Revisit of identities for Apostol-Euler and Frobenius-Euler numbers arising from differential equation

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Abstract

In this paper, we study differential equation arising from the generating function of Apostol-Euler and Frobenius-Euler numbers. In addition, we revisit some identities of Apostol-Euler and Frobenius-Euler numbers which are derived from differential equations. \copyright 2017 all rights reserved.

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1. Introduction

The Apostol-Euler numbers are defined by the generating function to be

\[
\frac{2}{\lambda e^t + 1} = \sum_{n=0}^{\infty} E_{n,\lambda} \frac{t^n}{n!}, \quad (\lambda \neq 0) \quad \text{(see [1, 2])}. \tag{1.1}
\]

From (1.1), we note that

\[
\lambda (E_{\lambda} + 1)^n + E_{n,\lambda} \begin{cases} 
2, & \text{if } n = 0, \\
0, & \text{if } n > 0,
\end{cases}
\]

with the usual convention about replacing \( E_{\lambda}^n \) by \( E_{n,\lambda} \). For \( u \in \mathbb{C} \) with \( u \neq 1 \), the Frobenius-Euler numbers are defined by the generating function to be

\[
\frac{1 - u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, \quad \text{(see [3–14])}. \tag{1.2}
\]

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Thus, we note that $H_n(-1) = E_n$ are ordinary Euler numbers which are defined by the generating function to be

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} \frac{E_n}{n!}, \quad \text{(see [6–11]).}$$

From (1.1) and (1.2), we note that

$$\frac{2}{\lambda e^t + 1} = \left(\frac{2}{\lambda(1 + \lambda^{-1})}\right) \left(\frac{1 + \lambda^{-1}}{e^t + \lambda^{-1}}\right) = \frac{2}{\lambda + 1} \sum_{n=0}^{\infty} H_n(-\lambda^{-1}) \frac{t^n}{n!}. \quad (1.3)$$

By (1.1) and (1.3), we get

$$E_{n,\lambda} = \frac{2}{\lambda + 1} H_n(-\lambda^{-1}), \quad (n \geq 0).$$

Let $k$ be the positive integer. Then the higher-order Apostol-Euler numbers are defined by the generating function as follows:

$$\left(\frac{2}{\lambda e^t + 1}\right)^k = \sum_{n=0}^{\infty} \frac{E^{(k)}_{n,\lambda}}{n!}, \quad \text{(see [1, 2]).} \quad (1.4)$$

For $u \in \mathbb{C}$ with $u \neq 1$, the higher-order Frobenius-Euler numbers are also given by

$$\left(\frac{1 - u}{e^t - u}\right)^k = \left(\frac{1 - u}{e^t - u}\right) \times \cdots \times \left(\frac{1 - u}{e^t - u}\right) = \sum_{n=0}^{\infty} H^{(k)}_n(u) \frac{t^n}{n!}. \quad (1.5)$$

From (1.1), (1.2), (1.4), and (1.5), we have

$$\sum_{l_1 + \cdots + l_r = n} \binom{n}{l_1, l_2, \ldots, l_r} E_{l_1,\lambda} E_{l_2,\lambda} \cdots E_{l_r,\lambda} = E^{(r)}_{n,\lambda}$$

and

$$\sum_{l_1 + \cdots + l_r = n} \binom{n}{l_1, l_2, \ldots, l_r} H_{l_1}(u) H_{l_2}(u) \cdots H_{l_r}(u) = H^{(r)}_n(u),$$

where $n \geq 0$ and $r \in \mathbb{N}$.

In this paper, we study some differential equations which are derived from the generating function of Apostol-Euler and Frobenius-Euler numbers and we revisit some identities of Apostol-Euler and Frobenius-Euler numbers arising from differential equations.

2. Revisit some identities for Apostol-Euler and Frobenius-Euler numbers

Let

$$F = F(t, \lambda) = \frac{1}{e^t + \lambda}, \quad (\lambda \neq 0).$$

Then we have

$$F^{(1)} = \frac{d}{dt} F(t, \lambda) = -\frac{1}{e^t + \lambda} + \frac{\lambda}{(e^t + \lambda)^2} = -F + F^2. \quad (2.1)$$
From (2.1), we have
\[ F^{(2)} = \frac{d}{dt} F^{(1)} = \left( \frac{d}{dt} \right)^2 F(t, \lambda) = F - 3\lambda F^2 + 2\lambda^2 F^3, \]
and
\[ F^{(3)} = \left( \frac{d}{dt} \right)^3 F(t, \lambda) = -F + 7\lambda F^2 - 12\lambda^2 F^3 + 6\lambda^3 F^4. \]
Continuing this process, we have
\[ F^{(N)} = \left( \frac{d}{dt} \right)^N F(t, \lambda) = \sum_{k=0}^{N} (-1)^{N-k}\lambda^k b_k(N, \lambda) F^{k+1}, \quad (n \in \mathbb{N}). \]
(2.2)

From (2.2), we note that
\[ F^{(N+1)} = \left( \frac{d}{dt} \right)^{N+1} F^{(N)} = \frac{d}{dt} \sum_{k=0}^{N} (-1)^{N-k} b_k(N, \lambda) \lambda^k F^{k+1} \]
\[ = \sum_{k=1}^{N+1} (-1)^{N-k} b_{k-1}(N, \lambda) \lambda^k F^{k+1} + \sum_{k=0}^{N} (-1)^{N-k} b_k(N, \lambda) (k+1) \lambda^k F^{k+1}. \]
(2.3)

By replacing \( N \) by \( N + 1 \) in (2.2), we get
\[ F^{(N+1)} = \sum_{k=0}^{N+1} (-1)^{N-k} b_k(N + 1, \lambda) \lambda^k F^{k+1}. \]
(2.4)

Comparing the coefficients on the both sides of (2.3) and (2.4), we obtain
\[ b_0(N, \lambda) = b_0(N + 1, \lambda), \lambda^{N+1}(N + 1)b_N(N, \lambda) = \lambda^{N+1}b_{N+1}(N + 1, \lambda), \]
(2.5)
and
\[ \lambda^k b_k(N + 1, \lambda) = \lambda^k b_{k-1}(N, \lambda) + \lambda^k (k+1)b_k(N, \lambda), \quad \text{where} \ 1 \leq k \leq N. \]
(2.6)

By (2.5) and (2.6), we get
\[ b_0(N + 1, \lambda) = b_0(N, \lambda) = b_0(N - 1, \lambda) = \cdots = b_1(1, \lambda) = 1, \]
and
\[ b_{N+1}(N + 1, \lambda) = (N + 1)b_N(N, \lambda) = (N + 1)N\cdot b_{N-1}(N - 1, \lambda) = \cdots = (N + 1)\cdot 2^{N-1}b_1(1, \lambda) = (N + 1).! \]

Since
\[ -F + \lambda F^2 = -b_0(1, \lambda) + \lambda b_1(1, \lambda) F^2. \]

Thus, \( b_0(1, \lambda) = 1 \) and \( b_1(1, \lambda) = 1 \). From (2.6), we note that
\[ b_1(N + 1, \lambda) = 2b_1(N, \lambda) + b_0(N, \lambda) \]
\[ \vdots \]
\[ = 2^N b_1(1, \lambda) + 2^{N-1}b_0(2, \lambda) + \cdots + 2b_0(N - 1, \lambda) + b_0(N, \lambda) \]
\[ = \sum_{i_1=0}^{N} 2^{i_1}. \]
Therefore, we obtain the following theorem.

**Theorem 2.1.** For \( N \in \mathbb{N} \), the following differential equation

\[
F^{(N)} = \left( \frac{d}{dt} \right)^N F(t, \lambda) = \sum_{k=0}^{N} (-1)^{N-k} b_k(N, \lambda) \lambda^k F^{k+1}
\]

has a solution \( F = F(t, \lambda) = \frac{1}{e^t + \lambda} \), where

\[
b_0(N, \lambda) = 1, \quad b_N(N, \lambda) = N!
\]

and

\[
b_k(N, \lambda) = k! \sum_{i_k=0}^{N-k} \cdots \sum_{i_2=0}^{N-i_k-k} \sum_{i_1=0}^{N-i_{k-1}-i_k-k} 2^{i_1} 3^{i_2} \cdots (k+1)^{i_k}, \quad (1 \leq k \leq N).
\]

Now, we observe that

\[
F^{(N)} = \left( \frac{d}{dt} \right)^N F(t, \lambda) = \left( \frac{d}{dt} \right)^N \left( \frac{1}{e^t + \lambda} \right) = \frac{1}{2\lambda} \left( \frac{d}{dt} \right)^N \left( \frac{2}{\lambda^{-1} e^t + 1} \right) = \frac{1}{2\lambda} \left( \frac{d}{dt} \right)^N \sum_{n=0}^{\infty} E_{n, \lambda^{-1}} \frac{t^n}{n!}
\]

and

\[
p^{k+1} = \left( \frac{1}{e^t + \lambda} \right) \times \left( \frac{1}{e^t + \lambda} \right) \times \cdots \times \left( \frac{1}{e^t + \lambda} \right)
\]

\[
= \frac{1}{2^{k+1} \lambda^{k+1}} \left( \frac{2}{\lambda^{-1} e^t + 1} \right) \times \left( \frac{2}{\lambda^{-1} e^t + 1} \right) \times \cdots \times \left( \frac{2}{\lambda^{-1} e^t + 1} \right) = \frac{1}{2^{k+1} \lambda^{k+1}} \sum_{n=0}^{\infty} E_{n, \lambda^{-1}} \frac{t^n}{n!}.
\]

Therefore, we obtain the following theorem.
Theorem 2.2. For $n \geq 0$, $N \in \mathbb{N}$, we have

$$E_{n+N,\lambda} = \sum_{k=1}^{N} (-1)^{N-k} k! 2^{-k} e^{-k} e^{(k+1)} E_{n+N,\lambda} - \sum_{i_k=0}^{N-k} \sum_{i_{k-1}=0}^{N-i_k-k} \ldots \sum_{i_1=0}^{N-i_k-\cdots-i_2-k} 2^{i_1} 3^{i_2} \cdots (k+1)^{i_k} (-1)^N E_{n+N,\lambda} - 1.$$  

For $\lambda \neq 0, -1$, by (1.2), we get

$$F^{(N)} = \left( \frac{d}{dt} \right)^N F(t, \lambda) = \frac{1}{1 + \lambda} \left( \frac{1 + \lambda}{e^t + \lambda} \right)^N = \frac{1}{1 + \lambda} \sum_{n=0}^{\infty} H_n(-\lambda) \frac{t^n}{n!}.$$  

From (1.5), we can easily derive the following equation:

$$F^{k+1} = \left( \frac{1}{e^t + \lambda} \right) \times \left( \frac{1}{e^t + \lambda} \right) \times \cdots \times \left( \frac{1}{e^t + \lambda} \right) = \left( \frac{1}{1 + \lambda} \right)^{k+1} \sum_{n=0}^{\infty} H_n^{(k+1)}(-\lambda) \frac{t^n}{n!}.$$  

Therefore, by Theorem 2.1, (2.7) and (2.8), we obtain the following theorem.

Theorem 2.3. For $n \geq 0$, $N \in \mathbb{N}$, we have

$$H_{n+N}(-\lambda) = \sum_{k=1}^{N} (-1)^{N-k} \left( \frac{\lambda}{1+\lambda} \right)^k H_n^{(k+1)}(-\lambda) \sum_{i_k=0}^{N-k} \sum_{i_{k-1}=0}^{N-i_k-k} \ldots \sum_{i_1=0}^{N-i_k-\cdots-i_2-k} 2^{i_1} 3^{i_2} \cdots (k+1)^{i_k} (-1)^N H_{n+N}(-\lambda),$$

where $\lambda \in \mathbb{C}$ with $\lambda \neq 0, -1$.

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