Abstract

In this paper, we introduce and investigate Jensen $\rho$-functional inequalities associated with the following Jensen functional equations

$$f(x + y + z) + f(x + y - z) - 2f(x) - 2f(y) = 0,$$

$$f(x + y + z) - f(x - y - z) - 2f(y) - 2f(z) = 0.$$

We prove the Hyers-Ulam-Rassias stability of the Jensen $\rho$-functional inequalities in complex Banach spaces and prove the Hyers-Ulam-Rassias stability of the Jensen $\rho$-functional equations associated with the $\rho$-functional inequalities in complex Banach spaces. ©2016 All rights reserved.

Keywords: Jensen functional inequalities, Hyers-Ulam-Rassias stability, complex Banach spaces.


1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [28] concerning the stability of group homomorphisms. The functional equation

$$f(x + y) = f(x) + f(y),$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is called to be an additive mapping. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ theorem was generalized by Aoki [11] for additive mappings and by Rassias [25] for linear...
mappings by considering an unbounded Cauchy difference. The paper of Rassias [25] has provided a lot of influence in the development of what we call generalized Hyers-Ulam stability of functional equations. A generalization of the Rassias theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach. The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [21, 9, 12, 16, 18, 27, 29]).

In [17], Park et al. investigated the following inequalities

\[ \|f(x) + f(y) + f(z)\| \leq \left\| 2f \left( \frac{x + y + z}{2} \right) \right\|, \]
\[ \|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\|, \]
\[ \|f(x) + f(y) + 2f(z)\| \leq \left\| 2f \left( \frac{x + y + z}{2} \right) \right\|, \]

in Banach spaces. Recently, Cho et al. [5] investigated the following functional inequality

\[ \|f(x) + f(y) + f(z)\| \leq \left\| Kf \left( \frac{x + y + z}{K} \right) \right\|, \quad (0 < |K| < |3|), \]

in non-Archimedean Banach spaces.

The function equations

\[ f(x + y + z) + f(x + y - z) - 2f(x) = 0, \quad (1.1) \]
\[ f(x + y + z) - f(x - y - z) - 2f(y) - 2f(z) = 0, \quad (1.2) \]

is called 3-variable Jensen. In this paper, we investigate the 3-variable Jensen functional equations and prove the Hyers-Ulam-Rassias stability of the functional inequalities in complex Banach spaces.

Throughout this paper, assume that \( X \) is a complex normed vector space with norm \( \| \cdot \| \) and that \( (Y, \| \cdot \|) \) is a complex Banach space.

2. Hyers-Ulam-Rassias stability of \( (1.1) \)

In this section, we prove that the Hyers-Ulam-Rassias stability of the 3-variable functional inequality

\[ \|f(x + y + z) + f(x + y - z) - 2f(x) - 2f(y)\| \leq \|\rho_1(f(x + y + z) - f(x) - f(y) - f(z))\| \]
\[ + \|\rho_2(f(x + y - z) - f(x) - f(y) + f(z))\|, \quad (2.1) \]

in the complex Banach space, where \( \rho_1 \) and \( \rho_2 \) are the fixed complex numbers with \( \|\rho_1\| < \frac{1}{2}, \|\rho_2\| < \frac{1}{2}. \)

Lemma 2.1. Let \( f : X \to Y \) be a mapping. If it satisfies \( (2.1) \) for all \( x, y, z \in X \), then \( f \) is additive.

Proof. By letting \( x = y = z = 0 \) in \( (2.1) \) for all \( x, y, z \in X \), we get

\[ \|2f(0)\| \leq 2\rho_1 f(0), \]

thus \( f(0) = 0. \)

By letting \( x = y = 0 \) in \( (2.1) \), we get

\[ \|f(z) + f(-z)\| \leq \|\rho_2(f(-z) + f(z))\|, \]

and so \( f(-x) = -f(x) \) for all \( x \in X. \)

Let \( z = 0 \) in \( (2.1) \), so we have

\[ \|2f(x + y) - 2f(x) - 2f(y)\| \leq \|\rho_1(f(x + y) - f(x) - f(y))\| \]
\[ + \|\rho_2(f(x + y) - f(x) - f(y))\| \]
\[ = (|\rho_1| + |\rho_2|)\|f(x + y) - f(x) - f(y)\|, \]

and so \( f(x + y) = f(x) + f(y) \) for all \( x, y \in X \). Hence \( f : X \to Y \) is additive. \( \square \)
Corollary 2.2. Let \( f : X \to Y \) be a mapping satisfying
\[
\|f(x + y + z) + f(x + y - z) - 2f(x) - 2f(y)\| = \|\rho_1(f(x + y + z) - f(x) - f(y) - f(z))\|
+ \|\rho_2(f(x + y - z) - f(x) - f(y) + f(z))\|
\]
for all \( x, y, z \in X \). Then \( F : X \to Y \) is additive.

We prove the Hyers-Ulam-Rassias stability of the additive functional inequality \([2,1]\) in complex Banach spaces.

Theorem 2.3. Let \( f : X \to Y \) be a mapping. If there is a function \( \varphi : X^3 \to [0, \infty) \) with \( \varphi(0, 0, 0) = 0 \) such that
\[
\|f(x + y + z) + f(x + y - z) - 2f(x) - 2f(y)\| \leq \|\rho_1(f(x + y + z) - f(x) - f(y) - f(z))\|
+ \|\rho_2(f(x + y - z) - f(x) - f(y) + f(z))\| + \varphi(x, y, z),
\]
and
\[
\bar{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, 2^j z) < \infty
\]
for all \( x, y, z \in X \). Then there exists a unique additive mapping \( A : X \to Y \) such that
\[
\|f(x) - A(x)\| \leq \bar{\varphi}(x, x, 0) \tag{2.3}
\]
for all \( x \in X \).

Proof. By letting \( x = y = z = 0 \) in (2.2), we get
\[
\|2f(0)\| \leq 2\rho_1 f(0),
\]
so \( f(0) = 0 \). Let \( y = x \) and \( z = 0 \) in (2.2), so we get
\[
\|2f(2x) - 4f(x)\| \leq |\rho_1|\|f(2x) - 2f(x)\| + |\rho_2|\|f(2x) - 2f(x)\| + \varphi(x, x, 0)
\]
for all \( x \in X \). Thus
\[
\left\| f(x) - \frac{f(2x)}{2} \right\| \leq \frac{1}{2 - |\rho_1| - |\rho_2|} \frac{1}{2} \bar{\varphi}(x, x, 0)
\leq \varphi(x, x, 0)
\]
for all \( x \in X \).

Hence one may have the following formula for positive integers \( m, l \) with \( m > l \),
\[
\left\| \left( \frac{1}{2^l} f\left( \frac{(2^l x)}{x} \right) - \frac{1}{(2^l)^m} f\left( \frac{(2^m x)}{x} \right) \right) \right\| \leq \sum_{i=l}^{m-1} \frac{1}{2^i} \varphi(2^i x, 2^i x, 0) \tag{2.4}
\]
for all \( x \in X \).

It follows from (2.4) that the sequence \( \left\{ \frac{f(2^k x)}{2^k} \right\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is a Banach space, the sequence \( \left\{ \frac{f(2^k x)}{2^k} \right\} \) converges. So one may define the mapping \( A : X \to Y \) by
\[
A(x) := \lim_{k \to \infty} \left\{ \frac{f(2^k x)}{2^k} \right\}, \quad \forall x \in X.
\]

By taking \( m = 0 \) and letting \( l \to \infty \) in (2.4), we get (2.3).
It follows from (2.2) that
\[
\|A(x + y + z) + A(x + y - z) = 2A(x) - 2A(y)\|
\leq \lim_{n \to \infty} 2^n \left\| f \left( \frac{x + y + z}{2^n} \right) - f \left( \frac{x + y - z}{2^n} \right) - f \left( \frac{x}{2^n} \right) - f \left( \frac{y}{2^n} \right) \right\|
\]
for all \(x, y, z \in X\). One can see that \(A\) satisfies the inequality (2.1) and so it is additive by Lemma 2.1.

Now, we show the uniqueness of \(A\). Let \(T : X \to Y\) be another additive mapping satisfying (2.2). Then one has
\[
\|A(x) - T(x)\| = \left\| \frac{1}{2^k} A(2^k x) - \frac{1}{2^k} T(2^k x) \right\|
\leq \frac{1}{2^k} \left( \| A(2^k x) - f(2^k x) \| + \| T(2^k x) - f(2^k x) \| \right)
\leq \frac{1}{2^k} \sim(2^k x, 2^k x, 0),
\]
which tends to zero as \(k \to \infty\) for all \(x \in X\). So we can conclude that \(A(x) = T(x)\) for all \(x \in X\).

**Corollary 2.4.** Let \(r < 1\) and \(\theta\) be nonnegative real numbers, and let \(f : X \to Y\) be a mapping such that
\[
\|f(x + y + z) + f(x + y - z) - 2f(x) - 2f(y)\|
\leq \|\rho_1(f(x + y + z) - f(x) - f(y) - f(z))\|
+ \|\rho_2(f(x + y - z) - f(x) - f(y) + f(z))\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\]
for all \(x, y, z \in X\). Then there exists a unique additive mapping \(A : X \to Y\) such that
\[
\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^r}\|x\|^r
\]
for all \(x \in X\).

**Theorem 2.5.** Let \(f : X \to Y\) be a mapping with \(f(0) = 0\). If there is a function \(\varphi : X^3 \to [0, \infty)\) satisfying (2.2) such that
\[
\tilde{\varphi}(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j} \right) < \infty
\]
for all \(x, y, z \in X\), then there exists a unique additive mapping \(A : X \to Y\) such that
\[
\|f(x) - A(x)\| \leq \tilde{\varphi} \left( \frac{x}{2}, \frac{x}{2}, 0 \right)
\]
for all \(x \in X\).
Proof. The proof is similar to Theorem 2.3, we can get
\[ \|f(x) - 2f\left(\frac{x}{2}\right)\| \leq \varphi\left(\frac{x}{2}, 0\right) \]
for all \( x \in X \).

Next, we can prove that the sequence \( \{2^n f\left(\frac{x}{2^n}\right)\} \) is a Cauchy sequence for all \( x \in X \), and define a mapping \( A : X \to Y \) by
\[ A(x) := \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right) \]
for all \( x \in X \) that is similar to the corresponding part of the proof of Theorem 2.3.

\[ \]

**Corollary 2.6.** Let \( r < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \to Y \) be a mapping such that
\[ \|f(x + y + z) + f(x + y - z) - 2f(x) - 2f(y)\| \leq \|\rho_1(f(x + y + z) - f(x) - f(y) - f(z))\| \]
\[ + \|\rho_2(f(x + y - z) - f(x) - f(y) + f(z))\| + \theta(\|x\| \|y\| + \|y\| \|z\|) \]
for all \( x, y, z \in X \). Then there exists a unique additive mapping \( A : X \to Y \) such that
\[ \|f(x) - A(x)\| \leq \frac{2^{1+r}}{2^r - 1} \|x\|^r \]
for all \( x \in X \).

\[ \]

3. Hyers-Ulam-Rassias stability of (1.2)

In this section, we prove that the Hyers-Ulam-Rassias stability of the 3-variable functional inequality
\[ \|f(x + y + z) - f(x - y - z) - 2f(x) - 2f(z)\| \leq \|\rho_1(f(x + y + z) - f(x + y) - f(z))\| \]
\[ + \|\rho_2(f(x + y - z) - f(x) - f(y) + f(z))\| , \quad (3.1) \]

in the complex Banach space, where \( \rho_1 \) and \( \rho_2 \) are the fixed complex numbers with \( \|\rho_1\| < \frac{1}{2}, \|\rho_2\| < \frac{1}{2} \).

**Lemma 3.1.** Let \( f : X \to Y \) be a mapping. If it satisfies (3.1) for all \( x, y, z \in X \), then \( f \) is additive.

**Proof.** By letting \( x = y = z = 0 \) in (3.1) for all \( x, y, z \in X \), we get
\[ \|4f(0)\| \leq \|\rho_1 f(0)\| , \]
thus \( f(0) = 0 \) and by letting \( x = y = 0 \) in (3.1), we get
\[ (1 - |\rho_2|)\|f(z) + f(-z)\| \leq 0 , \]
and so \( f(-z) = -f(z) \) for all \( z \in X \).

Let \( x = 0 \) in (3.1), so we have
\[ \|f(y + z) - f(-y - z) - 2f(y) - 2f(z)\| \leq \|\rho_1(f(y + z) - f(y) - f(z))\| \]
\[ + \|\rho_2(f(y - z) - f(y) + f(z))\| \]
for all \( y, z \in X \).

Thus
\[ (2 - |\rho_1|)\|f(y + z) - f(y) - f(z)\| \leq |\rho_2|\|f(y - z) - f(y) + f(z)\| \]
\[ (3.2) \]
for all \( y, z \in X \).
By replacing \( z \) by \(-z\) in (3.2), we have
\[
(2 - |\rho_1|)\|f(y - z) - f(y) + f(z)\| \leq |\rho_2|\|f(y + z) - f(y) - f(z)\|
\] (3.3)
for all \( y, z \in X \).

By (3.2) and (3.3), we get
\[
(2 - |\rho_1|)^2\|f(y + z) - f(y) - f(z)\| \leq |\rho_2|^2\|f(y + z) - f(y) - f(z)\|
\]
for all \( y, z \in X \).
Hence \( f : X \to Y \) is additive.

\[\square\]

**Corollary 3.2.** Let \( f : X \to Y \) be a mapping satisfying
\[
\|f(x + y + z) - f(x - y - z) - 2f(y) - 2f(z)\| = \|\rho_1(f(x + y + z) - f(x + y) - f(z))\|
\]
\[
+ \|\rho_2(f(x + y - z) - f(x) - f(y) + f(z))\|
\]
for all \( x, y, z \in X \). Then \( f : X \to Y \) is additive.

We prove the Hyers-Ulam-Rassias stability of the additive functional inequality (3.1) in complex Banach spaces.

**Theorem 3.3.** Let \( f : X \to Y \) be a mapping. If there is a function \( \varphi : X^3 \to [0, \infty) \) with \( \varphi(0,0,0) = 0 \) such that
\[
\|f(x + y + z) - f(x - y - z) - 2f(y) - 2f(z)\| \leq \|\rho_1(f(x + y + z) - f(x + y) - f(z))\|
\]
\[
+ \|\rho_2(f(x + y - z) - f(x) - f(y) + f(z))\| + \varphi(x,y,z),
\]
and
\[
\tilde{\varphi}(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, 2^j z) < \infty
\]
for all \( x, y, z \in X \), then there exists a unique additive mapping \( A : X \to Y \) such that
\[
\|f(x) - A(x)\| \leq \tilde{\varphi}(x,x,0)
\] (3.5)
for all \( x \in X \).

**Proof.** By letting \( x = y = z = 0 \) in (3.4), we get
\[
\|4f(0)\| \leq \|\rho_1 f(0)\|.
\]
So \( f(0) = 0 \).

Let \( y = x \) and \( z = 0 \) in (3.4), so we get
\[
\|f(2x) - 2f(x)\| \leq |\rho_2|\|f(2x) - 2f(x)\| + \varphi(x,x,0)
\]
for all \( x \in X \). Thus
\[
\left\|f(x) - \frac{f(2x)}{2}\right\| \leq \frac{1}{1 - |\rho_2|} \frac{1}{2} \varphi(x,x,0) \leq \varphi(x,x,0)
\]
for all \( x \), since \( |\rho_2| < \frac{1}{2}, \frac{1}{1 - |\rho_2|} < 2 \).
Hence one may have the following formula for positive integers \( m, l \) with \( m > l \),

\[
\left\| \frac{1}{(2)^l} f \left( (2)^l x \right) - \frac{1}{(2)^m} f \left( (2)^m x \right) \right\| \leq \sum_{i=l}^{m-1} \frac{1}{2^i} \varphi \left( 2^i, 2^i x, 0 \right)
\] (3.6)

for all \( x \in X \).

It follows from (3.6) that the sequence \( \left\{ f(2^k x) \right\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is a Banach space, the sequence \( \left\{ f(2^k x) \right\} \) converges. So one may define the mapping \( A : X \to Y \) by

\[
A(x) := \lim_{k \to \infty} \left\{ f \left( \frac{2^k x}{2^k} \right) \right\}, \quad \forall x \in X.
\]

By taking \( m = 0 \) and letting \( l \to \infty \) in (3.6), we get (3.5).

It follows from (3.4) that

\[
\| A(x + y + z) - A(x - y - z) - 2A(y) - 2A(z) \|
\]

\[
= \lim_{n \to \infty} 2^n \left\| f \left( \frac{x + y + z}{2^n} \right) - f \left( \frac{x - y - z}{2^n} \right) - 2f \left( \frac{y}{2^n} \right) - 2f \left( \frac{z}{2^n} \right) \right\|
\]

\[
\leq \lim_{n \to \infty} 2^n \left\| \rho_1 \left( f \left( \frac{x + y + z}{2^n} \right) - f \left( \frac{x}{2^n} \right) \right) \right\|
\]

\[
+ \lim_{n \to \infty} 2^n \left\| \rho_2 \left( f \left( \frac{x + y - z}{2^n} \right) - f \left( \frac{x}{2^n} \right) + f \left( \frac{y}{2^n} \right) + f \left( \frac{z}{2^n} \right) \right) \right\|
\]

\[
= \| \rho_1(A(x + y + z) - A(x + y) - A(z)) \|
\]

\[
+ \| \rho_2(A(x + y - z) - A(x) - A(y) + A(z)) \|
\]

for all \( x, y, z \in X \). One can see that \( A \) satisfies the inequality (3.1) and so it is additive by Lemma 3.1.

Now, we show the uniqueness of \( A \). Let \( T : X \to Y \) be another additive mapping satisfying (3.4). Then one has

\[
\| A(x) - T(x) \| = \left\| \frac{1}{2^k} A \left( \frac{2^k x}{2^k} \right) - \frac{1}{2^k} T \left( \frac{2^k x}{2^k} \right) \right\|
\]

\[
\leq \frac{1}{2^k} \left( \| A \left( \frac{2^k x}{2^k} \right) - f \left( \frac{2^k x}{2^k} \right) \| + \| T \left( \frac{2^k x}{2^k} \right) - f \left( \frac{2^k x}{2^k} \right) \| \right)
\]

\[
\leq 2 \cdot \frac{1}{2^k} \varphi (2^k x, 2^k x, 0),
\]

which tends to zero as \( k \to \infty \), for all \( x \in X \). So we can conclude that \( A(x) = T(x) \) for all \( x \in X \).

\[\Box\]

**Corollary 3.4.** Let \( r < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \to Y \) be a mapping such that

\[
\| f(x + y + z) - f(x - y - z) - 2f(y) - 2f(z) \|
\]

\[
\leq \| \rho_1(f(x + y + z) - f(x + y) - f(z)) \|
\]

\[
+ \| \rho_2(f(x + y - z) - f(x) - f(y) + f(z)) \| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\]

for all \( x, y, z \in X \). Then there exists a unique additive mapping \( A : X \to Y \) such that

\[
\| f(x) - A(x) \| \leq \frac{2\theta}{2 - 2^r} \|x\|^r
\]

for all \( x \in X \).
Theorem 3.5. Let \( f : X \rightarrow Y \) be a mapping with \( f(0) = 0 \). If there is a function \( \varphi : X^3 \rightarrow [0, \infty) \) satisfying (3.4) such that
\[
\tilde{\varphi}(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j} \right) < \infty
\]
for all \( x, y, z \in X \), then there exists a unique additive mapping \( A : X \rightarrow Y \) such that
\[
\| f(x) - A(x) \| \leq \tilde{\varphi} \left( \frac{x}{2}, \frac{x}{2}, 0 \right)
\]
for all \( x \in X \).

Proof. The proof is similar to Theorem 3.3, we can get
\[
\| f(x) - 2f \left( \frac{x}{2} \right) \| \leq \varphi \left( \frac{x}{2}, \frac{y}{2}, 0 \right)
\]
for all \( x \in X \).

Next, we can prove that the sequence \( \{2^n f \left( \frac{x}{2^n} \right)\} \) is a Cauchy sequence for all \( x \in X \), and define a mapping \( A : X \rightarrow Y \) by
\[
A(x) := \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right)
\]
for all \( x \in X \), that is similar to the corresponding part of the proof of Theorem 3.3.

Corollary 3.6. Let \( r < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \rightarrow Y \) be a mapping such that
\[
\| f(x + y + z) - f(x - y - z) - 2f(y) - 2f(z) \|
\leq \| \rho_1(f(x + y + z) - f(x + y) - f(z)) \|
+ \| \rho_2(f(x + y + z) - f(x) - f(y) - f(z)) \| + \theta(\| x \|^r + \| y \|^r + \| z \|^r)
\]
for all \( x, y, z \in X \). Then there exists a unique additive mapping \( A : X \rightarrow Y \) such that
\[
\| f(x) - A(x) \| \leq \frac{2^{1+r} \theta}{2^r - 1} \| x \|^r
\]
for all \( x \in X \).

Acknowledgment

G. Lu was supported by Doctoral Science Foundation of Shenyang University of Technology (No. 521101302) and the Project Sponsored by the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry. Y. Jin was supported by National Natural Science Foundation of China (11361066) The study of high-precision algorithm for high dimensional delay partial differential equations 2014-2017.

References


