t_k(D^m x_k)\|}{\rho_1} \right) \right] \leq 1,$$

and

$$\rho_2 < \|y\|_{p,q,M,u,A,A} + \frac{\varepsilon}{2} \quad \text{with} \quad \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\|A k^{\frac{1}{p} - \frac{1}{q}} t_k(D^m y_k)\|}{\rho_2} \right) \right] \leq 1.$$

If $\frac{1}{p} - \frac{1}{q} > 0$, then via properties (i) and (ii) of $\{t_k(D^m x_k)\}$, we get

$$\sum_{k \geq 1} u_k \left[ M_k \left( \frac{\|A k^{\frac{1}{p} - \frac{1}{q}} t_k(D^m x_k + y_k)\|}{2^{\frac{1}{p} - \frac{1}{q} + 1} \rho_1 + \rho_2} \right) \right] = \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\|A (2k)^{\frac{1}{p} - \frac{1}{q}} t_{2k}(D^m(x_k + y_k))\|}{2^{\frac{1}{p} - \frac{1}{q} + 1} \rho_1 + \rho_2} \right) \right] + \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\|A (2k - 1)^{\frac{1}{p} - \frac{1}{q}} t_{2k-1}(D^m(x_k + y_k))\|}{2^{\frac{1}{p} - \frac{1}{q} + 1} \rho_1 + \rho_2} \right) \right] \leq 2 \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\|A k^{\frac{1}{p} - \frac{1}{q}} t_k(D^m x_k) + t_k(D^m y_k))\|}{2 \rho_1 + \rho_2} \right) \right] \leq \frac{\rho_1}{\rho_1 + \rho_2} \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \left( \frac{\|A k^{\frac{1}{p} - \frac{1}{q}} t_k(D^m x_k)\|}{\rho_1} \right) + \frac{\rho_2}{\rho_1 + \rho_2} \left( \frac{\|A k^{\frac{1}{p} - \frac{1}{q}} t_k(D^m y_k)\|}{\rho_2} \right) \leq 1.$$

Hence,

$$\|x + y\|_{p,q,M,u,A,A} \leq 2^{\frac{1}{p} - \frac{1}{q} + 1} \rho_1 + \rho_2.$$
Theorem 2.2. Let \( h_{p,q,M,u,A,A} \) be a Musielak-Orlicz function, \( u = (u_k) \) be a sequence of strictly positive real numbers and \( A = (a_{nk}) \) be a nonnegative two-dimensional bounded-regular matrix. Then \( h_{p,q,M,u,A,A}(X) \) is a closed subspace of \( l_{p,q,M,u,A,A}(X) \) and \( \|x\|_{h_{p,q,M,u,A,A}} \to 0 \) as \( k \to \infty \). Also, inequality (2.1) is directly obtained from the definition of the quasi-norm \( \|\cdot\|_{p,q,M,u,A,A} \). This completes the proof. \( \square \)

**Theorem 2.2.** Let \( M = (M_k) \) be a Musielak-Orlicz function, \( u = (u_k) \) be a sequence of strictly positive real numbers and \( A = (a_{nk}) \) be a nonnegative two-dimensional bounded-regular matrix. Then \( h_{p,q,M,u,A,A}(X) \) is a closed subspace of \( l_{p,q,M,u,A,A}(X) \). Moreover, if \( M = (M_k) \) satisfies \( \Delta_2 \)-condition at 0, then \( l_{p,q,M,u,A,A}(X) = h_{p,q,M,u,A,A}(X) \).

**Proof.** First of all it is without a doubt that \( h_{p,q,M,u,A,A}(X) \) is a subspace of \( l_{p,q,M,u,A,A}(X) \). Now we prove that \( h_{p,q,M,u,A,A}(X) \) is closed in \( l_{p,q,M,u,A,A}(X) \). Suppose \( \bar{x} = \{x_k\} \in \bar{h}_{p,q,M,u,A,A}(X) \), the closure of \( h_{p,q,M,u,A,A}(X) \) in \( l_{p,q,M,u,A,A}(X) \). So there exists a sequence \( \{\gamma_k\} \in h_{p,q,M,u,A,A}(X) \), \( k \geq 1 \) and we have \( \|\gamma_k - \bar{x}\|_{p,q,M,u,A,A} \to 0 \) as \( k \to \infty \). Take any \( \delta > 0 \). Thus for \( \delta_1 = \min(2^\frac{1}{p} - \frac{1}{q}, \delta) \), we get \( k_0 \in \mathbb{N} \) such that

\[
\|\gamma_k - \bar{x}\|_{p,q,M,u,A,A} < \frac{\delta_1}{2}, \quad \text{for all } k \geq k_0. \tag{2.2}
\]

When \( \frac{1}{p} - \frac{1}{q} \geq 0 \), we have

\[
\sum_{n \geq 1} u_n \left[ M_n \left( \frac{\|A_{n\frac{1}{p} - \frac{1}{q}} t_n(D^n(x))\|}{\delta} \right) \right] 
\leq 2 \sum_{n \geq 1} u_n \left[ M_n \left( \frac{\|A_{n\frac{1}{p} - \frac{1}{q}} (t_n(D^n(x)) + t_n(D^n(\bar{y}_{k_0}))\|}{\delta_1} \right) \right]
\leq \sum_{n \geq 1} u_n \left[ M_n \left( \frac{\|A_{n\frac{1}{p} - \frac{1}{q}} t_n(D^n(x))\|}{\delta_1/2} \right) \right]
+ \sum_{n \geq 1} u_n \left[ M_n \left( \frac{\|A_{n\frac{1}{p} - \frac{1}{q}} t_n(D^n(\bar{y}_{k_0}))\|}{\delta_1/2} \right) \right]
\leq \sum_{n \geq 1} u_n \left[ M_n \left( \frac{\|A_{n\frac{1}{p} - \frac{1}{q}} t_n(D^n(x))\|}{\|x - \bar{y}_{k_0}\|_{p,q,M,u,A,A}} \right) \right]
\]
Now presume that $M$ satisfies $\Delta_2$-condition at 0. Let $\bar{x} \in l_{p,q,M,u,\Delta,A}(X)$, we have
\[
\sum_{k \geq k_0} u_k \left[ M_k \left( \frac{\|A k^{\frac{1}{p} - \frac{1}{q}} t_k (\Delta^m x_k)\|}{\rho_0} \right) \right] < \infty,
\]
for some $k_0 \in \mathbb{N}$. Hence
\[
\sum_{k \geq k_0} u_k \left[ M_k \left( \frac{\|A k^{\frac{1}{p} - \frac{1}{q}} t_k (\Delta^m x_k)\|}{\rho_0} \right) \right] < \infty,
\]
and so $\bar{x} \in h_{p,q,M,u,\Delta,A}(X)$, we have $h_{p,q,M,u,\Delta,A}(X) = l_{p,q,M,u,\Delta,A}(X)$. \hfill \Box

**Proposition 2.3.** Let $M = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $A = (a_{nk})$ be a nonnegative two-dimensional bounded-regular matrix. If $Y = h_{p,q,M,u,\Delta,A}(X) \cap c_0(X), 0 < p, q < \infty$. Then $Y$ equipped with the subspace topology of $h_{p,q,M,u,\Delta,A}(X)$ is a GAD-space.

**Proof.** Obviously $\phi(X) \subset Y$. Suppose $\bar{x} \in Y$. Now for any $\varepsilon > 0$, we can find $k_0 \in \mathbb{N}$, such that
\[
\sum_{k \geq k_0} u_k \left[ M_k \left( \frac{\|A k^{1/p - 1/q} t_k (\Delta^m x_k)\|}{\varepsilon} \right) \right] \leq 1.
\]
Let $I_k = \{ i \in \mathbb{N} : \|x_i\| > \frac{1}{k} \}$, $k \in \mathbb{N}$ and $\bar{r}_k = \sum_{i \in I_k} \delta x_i$. Since $\bar{x} \in c_0(X)$, $I_k$ is finite and so $\bar{r}_k \in \phi(X)$. Set $n_k = \text{card} I_k$. Then take $m_0 \in \mathbb{N}$ such that
\[
\sum_{k \geq m_0} u_k \left[ M_k \left( \frac{\|A k^{1/p - 1/q} t_k (\Delta^m x_k)\|}{\varepsilon} \right) \right] \leq \frac{1}{2}.
\]
Take k so large that
\[ \frac{1}{k} \sum_{i=1}^{m_0} u_k \left[ M_k \left( \frac{\| A^{i/p - 1/q} \|_\varepsilon}{\varepsilon} \right) \right] \leq \frac{1}{2}. \]

Thus,
\begin{align*}
\sum_{i \geq 1} u_k \left[ M_k \left( \frac{\| A^{i/p - 1/q} t_{n_i + i} (\Delta^m x_k) \|_\varepsilon}{\varepsilon} \right) \right] &\leq \frac{1}{k} \sum_{i=1}^{m_0} u_k \left[ M_k \left( \frac{\| A^{i/p - 1/q} \|_\varepsilon}{\varepsilon} \right) \right] \\
&\quad + \sum_{i \geq m_0 + 1} u_k \left[ M_k \left( \frac{\| A^{i/p - 1/q} t_i (\Delta^m x_k) \|_\varepsilon}{\varepsilon} \right) \right] \\
&\leq \frac{1}{2} + \frac{1}{2} = 1.
\end{align*}

This implies that \( \| x - \pi_k \|_{p,q,M,u,\Delta,A} \leq \varepsilon, \) for sufficiently large k. Hence Y is a GAD-space.

**Proposition 2.4.** Let \( M = (M_k) \) be a Musielak-Orlicz function satisfying \( \Delta_2 \)-condition at 0, \( u = (u_k) \) be a sequence of strictly positive real numbers and \( A = (a_{nk}) \) be a nonnegative two-dimensional bounded-regular matrix. If \( l_{p,q,M,u,\Delta,A}(X) \subset c_0(X), \) \( 0 < p \leq q \leq \infty. \) Then \( l_{p,q,M,u,\Delta,A} \) is a GAD-space.

**Remark 2.5.** It is very motivating to know whether the space \( h_{p,q,M,u,\Delta,A}(X) \) is a GAK-space, this means that the kth section \( \bar{x}^k = (x_1, x_2, \ldots, x_k, 0, 0, 0, \ldots) \) of an element \( x = (x_1) \) of \( h_{p,q,M,u,\Delta,A} \) converges to \( \bar{x} \) with respect to its quasi-norm. Whenever, if \( p, q > 0 \) with \( \frac{1}{p} - \frac{1}{q} \geq 0 \) and \( x \in h_{p,q,M,u,\Delta,A}(X) \) such that \( \|x_1\| > \|x_2\| > \|x_3\| > \cdots, \) then \( t_k(\bar{x}) = \|x_k\| \) and in this case, one can easily show that \( \|\bar{x} - \bar{x}^k\|_{p,q,M,u,\Delta,A} \rightarrow 0 \) as \( k \rightarrow \infty. \)

### 3. Duals of the space \( l_{p,q,M,u,\Delta,A}(X), 1 \leq p \leq q \leq \infty \)

Suppose that the spaces \( l_{p,q,M,u,\Delta,A}(X) \) are symmetric sequence spaces, since the decreasing rearrangement of \( x \) would be the same as that of \( x_\pi \) for any permutation \( \pi \) of \( N \) and \( M = (M_k) \) is an increasing function. Thus the \( \delta \)-dual of the scalar-valued sequence space \( l_{p,q,M,u,\Delta,A} \) would coincide with its cross-dual.

**Theorem 3.1.** Let \( M = (M_k) \) and \( N = (N_k) \) be two mutually complementary Musielak-Orlicz functions such that \( M \) satisfies \( \Delta_2 \)-condition at 0, \( u = (u_k) \) be a sequence of strictly positive real numbers and \( A = (a_{nk}) \) be a nonnegative two-dimensional bounded-regular matrix. Then \( (l_{p_1,q_1,M,u,\Delta,A})^\times \supset l_{p_2,q_2,N,u,\Delta,A}, \) where \( 1/p_1 + 1/p_2 = 1 \) and \( 1/q_1 + 1/q_2 = 1. \) Moreover, \( (l_{p_1,q_1,M,u,\Delta,A})^\times \supset l_{p_2,q_2,N,u,\Delta,A} \) when \( 1/p_1 - 1/q_1 \geq 0. \)

**Proof.** To show that \( l_{p_2,q_2,N,u,\Delta,A} \subset (l_{p_1,q_1,M,u,\Delta,A})^\times, \) suppose \( \overline{\beta} \in l_{p_2,q_2,N,u,\Delta,A}. \) Then, we have
\[
\sum_{k \geq 1} u_k \left[ N_k \left( \frac{\| A^{1/p} \overline{\beta} \|_{\delta_0}}{\delta_0} \right) \right] < \infty, \text{ for some } \delta_0 > 0.
\]

Let \( \overline{\alpha} \in l_{p_1,q_1,M,u,\Delta,A}. \) Then
\[
\sum_{k \geq 1} u_k \left[ M_k \left( \frac{\| A^{1/p_1} \overline{\alpha} \|_{\rho}}{\rho} \right) \right] < \infty, \text{ for all } \rho > 0. \]

Thus,
\[
\sum_{k \geq 1} |\alpha_k \beta_k| \leq \sum_{k \geq 1} t_k(\Delta^m \overline{\alpha}) t_k(\Delta^m \overline{\beta})
\]
\[
\leq \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\| A^{1/p_1} \overline{\alpha} \|_{\delta_0}}{\delta_0} \right) \right] + \sum_{k \geq 1} u_k \left[ N_k \left( \frac{\| A^{1/p_2} \overline{\beta} \|_{\delta_0}}{\delta_0} \right) \right] < \infty.
\]
Hence $\beta \in (l_{p_{1}, q_{1}, M, u, \Delta, A})^\times$. Now to prove $(l_{p_{1}, q_{1}, M, u, \Delta, A})^\times = l_{p_{2}, q_{2}, N, u, \Delta, A}$, suppose $\beta \in (l_{p_{1}, q_{1}, M, u, \Delta, A})^\times$, then $\sum_{i \geq 1} |a_i| \beta_i < \infty$, for all $a_i \in l_{p_{1}, q_{1}, M, u, \Delta, A}$. Since $l_{p_{1}, q_{1}, M, u, \Delta, A}$ and $(l_{p_{1}, q_{1}, M, u, \Delta, A})^\times$ both are symmetric sequence spaces, $\{t_k(\Delta^m \alpha)\} \subseteq l_{p_{1}, q_{1}, M, u, \Delta, A}$, for $\alpha \subseteq l_{p_{1}, q_{1}, M, u, \Delta, A}$ and $\{t_k(\Delta^m \beta)\} \subseteq (l_{p_{1}, q_{1}, M, u, \Delta, A})^\times$, for $\beta \in (l_{p_{1}, q_{1}, M, u, \Delta, A})^\times$. Hence $\sum_{k \geq 1} t_k(\Delta^m \alpha) t_k(\Delta^m \beta) < \infty$, for all $\alpha \subseteq l_{p_{1}, q_{1}, M, u, \Delta, A}$.

Again if $\gamma \in l_{M}$, then $\{t_k(\Delta^m \gamma)\} \subseteq l_{M}$ as $l_{M}$ is symmetric and normal and so

$$\{A_k \frac{1}{p} - \frac{1}{q} t_k(\Delta^m \gamma)\} \subseteq l_{p_{1}, q_{1}, M, u, \Delta, A}.$$

Hence

$$\sum_{k \geq 1} \left\| A_k \frac{1}{p} - \frac{1}{q} t_k(\Delta^m \gamma) t_k(\Delta^m \beta) \right\| < \infty, \quad \text{for all } \gamma \in l_{M}.$$

This implies that $\{A_k \frac{1}{p} - \frac{1}{q} t_k(\Delta^m \beta)\} \subseteq l_{M}^\times = l_{N}$ and so $\beta \subseteq l_{p_{2}, q_{2}, N, u, \Delta, A}$. Thus, we have

$$(l_{p_{1}, q_{1}, M, u, \Delta, A})^\times = l_{p_{2}, q_{2}, N, u, \Delta, A}.$$

\[\square\]

**Proposition 3.3.** Let $M = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $A = (a_{nk})$ be a nonnegative two-dimensional bounded-regular matrix. For positive reals $p_1, p_2, q_1, q_2$ with $1/p_1 + 1/p_2 = 1, 1/q_1 + 1/q_2 = 1$ such that $q_1 < p_1$, the spaces $l_{p_{1}, q_{1}, M, u, \Delta, A}$ are perfect sequences spaces.

**Proof.** In fact in this case $(l_{p_{1}, q_{1}, M, u, \Delta, A})^\times = l_{p_{1}, q_{1}, M, u, \Delta, A}$ and $l_{p_{2}, q_{2}, N, u, \Delta, A} \subseteq (l_{p_{1}, q_{1}, M, u, \Delta, A})^\times$. So we have $l_{p_{1}, q_{1}, M, u, \Delta, A} \subseteq (l_{p_{1}, q_{1}, M, u, \Delta, A})^\times \subseteq (l_{p_{2}, q_{2}, N, u, \Delta, A})^\times = l_{p_{2}, q_{2}, N, u, \Delta, A}$.

\[\square\]

**Theorem 3.4.** Let $M = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $A = (a_{nk})$ be a nonnegative two-dimensional bounded-regular matrix. Suppose $p_1, p_2, q_1, q_2$ are real numbers with $1 < p_1, q_1, p_2, q_2 < \infty$ and $1/p_1 + 1/p_2 = 1, 1/q_1 + 1/q_2 = 1$. Then the dual of $l_{p_{1}, q_{1}, M, u, \Delta, A}(X)$ is topologically isomorphic to $l_{p_{2}, q_{2}, N, u, \Delta, A}(X^*)$, if and only if the sequence $(f_i) \subseteq l_{p_{2}, q_{2}, N, u, \Delta, A}(X^*)$ is identified with the linear functional $F$ given by

$$F([x_1]) = \sum_{i \geq 1} \langle x_i, f_i \rangle, \quad \text{for each } x_i \in l_{p_{1}, q_{1}, M, u, \Delta, A}(X). \quad (3.1)$$

**Proof.** Subsequently for $(f_i) \subseteq l_{p_{2}, q_{2}, N, u, \Delta, A}(X^*)$, we define a linear functional $F$ on $l_{p_{1}, q_{1}, M, u, \Delta, A}(X)$ as in $(\ast)$ where convergence of the series is being guaranteed by Proposition 3.3. For $k \in \mathbb{N}$, let

$$F_k([x_i]) = \sum_{i = 1}^{k} \langle x_i, f_i \rangle, \quad x_i \in l_{p_{1}, q_{1}, M, u, \Delta, A}(X).$$

Obviously, $\{F_k\}$ is a sequence of continuous linear functionals on $l_{p_{1}, q_{1}, M, u, \Delta, A}(X)$ converging pointwise to $F$. Thus $F$ is continuous by Banach-Steinhaus Theorem (see [19]). Hence, $F \in (l_{p_{1}, q_{1}, M, u, \Delta, A}(X))^*$. Next, for $x \in l_{p_{1}, q_{1}, M, u, \Delta, A}(X)$, we get
\[ |f(x)| \leq \sum_{i \geq 1} |\langle x_i, f_i \rangle| \]
\[ \leq \sum_{i \geq 1} t_i(\overline{x}) t_i(\overline{f}) \]
\[ \leq \left\| \overline{f} \right\|_{p_2,q_2,N,u,A,A} \sum_{i \geq 1} u_i \left[ M_i \left( |A(1/p_1-1/q_1)t_i(\Delta^m x)| \left\| \left(1/p_2-1/q_2t_i(\Delta^m f) \right) \right\|_{(M)} \right) \right] \]
\[ \leq \left\| \overline{f} \right\|_{p_2,q_2,N,u,A,A} \left\| u_i A(1/p_1-1/q_1)t_i(\Delta^m x) \right\|_{(M)}, \]

since \( \sum_{i \geq 1} u_i \left[ N_i \left( \left\| A \right\|_{1/p_1-1/q_1/t_i(\Delta^m x)} \right) \right] \leq 1. \) Therefore,
\[ |F(\overline{x})| \leq 2\left\| \overline{f} \right\|_{p_2,q_2,N,u,A,A} \left\| \overline{x} \right\|_{p_1,q_1,M,u,A,A}, \]

for any \( \overline{x} \in l_{p_1,q_1,M,u,A,A}(X). \) Thus,
\[ \|f\| \leq 2\|\overline{f}\|_{p_2,q_2,N,u,A,A}. \] \hfill (3.2)

Conversely, suppose \( F \in l_{p_1,q_1,M,u,A,A}(X)^*. \) Define \( f_i \in X^*, i \in \mathbb{N} \) as \( f_i(x) = F(\delta_i^x). \) Now to prove \( \{f_i\} \in l_{p_2,q_2,N,u,A,A}(X^*) \) we choose \( \{\alpha_i\} \in l_{p_1,q_1,M,u,A,A}. \) Take \( \{x_i\} \subseteq X \) with \( \|x_i\| = 1 \) and \( \|f_i\| < f_i(x_i) + 1/2^i \), for all \( i \in \mathbb{N}. \) Let \( \{\beta_i\} \subset C \) be such that \( f_i(\alpha_i x_i) = f_i(\alpha_i \beta_i x_i), \) for all \( i \in \mathbb{N}. \) Obviously, \( |\beta_i| = 1 \), for all \( i \in \mathbb{N}, \) and so \( \{\alpha_i \beta_i x_i\} \in l_{p_1,q_1,M,u,A,A}(X). \) Suppose
\[ \sum_{i \geq 1} |\alpha_i| \|f_i\| < \sum_{i \geq 1} f_i(\alpha_i \beta_i x_i) + \sum_{i \geq 1} \frac{\alpha_i}{2^i} \]
\[ = \sum_{i \geq 1} F(\delta_{\alpha_i \beta_i x_i}) + K \]
\[ = \lim_{k \to \infty} \sum_{i=1}^k F(\delta_{\alpha_i \beta_i x_i}) + K \]
\[ = F((\delta_{\alpha_i \beta_i x_i})) + K, \]

where \( K = \sum_{i \geq 1} \frac{\alpha_i}{2^i}. \) Therefore, \( \sum_{i \geq 1} |\alpha_i| \|f_i\| < \infty, \) for all \( \{\alpha_i\} \in l_{p_1,q_1,M,u,A,A} \) and hence \( \{f_i\} \in l_{p_2,q_2,N,u,A,A}(X^*) \) by Theorem 3.1. To prove that \( F \) has the form as given in equation (3.1), suppose for \( \{x_i\} \in l_{p_1,q_1,M,u,A,A}(X) \)
\[ \sum_{i \geq 1} |\langle x_i, f_i \rangle| = \sum_{i \geq 1} |F(\delta_{x_i})| \]
\[ = \lim_{k \to \infty} \sum_{i=1}^k F(\delta_{\beta_i x_i}) = F(\{\beta_i x_i\}), \]

where \( \beta_i \) are taken as above. Thus \( \sum_{i \geq 1} |\langle x_i, f_i \rangle| < \infty. \) Hence,
\[ \sum_{i \geq 1} |\langle x_i, f_i \rangle| \text{ is unconditionally convergent.} \] \hfill (3.3)

Now if \( 0 < p_1 < q_1 \leq \infty, \) \( l_{p_1,q_1,M,u,A,A}(X) \) is a GAD-space. We write \( t_i(\overline{x}) = \|x_i f_i(\overline{x})\|, \) for some \( f_i \in \pi \) and \( \overline{u}_k = \sum_{i=1}^k \delta_{\phi(i)} x_i, \) for \( k \in \mathbb{N}. \) Therefore \( \overline{u}_k \in \phi(X) \) and \( \|X - \overline{u}_k\|_{p_1,q_1,M,u,A,A} \to 0 \) as \( k \to \infty \) by Proposition 3.3. Then,
\[ F(\{x_i\}) = F(k \to \infty \overline{u}_k) = \sum_{i \geq 1} \langle x_i, f_i \rangle, \]
by (3.3). Hence the mapping \( R : l_{p,q,M,u,A}\{X^*\} \rightarrow (l_{p,q,M,u,A}\{X\})^* \) defined by \( R(\vec{t}) = F \), with \( \vec{t} = \{ f_i \}, f_i(x) = f(\delta^i x), i \in \mathbb{N} \) is a topological isomorphism from equations (3.1), (3.2) and the open mapping theorem (see [19]). This completes the proof. \( \square \)

4. The operator ideals \( L^{(s)}_{p,q,M,u,A}, 0 < p, q \leq \infty \)

Let \( M = (M_k) \) be a Musielak-Orlicz function and \( X, Y \) are Banach spaces.

**Definition 4.1.** Let \( T : X \rightarrow Y \) be a bounded linear operator. Then \( T \) is said to be of type \( l_{p,q,M,u,A} \), if \( \{s_k(T)\} \in l_{p,q,M,u,A} \). We shall denote the set of all above mappings by \( L^{(s)}_{p,q,M,u,A} \) where \( L^{(s)}_{p,q,M,u,A} = \{ T \in L : \{s_k(T)\} \in l_{p,q,M,u,A} \} \).

We define the norm for any \( T \in L^{(s)}_{p,q,M,u,A} \) as

\[
\|T\|_{p,q,M,u,A} = \inf \left\{ \rho > 0 : \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\|A k^\frac{1}{p} - \frac{1}{q} \Delta^m s_k(T)\|}{\rho} \right) \right] \leq 1 \right\}.
\]

**Theorem 4.2.** Let \( M = (M_k) \) be a Musielak-Orlicz function, \( u = (u_k) \) be a sequence of strictly positive real numbers and \( A = (a_{nk}) \) be a nonnegative two-dimensional bounded regular matrix. Then for \( p < q \), \( L^{(s)}_{p,q,M,u,A} \) equipped with \( \| \cdot \|_{p,q,M,u,A} \) is a quasi-Banach operator ideal and for \( p \geq q \) it is a Banach ideal.

**Proof.** To show that \( L^{(s)}_{p,q,M,u,A} \) is an operator ideal, firstly note that all finite rank operators are contained in \( L^{(s)}_{p,q,M,u,A} \), since \( s_k(T) = 0 \) for \( k \geq k_0 \), if rank \( T < k_0 \). For \( T_1, T_2 \in L^{(s)}_{p,q,M,u,A}(X, Y) \), we have

\[
\sum_{k \geq 1} u_k \left[ M_k \left( \frac{\|A k^\frac{1}{p} - \frac{1}{q} \Delta^m s_k(T_1)\|}{\rho_1} \right) \right] < \infty,
\]

and

\[
\sum_{k \geq 1} u_k \left[ M_k \left( \frac{\|A k^\frac{1}{p} - \frac{1}{q} \Delta^m s_k(T_2)\|}{\rho_2} \right) \right] < \infty,
\]

for some \( \rho_1, \rho_2 > 0 \). Firstly, we consider the condition when \( \frac{1}{p} - \frac{1}{q} \geq 0 \)

\[
\sum_{k \geq 1} u_k \left[ M_k \left( \frac{\|A k^\frac{1}{p} - \frac{1}{q} \Delta^m s_k(T_1 + T_2)\|}{\rho_1 + \rho_2} \right) \right] \leq \sum_{k \geq 1} \frac{\rho_1}{\rho_1 + \rho_2} u_k \left[ M_k \left( \frac{\|A k^\frac{1}{p} - \frac{1}{q} \Delta^m s_k(T_1)\|}{\rho_1} \right) \right]
\]

\[
+ \sum_{k \geq 1} \frac{\rho_2}{\rho_1 + \rho_2} u_k \left[ M_k \left( \frac{\|A k^\frac{1}{p} - \frac{1}{q} \Delta^m s_k(T_2)\|}{\rho_2} \right) \right] < \infty.
\]

Again, if \( \frac{1}{p} - \frac{1}{q} < 0 \), then

\[
\sum_{k \geq 1} u_k \left[ M_k \left( \frac{\|A k^\frac{1}{p} - \frac{1}{q} \Delta^m s_k(T_1 + T_2)\|}{\rho_1 + \rho_2} \right) \right] \leq \sum_{k \geq 1} \frac{\rho_1}{\rho_1 + \rho_2} u_k \left[ M_k \left( \frac{\|A k^\frac{1}{p} - \frac{1}{q} \Delta^m s_k(T_1)\|}{\rho_1} \right) \right]
\]

\[
+ \sum_{k \geq 1} \frac{\rho_2}{\rho_1 + \rho_2} u_k \left[ M_k \left( \frac{\|A k^\frac{1}{p} - \frac{1}{q} \Delta^m s_k(T_2)\|}{\rho_2} \right) \right] < \infty.
\]

This implies that \( T_1 + T_2 \in L^{(s)}_{p,q,M,u,A}(X, Y) \). Now, we want to show that for \( T \in L^{(s)}_{p,q,M,u,A}(E, F) \),
\[ R \in L(F, Y) \text{ and } S \in L(X, E), \text{ RTS} \in \mathcal{L}_{p, q, M, u, \Delta, A}^{(s)}(X, Y). \text{ Thus for } T \in \mathcal{L}_{p, q, M, u, \Delta, A}^{(s)}(E, F), \text{ we have} \]

\[
\sum_{k \geq 1} u_k \left[ M_k \left( \frac{\|A k^{\frac{1}{p'}} - \frac{1}{q} s_k(T)\|}{\rho_0} \right) \right] < \infty,
\]

for some \( \rho_0 > 0 \) and hence

\[
\sum_{k \geq 1} u_k \left[ M_k \left( \frac{\|A k^{\frac{1}{p'}} - \frac{1}{q} s_k(\text{RTS})\|}{\|S\|\rho_0} \right) \right] < \infty,
\]

by the property (iii) of \( s \)-number function. Thus RTS \( \in \mathcal{L}_{p, q, M, u, \Delta, A}^{(s)}(X, Y) \). Therefore, \( \mathcal{L}_{p, q, M, u, \Delta, A}^{(s)} \) is an operator ideal.

The proof of the function \( \| \cdot \|_{p, q, M, u, \Delta, A}^{(s)} \) is a quasi-norm (or, norm) defined on \( \mathcal{L}_{p, q, M, u, \Delta, A}^{(s)} \) is similar to one defined on \( \mathcal{L}_{p, q, M, u, \Delta, A}(X) \) and so excluded. To prove the completeness, suppose \( \{T_k\} \) is a Cauchy sequence in component of \( \mathcal{L}_{p, q, M, u, \Delta, A}^{(s)}(X, Y) \) of \( \mathcal{L}_{p, q, M, u, \Delta, A}^{(s)} \). Thus for \( \varepsilon > 0 \), there exists \( k_0 \in \mathbb{N} \) such that

\[
\|T_{k+j} - T_k\|_{p, q, M, u, \Delta, A} < \varepsilon, \quad \text{for all } k \geq k_0 \text{ and } j \in \mathbb{N}.
\]

This implies that there exists \( \rho > 0 \) such that \( \rho < \varepsilon \) and

\[
\sum_{n \geq 1} u_n \left[ M_n \left( \frac{\|A n^{\frac{1}{p'}} - \frac{1}{q} s_n(T_{k+j} - T_k)\|}{\varepsilon} \right) \right] \leq 1, \quad \text{for all } k \geq k_0, j \in \mathbb{N}. \tag{4.1}
\]

Thus, \( \left\{ A n^{\frac{1}{p'}} - \frac{1}{q} s_n(T_{k+j} - T_k) \right\}_{n \geq 1} \) is a bounded sequence for each \( k \geq k_0 \) and \( j \in \mathbb{N} \). Therefore for some constant \( K > 0 \), we get

\[
\|T_{k+j} - T_k\| < \varepsilon K, \quad \text{for all } k \geq k_0, j \in \mathbb{N}.
\]

Thus, \( \{T_k\} \) is a Cauchy sequence in \( L(X, Y) \). So there exists a \( T \in L(X, Y) \) such that \( \|T_k - T\| \to 0 \) as \( k \to \infty \). As \( s_n(T_k - T) \leq \|T_k - T\| \), for all \( k \geq 1 \), we have \( s_n(T_k - T) \to 0 \) as \( k \to \infty \). Also,

\[
|s_n(T_{k+j} - T_k) - s_n(T - T_k)| \leq \|T_{k+j} - T_k\|,
\]

which implies

\[
\lim_{n \to \infty} s_n(T_{k+j} - T_k) = s_n(T - T_k) \quad \text{as } j \to \infty.
\]

Now, we have from (4.1),

\[
\sum_{n \geq 1} u_n \left[ M_n \left( \frac{\|A n^{\frac{1}{p'}} - \frac{1}{q} s_n(T - T_k)\|}{\varepsilon} \right) \right] \leq 1, \quad \text{for all } k \geq k_0.
\]

This implies that \( T - T_k \in \mathcal{L}_{p, q, M, u, \Delta, A}^{(s)}(X, Y) \) and \( \|T - T_k\|_{p, q, M, u, \Delta, A} < \varepsilon, \quad \text{for all } k \geq k_0. \) Therefore, \( T \in \mathcal{L}_{p, q, M, u, \Delta, A}^{(s)} \) and \( T_k \to T \in \mathcal{L}_{p, q, M, u, \Delta, A}^{(s)} \), which shows that \( \mathcal{L}_{p, q, M, u, \Delta, A}^{(s)} \) is a quasi-Banach operator ideal.

**Theorem 4.3.** Let \( M = (M_k) \) and \( N = (N_k) \) be two complementary Musielak-Orlicz functions, \( u = (u_k) \) be a sequence of strictly positive real numbers, \( A = (a_{nk}) \) be a nonnegative two-dimensional bounded-regular matrix and \( s \) is a multiplicative \( s \)-number function. If \( 0 < p_1, p_2, p, q_1, q_2, q < \infty \) are such that \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} \), then

\[
\mathcal{L}_{p_1, q_1, M, u, \Delta, A}^{(s)} \circ \mathcal{L}_{p_2, q_2, M, u, \Delta, A}^{(s)} \subseteq \mathcal{L}_{p, q, 1, M, u, \Delta, A}^{(s)}.
\]
where
\[ L_{p,q,l,u,\Delta,A}^{(s)} = \left\{ T \in L : \sum_{k \geq 1} u_k \left[ \| A k^{\frac{1}{p}} k^{\frac{1}{q}} \frac{1}{p} \Delta^m s_k T \| \right] \in l_1 \right\}. \]

**Proof.** Suppose \( T \in L_{p_1,q_1,m,u,\Delta,A}^{(s)} \cap L_{p_2,q_2,m,u,\Delta,A}^{(s)}(X,Y). \) Then \( T = T_1 T_2, \) where \( T_1 \in L_{p_1,q_1,m,u,\Delta,A}^{(s)}(Z,Y) \) and \( T_2 \in L_{p_2,q_2,m,u,\Delta,A}^{(s)}(X,Z) \) and here \( Z \) is a Banach space. Thus
\[
\sum_{k \geq 1} u_k \left[ M_k \left( \frac{\| A k^{\frac{1}{p}} k^{\frac{1}{q}} \frac{1}{p} \Delta^m s_k (T_1) \|}{\rho_1} \right) \right] < \infty, \ 	ext{for some } \rho_1 > 0,
\]
\[
\sum_{k \geq 1} u_k \left[ N_k \left( \frac{\| A k^{\frac{1}{p}} k^{\frac{1}{q}} \frac{1}{p} \Delta^m s_k (T_2) \|}{\rho_2} \right) \right] < \infty, \ 	ext{for some } \rho_2 > 0.
\]

If \( \frac{1}{p} - \frac{1}{q} \geq 0, \) we have
\[
\sum_{k \geq 1} u_k \left[ \frac{\| A k^{\frac{1}{p}} k^{\frac{1}{q}} \frac{1}{p} \Delta^m s_k (T_1 T_2) \|}{2^{\frac{1}{p}} - \frac{1}{p} \rho_1 \rho_2} \right] 
\leq 2 \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\| A k^{\frac{1}{p}} k^{\frac{1}{q}} \frac{1}{p} \Delta^m s_k (T_1) \|}{\rho_1} \right) \right] + \sum_{k \geq 1} u_k \left[ N_k \left( \frac{\| A k^{\frac{1}{p}} k^{\frac{1}{q}} \frac{1}{p} \Delta^m s_k (T_2) \|}{\rho_2} \right) \right] < \infty,
\]
and for \( \frac{1}{p} - \frac{1}{q} < 0, \) we have
\[
\sum_{k \geq 1} u_k \left[ \frac{\| A k^{\frac{1}{p}} k^{\frac{1}{q}} \frac{1}{p} \Delta^m s_k (T_1 T_2) \|}{\rho_1 \rho_2} \right] \leq 2 \sum_{k \geq 1} u_k \left[ M_k \left( \frac{\| A k^{\frac{1}{p}} k^{\frac{1}{q}} \frac{1}{p} \Delta^m s_k (T_1) \|}{\rho_1} \right) \right] 
+ \sum_{k \geq 1} u_k \left[ N_k \left( \frac{\| A k^{\frac{1}{p}} k^{\frac{1}{q}} \frac{1}{p} \Delta^m s_k (T_2) \|}{\rho_2} \right) \right] < \infty.
\]

This implies that \( \{ A k^{\frac{1}{p}} k^{\frac{1}{q}} \frac{1}{p} \Delta^m s_k (T_1 T_2) \} \in l_1 \) or \( T_1 T_2 \in L_{p,q,l,u,\Delta,A}^{(s)}. \) This completes the proof. \( \square \)

**Acknowledgment**

This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. (G/586/130/37). The authors, therefore, acknowledge with thanks DSR for technical and financial support.

**References**


