Hyers-Ulam stability of derivations in fuzzy Banach space

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Abstract

In this paper, we construct an additive functional equation, and use the fixed point alternative theorem to investigate the Hyers-Ulam stability of derivations fuzzy Banach space and fuzzy Lie Banach space associated with the following functional equation:

\[ f(2x - y - z) + f(x - z) + f(x + y + 2z) = f(4x). \]

Keywords: Fuzzy normed space, additive functional equation, Hyers-Ulam stability, fixed point alternative, fuzzy Banach space.


1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [46] concerning the stability of group homomorphisms. Hyers [22] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ Theorem was generalized by Aoki [2] for additive mappings and by Rassias [40] for linear mappings by considering an unbounded Cauchy difference. Those results have been recently complemented in [8]. A generalization of the Aoki and Rassias theorem was obtained by Găvruta [21], who used a more general function controlling the possibly unbounded Cauchy difference in the spirit of Rassias’ approach. The stability problems for several functional equations or inequalities have been extensively

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investigated by a number of authors and there are many interesting results concerning this problem (see [7, 10, 11, 14, 15, 23–28, 30, 31, 34, 36–39, 41–43]).

We recall a fundamental result in fixed point theory.

Let $X$ be a set. A function $d: X \times X \to [0, \infty]$ is called a generalized metric on $X$, if $d$ satisfies

1. $d(x, y) = 0$, if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

**Theorem 1.1** ([13, 18]). Let $(X, d)$ be a complete generalized metric space and let $J: X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers $n$, or there exists a positive integer $n_0$ such that

1. $d(J^n x, J^{n+1} x) < \infty$, for all $n \geq n_0$;
2. the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;
3. $y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;
4. $d(y, y^*) \leq \frac{1}{1 - L} d(y, Jy)$, for all $y \in Y$.

By using the fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors (see [1, 9, 12, 13, 17, 19, 28, 33, 35, 39]).

In 1984, Katsaras [27] defined a fuzzy norm on a linear space and at the same year Wu and Fang [47] also introduced a notion of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear space. In [6], Biswas defined and studied fuzzy inner product spaces in linear space. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [5, 20, 30, 45, 48]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [29]. In 2003, Bag and Samanta [3] and Saadati and Vaezpour [44] modified the definition of Cheng and Mordeson [16] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms (see [4]). Following [3], we give the employing notion of a fuzzy norm.

Let $X$ be a real linear space. A function $N: X \times \mathbb{R} \to [0, 1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on $X$, if for all $x, y \in X$ and all $a, b \in \mathbb{R}$:

1. $N(x, a) = 0$ for $a \leq 0$;
2. $x = 0$, if and only if $N(x, a) = 1$ for all $a > 0$;
3. $N(ax, b) = N(x, \frac{b}{|a|})$, if $a \neq 0$;
4. $N(x + y, a + b) \geq \min \{N(x, a), N(y, b)\}$;
5. $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{a \to \infty} N(x, a) = 1$;
6. For $x \neq 0$, $N(x, \cdot)$ is (upper semi) continuous on $\mathbb{R}$.

The pair $(X, N)$ is called a fuzzy normed linear space. One may regard $N(x, a)$ as the truth value of the statement the norm of $x$ is less than or equal to the real number $a'$. 
Example 1.2. Let $(X, \|\|)$ be a fuzzy normed space. Define

$$N(x, a) = \begin{cases} 
\frac{a}{a + \|x\|}, & a > 0, \\
0, & a \leq 0
\end{cases}, \quad x \in X.$$ 

Then $(X, N)$ is called the induced fuzzy normed space.

Definition 1.3. Let $(X, N)$ be a fuzzy normed linear space. Let $x_n$ be a sequence in $X$. Then $x_n$ is said to be convergent, if there exists $x \in X$ such that $\lim_{n \to \infty} N(x_n - x, a) = 1$ for all $a > 0$. In that case, $x$ is called the limit of the sequence $x_n$ and we denote it by $N$-$\lim_{n \to \infty} x_n = x$.

Definition 1.4. A sequence $x_n$ in $X$ is called Cauchy, if for each $\epsilon > 0$ and each $a > 0$ there exists $n_0$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_n + p - x_n, a) > 1 - \epsilon$.

It is known that every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Definition 1.5. Let $(X, N)$ and $(Y, N)$ be fuzzy normed algebras.

1. An $\mathbb{R}$-linear mapping $f : X \to Y$ is called a homomorphism, if

$$f(xy) = f(x)f(y)$$

for all $x, y \in X$.

2. An $\mathbb{R}$-linear mapping $f : X \to X$ is called a derivation, if

$$f(xy) = f(x)y + xf(y)$$

for all $x, y \in X$.

2. The stability of derivations on fuzzy C*-algebras

Throughout this section, assume that $A$ is a fuzzy C*-algebra with the fuzzy norm. For any mapping $f : A \to A$, we define

$$Df(x, y, z) := f(2x - y - z) + f(x - z) + f(x + y + 2z) - f(4x)$$

for all $x, y, z \in A$.

Firstly, we prove that $Df(x, y, z) = 0$ implies the additivity of $f$.

Lemma 2.1. Let $(Z, N)$ be a fuzzy normed vector space and $f : X \to Z$ be a mapping such that

$$N(f(2x - y - z) + f(x - z) + f(x + y + 2z), t) \geq N\left(f(4x), \frac{t}{2}\right)$$

(2.1)

for all $x, y, z \in X$ and all $t > 0$. Then $f$ is additive.

Proof. By letting $x = y = z = 0$ in (2.1), we get

$$N(3f(0), t) = N\left(f(0), \frac{t}{3}\right) \geq N\left(f(0), \frac{t}{2}\right)$$

for all $t > 0$. By $(N_5)$ and $(N_6)$, $N(f(0), t) = 1$ for all $t > 0$. It follows from $(N_2)$ that $f(0) = 0$. 

By letting $x = z = 0$ in (2.1), we get
\[ N(f(y) + f(-y) + f(0), t) \geq N\left(f(0), \frac{t}{2}\right) = 1 \]
for all $t > 0$. It follows from (N2) that $f(-y) + f(y) = 0$ for all $y \in X$. Thus
\[ f(-y) = -f(y) \]
for all $y \in X$.

By letting $x = 0$ and $l = y + z$ in (2.1), we get
\[ N(f(-l) + f(-z) + f(l + z), t) \geq N\left(f(0), \frac{t}{2}\right) = 1 \]
for all $t > 0$. It follows from (N2) that
\[ f(-l) + f(-z) + f(l + z) = 0 \]
for all $l, z \in X$. Thus
\[ f(l + z) = f(l) + f(z) \]
for all $l, z \in X$, as desired.

Now, we investigate the Hyers-Ulam stability of derivations on fuzzy Banach space for the functional equation
\[ Df(x, y, z) = 0 \]
for all $x, y, z \in A$.

**Theorem 2.2.** Let $\phi : A^3 \to [0, 1]$ be a function such that there exists an $L < 1$ with
\[ \phi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2} \phi(x, y, z) \quad (2.2) \]
for all $x, y, z \in A$. Let $f : A \to A$ be a mapping such that
\[ N(Df(x, y, z), t) \geq \frac{t}{t + \phi(x, y, z)}, \quad (2.3) \]
\[ N(f(xy) - f(x)y - xf(y), t) \geq \frac{t}{t + \phi(x, y, 0)} \quad (2.4) \]
for all $x, y, z \in A$, all $t > 0$. Then there exists a unique fuzzy derivation $\delta : A \to A$ such that
\[ N(f(x) - \delta(x), t) \geq \frac{2(1 - L)t}{2(1 - L)t + L\phi(x, 0, x)} \quad (2.5) \]
for all $x \in A$ and all $t > 0$.

**Proof.** By letting $\mu = 1, y = -x, z = x$ in (2.4), we have
\[ N(2f(x) - f(2x), t) \geq \frac{t}{t + \phi\left(\frac{x}{2}, \frac{x}{2}, 0\right)}, \quad (2.6) \]
and so
\[ N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t + \phi\left(\frac{x}{4}, \frac{x}{4}, 0\right)} \geq \frac{t}{t + \frac{L}{4}\phi(x, -x, 0)} \]
for all \( x \in A \). Thus

\[
N \left( 2f \left( \frac{x}{2} \right) - f(x), \frac{L}{4}t \right) \geq \frac{\frac{Lt}{4} + \frac{L}{4} \phi(x, -x, 0)}{\frac{Lt}{4} + \frac{L}{4} \phi(x, -x, 0)} = \frac{t}{t + \phi(x, -x, 0)} \tag{2.7}
\]

for all \( x \in A \).

Consider the set

\[
X := \{ g : A \to A \},
\]

and introduce the generalized metric on \( X \):

\[
d(g, h) := \inf \{ a \in \mathbb{R}^+ : N(g(x) - h(x), at) \geq \frac{t}{t + \phi(x, -x, 0)} \}
\]

for all \( x \in A \) and all \( t > 0 \), where \( a \in (0, \infty) \). It is easy to show that \( (X, d) \) is complete (see the proof of \cite[Lemma 2.1]{32}).

Now, we consider the linear mapping \( Q : X \to X \) such that

\[
Qg(x) := 2g \left( \frac{x}{2} \right)
\]

for all \( x \in A \).

Let \( g, h \in X \) be given such that \( d(g, h) = \epsilon \). Then

\[
N(g(x) - h(x), \epsilon t) \geq \frac{t}{t + \phi(x, -x, 0)}
\]

for all \( x \in A \) and all \( t > 0 \). Hence

\[
N(Qg(x) - Qh(x), Lt) = N \left( 2g \left( \frac{x}{2} \right) - 2h \left( \frac{x}{2} \right), Lt \right) = N \left( g \left( \frac{x}{2} \right) - h \left( \frac{x}{2} \right), \frac{L}{2}t \right)
\]

\[
\geq \frac{\frac{Lt}{2} + \phi \left( \frac{x}{2}, -x, 0 \right)}{\frac{Lt}{2} + \frac{L}{2} \phi(x, -x, 0)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2} \phi(x, -x, 0)}
\]

\[
= \frac{t}{t + \phi(x, -x, 0)}
\]

for all \( x \in A \) and all \( t > 0 \). Thus \( d(g, h) = \epsilon \) implies that \( d(Qg, Qh) \leq L \epsilon \). This means that

\[
d(Qg, Qh) \leq L d(g, h)
\]

for all \( g, h \in X \).

It follows from (2.7) that \( d(f, Qf) \leq \frac{L}{4} \).

By Theorem \cite{1,1} there exists a mapping \( \delta : A \to A \) satisfying the following:

1. \( \delta \) is a fixed point of \( Q \), i.e.,

\[
\delta \left( \frac{x}{2} \right) = \frac{1}{2} \delta(x), \tag{2.8}
\]

for all \( x \in A \). The mapping \( \delta \) is a unique fixed point of \( Q \) in the set

\[
M = \{ g \in G : d(f, g) < \infty \}.
\]

This implies that \( \delta \) is a unique mapping satisfying (2.8) such that there exists an \( a \in (0, \infty) \) satisfying

\[
N(f(x) - \delta(x), at) \geq \frac{t}{t + \phi(x, -x, 0)}
\]

for all \( x \in A \) and \( t > 0 \).
(2) \(d(Q^k f, \delta) \to 0\) as \(k \to \infty\). This implies the equality

\[
N - \lim_{k \to \infty} 2^k f \left( \frac{x}{2^k} \right) = \delta(x)
\]

for all \(x \in A\).

(3) \(d(f, \delta) \leq \frac{1}{1-L} d(f, Qf)\), which implies the inequality

\[
d(f, A) \leq \frac{L}{4(1-L)}.
\]

This implies that the inequality \((2.5)\) holds.

Next we show that \(\delta\) is additive. It follows from \((2.2)\) that

\[
\sum_{k=0}^{\infty} 2^k \phi \left( \frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k} \right) = \phi(x, y, z) + 2\phi \left( \frac{x}{2}, \frac{y}{2}, \frac{z}{2} \right) + 2^2 \phi \left( \frac{x}{2^2}, \frac{y}{2^2}, \frac{z}{2^2} \right) + \cdots
\]

\[
\leq \phi(x, y, z) + L\phi(x, y, z) + L^2 \phi(x, y, z) + \cdots
\]

\[
= \frac{1}{1-L} \phi(x, y, z) < \infty
\]

for all \(x, y, z \in A\).

By \((2.3)\),

\[
N \left( 2^k f \left( \frac{2x - y - z}{2^k} \right) \right) + 2^k f \left( \frac{x - z}{2^k} \right) + f \left( \frac{x + y + 2z}{2^k} \right) - 2^k f \left( \frac{4}{2^k} x, \right) - 2^{k-1} f \left( \frac{4}{2^k} x, \right) \geq \frac{t}{t + \phi \left( \frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k} \right)}
\]

and so

\[
N \left( 2^k f \left( \frac{2x - y - z}{2^k} \right) \right) + 2^k f \left( \frac{x - z}{2^k} \right) + 2^k f \left( \frac{x + y + 2z}{2^k} \right) - 2^k f \left( \frac{4}{2^k} x, \right) \geq \frac{t}{t + \phi \left( \frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k} \right)}
\]

for all \(x, y, z \in A\), all \(t > 0\) and all \(\mu \in \mathbb{T}^1\). Since \(\lim_{k \to \infty} \frac{t}{t + 2^{k+1} \phi \left( \frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k} \right)} = 1\), for all \(x, y, z \in A\) and all \(t > 0\),

\[
N \left( \delta \left( 2x - y - z \right) + \delta \left( x - z \right) + \delta \left( x + y + 2z \right) - \delta \left( 4x \right) \right), t = 1
\]

for all \(x, y, z \in X\), all \(t > 0\) and all \(\mu \in \mathbb{T}^1\). So

\[
\delta \left( 2x - y - z \right) + \delta \left( x - z \right) + \delta \left( x + y + 2z \right) = \delta \left( 4x \right)
\]

for all \(x, y, z \in A\), all \(t > 0\). By Lemma \((2.1)\) we get \(\delta(x)\) is an additive mapping.

It follows from \((2.4)\) that

\[
N \left( f \left( \frac{xy}{2^{2k}} \right) - \frac{y}{2^k} f \left( \frac{x}{2^k} \right) - \frac{x}{2^k} f \left( \frac{y}{2^k} \right), \right) = N \left( 2^k f \left( \frac{xy}{2^k} \right) - 2^{k-1} f \left( \frac{x}{2^k} \right) - 2^{k-1} f \left( \frac{y}{2^k} \right), \frac{t}{2^k} \right)
\]

\[
\geq \frac{t}{2^k} + \phi \left( \frac{x}{2^k}, \frac{y}{2^k}, 0 \right)
\]

\[
= \frac{t}{t + 2^{k+1} \phi \left( \frac{x}{2^k}, \frac{y}{2^k}, 0 \right)}
\]

\[
= \frac{t}{t + (2L)^k \phi \left( x, y, 0 \right)}
\]
for all \(x, y \in A\) and all \(t \geq 0\). Since \(\lim_{k \to \infty} \frac{t}{t + (2^n - 1) L \phi(x,y,0)} = 1\), for all \(x, y \in A\) and all \(t \geq 0\), we get
\[
\delta(xy) = y\delta(x) + x\delta(y)
\]
for all \(x, y \in A\).

**Corollary 2.3.** Let \(p\) be a real number with \(p > 1\), \(\theta \geq 0\), and \(X\) be a normed vector space with norm \(\|\cdot\|\). Let \(f : X \to X\) be a mapping satisfying
\[
N(Df(x,y,z),t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p + \|z\|^p)},
\]
\[
N(f(xy) - f(x)y - xf(y),t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}
\]
for all \(x, y, z \in A\), all \(t > 0\). Then there exists a unique derivation \(\delta : A \to A\) satisfying
\[
N(f(x) - \delta(x),t) \geq \frac{(2^p - 2)t}{(2^p - 2)t + \theta \|x\|^p}
\]
for all \(x \in X\) and all \(t > 0\).

**Proof.** The proof follows from Theorem 2.2 by taking
\[
\phi(x,y,z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p),
\]
and \(L = 3^{1-p}\).

### 3. Stability of derivations on fuzzy Lie Banach space

A fuzzy Banach space, endowed with the Lie product
\[
[x, y] := \frac{xy - yx}{2},
\]
on \(\mathbb{R}\), is called a fuzzy Lie Banach space.

**Definition 3.1.** Let \(A\) be a fuzzy Lie Banach space. An \(\mathbb{R}\)-linear mapping \(\delta : A \to A\) is called a fuzzy Lie derivation, if
\[
\delta([x,y]) = [\delta(x), y] + [x, \delta(y)]
\]
for all \(x, y \in A\).

In this section, suppose that \(A\) is a fuzzy Lie Banach space with norm \(N\). We prove the Hyers-Ulam stability of fuzzy Lie derivations on fuzzy Lie Banach space for the functional equation
\[
Df(x,y,z) = 0.
\]

**Theorem 3.2.** Let \(\phi : A^3 \to [0, \infty)\) be a function such that there exists an \(L < 1\) with
\[
3L\phi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \phi(x, y, z)
\]
for all \(x, y, z \in A\). Let \(f : A \to A\) be a mapping satisfying
\[
N(Df(x,y,z),t) \geq \frac{t}{t + \phi(x,y,z)},
\]
for all \(x, y, z \in A\).
and
\[ N(f([x, y]) - [f(x), y] - [x, f(y)], t) \geq \frac{t}{t + \phi(x, y, 0)}. \]

Then there exists a unique fuzzy Lie derivation \( \delta : A \to A \) satisfying
\[ N(f(x) - \delta(x), t) \geq \frac{2(1 - L)t}{2(1 - L)t + \phi(x, 0, x)} \quad (3.1) \]
for all \( x \in A \) and all \( t > 0 \).

Proof. Let \( (X, d) \) be the generalized metric space defined in the proof of Theorem 2.2. Consider the linear mapping \( Q : X \to X \) such that
\[ Qg(x) := \frac{1}{2} g(2x) \]
for all \( x \in A \).

It follow from (2.6) that
\[ N\left(f(x) - \frac{1}{2} f(2x), \frac{1}{2} t\right) \geq \frac{t}{t + \phi(x, 0, x)} \]
for all \( x \in A \) and all \( t > 0 \). Thus \( d(f, Qf) \leq \frac{1}{2} \). Hence
\[ d(f, A) \leq \frac{1}{2(1 - L)}, \]
which implies that the inequality (3.1) holds.

The rest of the proof is similar to the proof of Theorem 2.2.

Corollary 3.3. Let \( \theta \geq 0 \) and let \( p \) be a positive real number with \( p < 1 \). Let \( A \) be a fuzzy Lie Banach space with norm \( \| \cdot \| \). Let \( f : A \to A \) be a mapping satisfying
\[ N(Df(x, y, z), t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p + \|z\|^p)}, \]
and
\[ N(f([x, y]) - [f(x), y] - [x, f(y)], t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \]
for all \( x, y, z \in A \) and \( t > 0 \). Then there exists a unique fuzzy Lie derivation \( \delta : A \to A \) such that
\[ N(f(x) - \delta(x), t) \geq \frac{(2 - 2p)t}{(2 - 2p)t + \theta \|x\|^p} \]
for every \( x \in A \) and all \( t > 0 \).

Proof. The proof follows from Theorem 3.2 by taking
\[ \phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p), \]
and \( L = 3^{p-1} \).

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