Quadratic $\rho$-functional inequalities in $\beta$-homogeneous normed spaces

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Abstract

In this paper, we solve the quadratic $\rho$-functional inequalities

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \|\rho\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\|, \quad (1)$$

where $\rho$ is a fixed complex number with $|\rho| < 1$, and

$$\left\|4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right\| \leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\|, \quad (2)$$

where $\rho$ is a fixed complex number with $|\rho| < 1$.

Using the direct method, we prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequalities (1) and (2) in $\beta$-homogeneous complex Banach spaces. ©2017 all rights reserved.

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1. Introduction and preliminaries


The functional equation $f(x + y) = f(x) + f(y)$ is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. Hyers [4] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ Theorem was generalized by Aoki [1] for additive mappings and by Rassias [9] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Gavruta [3] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach.

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The functional equation \( f(x + y) + f(x - y) = 2f(x) + 2f(y) \) is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a \textit{quadratic mapping}. The stability of quadratic functional equation was proved by Skof [15] for mappings \( f : E_1 \rightarrow E_2 \), where \( E_1 \) is a normed space and \( E_2 \) is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain \( E_1 \) is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [5–8, 11–14]).

\textbf{Definition 1.1.} Let \( X \) be a linear space. A nonnegative valued function \( \| \cdot \| \) is an \( F \)-norm if it satisfies the following conditions:

\begin{enumerate}[(FN1)]
\item \( \| x \| = 0 \) if and only if \( x = 0 \);
\item \( \| \lambda x \| = \| x \| \) for all \( x \in X \) and all \( \lambda \) with \( |\lambda| = 1 \);
\item \( \| x + y \| \leq \| x \| + \| y \| \) for all \( x, y \in X \);
\item \( \| \lambda_n x \| \to 0 \) provided \( \lambda_n \to 0 \);
\item \( \| \lambda x_n \| \to 0 \) provided \( x_n \to 0 \).
\end{enumerate}

Then \( (X, \| \cdot \|) \) is called an \( F^* \)-space. An \( F \)-space is a complete \( F^* \)-space.

An \( F \)-norm is called \( \beta \)-homogeneous \((\beta > 0)\) if \( t \| x \| = |t|^\beta \| x \| \) for all \( x \in X \) and all \( t \in \mathbb{C} \) (see [10]). A \( \beta \)-homogeneous \( F \)-space is called a \( \beta \)-homogeneous complex Banach space.

In Section 2, we solve the quadratic \( \rho \)-functional inequality (1) and prove the Hyers-Ulam stability of the quadratic \( \rho \)-functional inequality (1) in \( \beta_2 \)-homogeneous complex Banach space.

In Section 3, we solve the quadratic \( \rho \)-functional inequality (2) and prove the Hyers-Ulam stability of the quadratic \( \rho \)-functional inequality (2) in \( \beta_2 \)-homogeneous complex Banach space.

Throughout this paper, let \( \beta_1, \beta_2 \) be positive real numbers with \( \beta_1 < 1 \) and \( \beta_2 < 1 \). Assume that \( X \) is a \( \beta_1 \)-homogeneous real or complex normed space with norm \( \| \cdot \| \) and that \( Y \) is a \( \beta_2 \)-homogeneous complex Banach space with norm \( \| \cdot \| \).

\section{Quadratic \( \rho \)-functional inequality (1) in \( \beta \)-homogeneous complex Banach spaces}

Throughout this section, assume that \( \rho \) is a complex number with \( |\rho| < \frac{1}{2} \). We solve and investigate the quadratic \( \rho \)-functional inequality (1) in complex normed spaces.

\textbf{Lemma 2.1.} If a mapping \( f : G \rightarrow Y \) satisfies

\begin{equation}
\| f(x + y) + f(x - y) - 2f(x) - 2f(y) \| \leq \| \rho \left( 4f \left( \frac{x + y}{2} \right) + f(x - y) - 2f(x) - 2f(y) \right) \|,
\end{equation}

for all \( x, y \in G \), then \( f : G \rightarrow Y \) is quadratic.

\textbf{Proof.} Assume that \( f : G \rightarrow Y \) satisfies (2.1).

Letting \( x = y = 0 \) in (2.1), we get \( \| 2f(0) \| \leq \| \rho \| \| f(0) \| \). So \( f(0) = 0 \).

Letting \( y = x \) in (2.1), we get \( \| f(2x) - 4f(x) \| \leq 0 \) and so \( f(2x) = 4f(x) \) for all \( x \in G \). Thus

\begin{equation}
 f \left( \frac{x}{2} \right) = \frac{1}{4} f(x),
\end{equation}

for all \( x \in G \).

It follows from (2.1) and (2.2) that

\begin{equation}
\| f(x + y) + f(x - y) - 2f(x) - 2f(y) \| \leq \| \rho \left( 4f \left( \frac{x + y}{2} \right) + f(x - y) - 2f(x) - 2f(y) \right) \| = |\rho| \| f(x + y) + f(x - y) - 2f(x) - 2f(y) \|.
\end{equation}
and so\[ f(x + y) + f(x - y) = 2f(x) + 2f(y),\]
for all \(x, y \in G.\]

Now, we prove the Hyers-Ulam stability of the quadratic \(\rho\)-functional inequality (2.1) in \(\beta\)-homogeneous complex Banach spaces.

**Theorem 2.2.** Let \(r > \frac{2\beta_2}{\beta_1}\) and \(\theta\) be nonnegative real numbers and let \(f : X \to Y\) be a mapping satisfying \(f(0) = 0\) and\[ \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \rho \left( 4f \left( \frac{x + y}{2} \right) + f \left( x - y \right) - 2f(x) - 2f(y) \right) + \theta(||x||^r + ||y||^r),\]for all \(x, y \in X.\) Then there exists a unique quadratic mapping \(Q : X \to Y\) such that\[ \|f(x) - Q(x)\| \leq \frac{2\theta}{2\beta_1 r - 4\beta_2} ||x||^r,\]for all \(x \in X.\)

**Proof.** Letting \(y = x\) in (2.3), we get\[ ||f(2x) - 4f(x)|| \leq 2\theta ||x||^r,\]for all \(x \in X.\) So\[ \left\| f(x) - 4f(x) \right\| \leq \frac{2\theta}{2\beta_1 r} ||x||^r,\]for all \(x \in X.\) Hence
\[
\left\| 4^l \left( \frac{x}{2^l} \right) - 4^m \left( \frac{x}{2^m} \right) \right\| \leq \sum_{j=1}^{m-1} \left( 4^l \left( \frac{x}{2^j} \right) - 4^{l+1} \left( \frac{x}{2^{j+1}} \right) \right) \leq \frac{2\beta_2^j}{2\beta_1 r} \sum_{j=1}^{m-1} \beta_2^j \theta ||x||^r, \tag{2.6}
\]
for all nonnegative integers \(m\) and \(l\) with \(m > l\) and all \(x \in X.\) It follows from (2.6) that the sequence \(\{4^k f(\frac{x}{2^k})\}\) is Cauchy for all \(x \in X.\) Since \(Y\) is a Banach space, the sequence \(\{4^k f(\frac{x}{2^k})\}\) is convergent. So one can define the mapping \(Q : X \to Y\) by\[ Q(x) := \lim_{k \to \infty} 4^k f \left( \frac{x}{2^k} \right),\]
for all \(x \in X.\) Moreover, letting \(l = 0\) and passing the limit \(m \to \infty\) in (2.6), we get (2.4). It follows from (2.3) that
\[
\|Q(x + y) + Q(x - y) - 2Q(x) - 2Q(y)\| = \lim_{n \to \infty} \left\| 4^n \left( f \left( \frac{x + y}{2^n} \right) + f \left( \frac{x - y}{2^n} \right) - 2f(x) - 2f(y) \right) \right\|
\leq \lim_{n \to \infty} 4^n \rho \left( 4f \left( \frac{x + y}{2^{n+1}} \right) + f \left( \frac{x - y}{2^{n+1}} \right) - 2f \left( \frac{x}{2^n} \right) - 2f \left( \frac{y}{2^n} \right) \right)
+ \lim_{n \to \infty} \frac{4\beta_2^n}{2\beta_1 r} \theta ||x||^r + ||y||^r
= \left\| \rho \left( 4Q \left( \frac{x + y}{2} \right) + Q(x - y) - 2Q(x) - 2Q(y) \right) \right\|,
\]
for all $x, y \in X$. So

$$\left\| Q \left( \frac{x+y}{2} \right) + Q \left( \frac{x-y}{2} \right) - 2Q(x) - 2Q(y) \right\| \leq \left\| \rho \left( 4Q \left( \frac{x+y}{2} \right) + Q \left( x-y \right) - 2Q(x) - 2Q(y) \right) \right\|,$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.4). Then we have

$$\left\| Q(x) - T(x) \right\| = \left\| 4^q Q \left( \frac{x}{2^q} \right) - 4^q T \left( \frac{x}{2^q} \right) \right\| \leq \left\| 4^q Q \left( \frac{x}{2^q} \right) - 4^q f \left( \frac{x}{2^q} \right) \right\| + \left\| 4^q T \left( \frac{x}{2^q} \right) - 4^q f \left( \frac{x}{2^q} \right) \right\| \leq 2 \theta \frac{4^q q}{2^{q+1}} \| x \|^r,$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $Q$, as desired. \qed

**Theorem 2.3.** Let $r < \frac{2\beta_2}{\beta_1}$ and $\theta$ be nonnegative real numbers and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\left\| f(x) - Q(x) \right\| \leq \frac{2\theta}{4\beta_2 - 2\beta_1 r} \| x \|^r,$$

for all $x \in X$.

**Proof.** It follows from (2.5) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{2\theta}{4\beta_2} \| x \|^r,$$

for all $x \in X$. Hence

$$\left\| \frac{1}{4^j} f(2^j x) - \frac{1}{4^m} f(2^m x) \right\| \leq \sum_{j=1}^{m-1} \left\| \frac{1}{4^j} f(2^j x) - \frac{1}{4^{j+1}} f(2^{j+1} x) \right\| \leq \frac{2\theta}{4\beta_2} \sum_{j=1}^{m-1} \frac{2^{j+1}}{4^{\beta_2 j}} \| x \|^r,$$  \(2.8\)

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.8) that the sequence \(\left\{ \frac{1}{4^n} f(2^n x) \right\}\) is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence \(\left\{ \frac{1}{4^n} f(2^n x) \right\}\) is convergent. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n \to \infty} \frac{1}{4^n} f(2^n x),$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.8), we get (2.7).

The rest of proof is similar to the proof of Theorem 2.2. \qed

**Remark 2.4.** If $\rho$ is a real number such that $-1 < \rho < 1$ and $Y$ is a $\beta$-homogeneous real Banach space, then all the assertions in this section remain valid.

3. **Quadratic $\rho$-functional inequality (2) in $\beta$-homogeneous complex Banach spaces**

Throughout this section, assume that $\rho$ is a complex number with $|\rho| < 1$.

We solve and investigate the quadratic $\rho$-functional inequality (2) in $\beta$-homogeneous complex normed spaces.

**Lemma 3.1.** If a mapping $f : G \to Y$ satisfies

$$\left\| 4f \left( \frac{x+y}{2} \right) + f(x) - f(x-y) - 2f(x) - 2f(y) \right\| \leq \left\| \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \right\|,$$  \(3.1\)

for all $x, y \in G$, then $f : G \to Y$ is quadratic.
Proof. Assume that \( f : G \to Y \) satisfies (3.1).
Letting \( x = y = 0 \) in (3.1), we get \( \|f(0)\| \leq |\rho|2f(0)\). So \( f(0) = 0 \).
Letting \( y = 0 \) in (3.1), we get \( \|4f(\frac{x}{2}) - f(x)\| \leq 0 \) and so
\[
4f\left(\frac{x}{2}\right) = f(x),
\] (3.2)
for all \( x \in G \).
It follows from (3.1) and (3.2) that
\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| = \left\|4f\left(\frac{x + y}{2}\right) + f(x - y) - 2f(x) - 2f(y)\right\|
\leq |\rho|\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\|
\]
and so
\[
f(x + y) + f(x - y) = 2f(x) + 2f(y),
\]
for all \( x, y \in G \).
\( \square \)

Now, we prove the Hyers-Ulam stability of the quadratic \( \rho \)-functional inequality (3.1) in \( \beta \)-homogeneous complex Banach spaces.

**Theorem 3.2.** Let \( r > \frac{2\beta_2}{\beta_1} \) and \( \theta \) be nonnegative real numbers and let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) and
\[
\left\|4f\left(\frac{x + y}{2}\right) + f(x - y) - 2f(x) - 2f(y)\right\| \leq |\rho|\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\|
+ \theta(||x||^r + ||y||^r),
\] (3.3)
for all \( x, y \in X \). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that
\[
\|f(x) - Q(x)\| \leq \frac{2\beta_1 r \theta}{2\beta_1 r - 4\beta_2} ||x||^r,
\] (3.4)
for all \( x \in X \).

Proof. Letting \( y = 0 \) in (3.3), we get
\[
\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \leq \left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \theta ||x||^r,
\] (3.5)
for all \( x \in X \). So
\[
\left\|4^j f\left(\frac{x}{2^j}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \leq \sum_{j=1}^{m-1} \left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \leq \sum_{j=1}^{m-1} \frac{4 \beta_2 j \theta}{2 \beta_1 r} ||x||^r,
\] (3.6)
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (3.6) that the sequence \( \{4^k f\left(\frac{x}{2^k}\right)\} \) is Cauchy for all \( x \in X \). Since \( Y \) is a Banach space, the sequence \( \{4^k f\left(\frac{x}{2^k}\right)\} \) is convergent. So one can define the mapping \( Q : X \to Y \) by

\[
Q(x) := \lim_{k \to \infty} 4^k f\left(\frac{x}{2^k}\right),
\]
for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.6), we get (3.4).

The rest of proof is similar to the proof of Theorem 2.2. \( \square \)
Theorem 3.3. Let $r < \frac{2\beta_1}{\beta_2}$ and $\theta$ be nonnegative real numbers and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.3). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$
\|f(x) - Q(x)\| \leq \frac{2^{\beta_1}r}{4^{\beta_2}} \|x\|^r,
$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$
\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{2^{\beta_1}r}{4^{\beta_2}} \|x\|^r,
$$

for all $x \in X$. Hence

$$
\left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| \leq \sum_{j=1}^{m-1} \left\| \frac{1}{4^j} f(2^j x) - \frac{1}{4^{j+1}} f(2^{j+1} x) \right\| \leq \sum_{j=1}^{m} \frac{2^{\beta_1}r}{4^{\beta_2}} \theta \|x\|^r,
$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is convergent. So one can define the mapping $Q : X \to Y$ by

$$
Q(x) := \lim_{n \to \infty} \frac{1}{4^n} f(2^n x),
$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.7).

The rest of proof is similar to the proof of Theorem 2.2. \hfill \Box

Remark 3.4. If $\rho$ is a real number such that $-1 < \rho < 1$ and $Y$ is a $\beta$-homogeneous real Banach space, then all the assertions in this section remain valid.

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