Geraghty and Ćirić type fixed point theorems in $b$-metric spaces

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Abstract

In this paper, we obtain some fixed point theorems for admissible mappings in $b$-metric spaces. Some useful examples are given to illustrate the facts. We also discuss an application to a nonlinear quadratic integral equation. Our results complement, extend and generalize a number of fixed point theorems including the well-known Geraghty [M. A. Geraghty, Proc. Amer. Math. Soc., 40 (1973), 604–608] and Ćirić [L. B. Ćirić, Proc. Amer. Math. Soc., 45 (1974), 267–273] theorems on $b$-metric spaces. ©2016 All rights reserved.

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1. Introduction

Geraghty [13] and Ćirić [9] obtained two important generalizations of the classical Banach contraction principle (BCP) as follows:

Theorem 1.1 (13). Let $(X, d)$ be a complete metric space and $T : X \to X$ be a self-mapping such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y),$$

where $\beta : [0, \infty) \to [0, 1)$ is a function satisfying $\beta(t_n) \to 1$ implies $t_n \to 0$ as $n \to \infty$. Then $T$ has a unique fixed point $z \in X$.

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Theorem 1.2 ([9]). Let $X$ be a $T$-orbitally complete metric space and $T : X \to X$ be a quasi-contraction, that is, there exists a real number $r \in [0,1)$ such that for all $x, y \in X$,
\[ d(Tx, Ty) \leq r \, m(x, y), \]
where $m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Tx), d(y, Ty)\}$. Then $T$ has a unique fixed point $z \in X$.

As per Rhoades [18], a quasi-contraction on a metric space is the most general among contractions. Recently, Kumam et al. [16] introduced the notion generalized quasi-contraction and obtained an interesting generalization of Theorem 1.2.

Definition 1.3. A self-mapping $T$ of a metric space $X$ is called a generalized quasi-contraction, if there exists a number $r \in [0,1)$ such that for all $x, y \in X$,
\[ d(Tx, Ty) \leq r \, M(x, y), \]
where
\[ M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(T^2 x, x), d(T^2 x, Tx), d(T^2 x, Ty)\}. \]

Theorem 1.4 ([16]). Let $T$ be a generalized quasi-contraction on a $T$-orbitally complete metric space $X$. Then $T$ has a unique fixed point $z \in X$.

On the other hand, Samet et al. [19] introduced the concept of $\alpha$-$\psi$ contractive type mappings as well as $\alpha$-admissible mappings and established various results in complete metric spaces. Indeed, they extended and generalized many existing fixed point results in the literature. Subsequently, a number of extensions and generalizations of their results have appeared in [2, 3, 7, 8, 15, 21] and elsewhere. Motivated by Ćirić [9], Geraghty [13], Kumam et al. [16] and Samet et al. [19], in this paper we obtain some fixed point theorems for admissible mappings in $b$-metric spaces. Besides presenting some useful examples, we discuss an application to a nonlinear quadratic integral equation.

2. Preliminaries

For the sake of completeness, we recall some basic definitions and results.

Definition 2.1 ([9, 16]). Let $X$ be a metric space and $T : X \to X$ be a self-mapping. For each $x \in X$ and $n \in \mathbb{N}$, define
\[ O(x; n) = \{x, Tx, \ldots, T^n x\} \] and \[ O(x; \infty) = \{x, Tx, \ldots, T^n x, \ldots\}. \]
The set $O(x; \infty)$ is called the orbit of $T$ and the metric space $X$ is said be $T$-orbitally complete, if every Cauchy sequence in $O(x; \infty)$ is convergent in $X$.

Every complete metric space is $T$-orbitally complete for all mappings $T : X \to X$ but the converse is not true.

Example 2.2 ([16]). Let $X$ be a metric space which is not complete and $T : X \to X$, a mapping defined by $Tx = x_0$ for all $x \in X$ and some $x_0 \in X$. Then $X$ is a $T$-orbitally complete metric space but not complete.

In [10-12], Czerwik et al. introduced a wider class of metric spaces namely $b$-metric spaces and extended some fixed point theorems from metric spaces to these spaces. In recent years, a number of fixed point results for single-valued and multi-valued operators in $b$-metric spaces have been studied extensively in [4, 6, 10-12, 17, 20] and elsewhere.

Definition 2.3 ([10-12]). Let $X$ be a non-empty set and $d : X \times X \to [0, \infty)$ be a functional. Then $d$ is called a $b$-metric on $X$, if
(1) \(d(x, y) = 0\), if \(x = y\);
(2) \(d(x, y) = d(y, x)\);
(3) \(d(x, y) \leq s[d(x, z) + d(y, z)]\), where \(s \geq 1\).

The pair \((X, d)\) is called a \(b\)-metric space or a generalized metric space.

If we take \(s = 1\), we get the usual definition of a metric space. However, a \(b\)-metric on \(X\) needs not to be a metric on \(X\). Therefore the class of \(b\)-metrics is larger than the class of metrics.

The following examples are some known \(b\)-metric spaces.

**Example 2.4.** Let \(X = \{x_1, x_2, x_3\}\) and \(d : X \times X \to [0, \infty)\) be a function such that
\[
\begin{align*}
  d(x_1, x_2) &= a > 2, \\
  d(x_1, x_3) &= d(x_2, x_3) = 1, \\
  d(x_n, x_n) &= 0, \\
  d(x_n, x_k) &= d(x_k, x_n), \\
  d(x_n, x_k) &\leq \frac{a}{2}[d(x_n, x_i) + d(x_i, x_k)], \quad n, k, i \in \{1, 2, 3\}.
\end{align*}
\]

Then \((X, d)\) is a \(b\)-metric space.

**Example 2.5**. Let \(\mathbb{R}\) be the set of reals and \(\ell_p(\mathbb{R}) = \left\{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\right\}\) with \(0 < p < 1\). The functional \(d : \ell_p(\mathbb{R}) \times \ell_p(\mathbb{R}) \to \mathbb{R}\) defined by
\[
d(x, y) := \left(\sum_{k=1}^{\infty} |x_k - y_k|^p\right)^{1/p}, \quad \text{for all } x = \{x_n\}, \ y = \{y_n\} \in \ell_p(\mathbb{R}),
\]
is a \(b\)-metric on \(\ell_p(\mathbb{R})\) with coefficient \(s = 2^{1/p} > 1\).

Notice that the above result holds for the general case \(\ell_p(X)\) with \(0 < p < 1\), where \(X\) is a Banach space.

**Definition 2.6.** Let \(X\) be a \(b\)-metric space and \(\{x_n\}\) a sequence in \(X\). Then
(a) the sequence \(\{x_n\}\) is convergent, if there exists \(z \in X\) such that \(\lim_{n \to \infty} d(x_n, z) = 0\);
(b) the sequence \(\{x_n\}\) is Cauchy, if \(\lim_{n,m \to \infty} d(x_n, x_m) = 0\);
(c) \(X\) is complete, if every Cauchy sequence in \(X\) is convergent.

**Remark 2.7.** Also note that,
(d) every convergent sequence \(\{x_n\}\) in \(X\) has a unique limit;
(e) every convergent sequence \(\{x_n\}\) in \(X\) is Cauchy.

In general, a \(b\)-metric needs not to be a continuous functional.

**Example 2.8**. Let \(X = \mathbb{N} \cup \{\infty\}\) and \(d : X \times X \to [0, \infty)\) be defined by
\[
d(m, n) = \begin{cases}
  0 & \text{if } m = n, \\
  |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\
  5 & \text{if one of } m, n \text{ is odd and the other is odd (and } m \neq n \text{) or } \infty, \\
  2 & \text{otherwise}.
\end{cases}
\]

Then \((X, d)\) is a \(b\)-metric space (with \(s = 5/2\)). Let \(x_n = 2n\) for each \(n \in \mathbb{N}\). Then
\[
\lim_{n \to \infty} d(x_n, \infty) = \lim_{n \to \infty} d(2n, \infty) = \lim_{n \to \infty} \frac{1}{2n} = 0,
\]
but \(\lim_{n \to \infty} d(x_n, 1) = 2 \neq 5 = d(\infty, 1)\).
Definition 2.9 ([19]). Let \( \alpha : X \times X \to [0, \infty) \) be a functional. A mapping \( T : X \to X \) is said to be \( \alpha \)-admissible, if for all \( x, y \in X \),
\[
\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1.
\]

Definition 2.10 ([14]). The mapping \( T : X \to X \) is said to be triangular \( \alpha \)-admissible, if for all \( x, y, z \in X \),
(i) \( \alpha(x, y) \geq 1 \) implies \( \alpha(Tx, Ty) \geq 1 \);
(ii) \( \alpha(x, z) \geq 1 \) and \( \alpha(z, y) \geq 1 \) implies \( \alpha(x, y) \geq 1 \).

3. Generalized \( \alpha \)-quasi contraction

In this section, we obtain a Ćirić type result for admissible mappings. Now onwards, \( \mathbb{N} \) denotes the set of natural numbers and \( X \) a \( b \)-metric space \( (X, d) \), where \( d \) is continuous.

Definition 3.1. Let \( X \) be a \( b \)-metric space. A mapping \( T : X \to X \) is said to be generalized \( \alpha \)-quasi contraction, if there exists a functional \( \alpha : X \times X \to [0, \infty) \) and \( q < \frac{1}{r} \) such that
\[
\alpha(x, y)d(Tx, Ty) \leq qM(x, y).
\]

Our main result of this section is prefaced by the following lemmas.

Lemma 3.2 ([14]). Let \( T \) be a triangular \( \alpha \)-admissible mapping. Assume that there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \). Define a sequence \( \{x_n\} \) by \( x_n = T^n x_0 \). Then \( \alpha(x_m, x_n) \geq 1 \) for all \( m, n \in \mathbb{N} \) with \( m < n \).

Lemma 3.3. Let \( X \) be a \( b \)-metric space and \( T : X \to X \) be a generalized \( \alpha \)-quasi contraction satisfying the following conditions:

(A) \( T \) is triangular \( \alpha \)-admissible;
(B) there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \).

Then for all positive integers \( i, j \in \{1, 2, \cdots, n\}, (i < j) \)
\[
d(T^i x_0, T^j x_0) \leq q.\delta[O(x_0, n)].
\]

Proof. By assumption, there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \). Define \( x_n = T^n x_0 \) for all \( n \in \mathbb{N} \). Since \( T \) is triangular \( \alpha \)-admissible, from Lemma 3.2 it follows that
\[
\alpha(T^i x_0, T^j x_0) = \alpha(x_i, x_j) \geq 1, \quad \text{for } i, j \in \mathbb{N} \cup \{0\} \text{ with } i < j.
\]

Let \( 1 \leq i \leq n - 1 \) and \( 1 \leq j \leq n \). Then \( T^{i-1} x_0, T^i x_0, T^{j-1} x_0, T^j x_0 \in O(x_0, n) \). Since \( T \) is a generalized \( \alpha \)-quasi contraction, we have
\[
d(T^i x_0, T^j x_0) = d(TT^{i-1} x_0, TT^{j-1} x_0) \leq q.\max\{d(T^{i-1} x_0, T^{j-1} x_0), d(T^{i-1} x_0, TT^{i-1} x_0), d(T^{j-1} x_0, TT^{j-1} x_0), d(T^{i-1} x_0, TT^{j-1} x_0), d(T^{j-1} x_0, T^{i-1} x_0), d(T^{j-1} x_0, TT^{i-1} x_0), d(T^{j-1} x_0, TT^{j-1} x_0), d(T^{i-1} x_0, TT^{j-1} x_0), d(T^{i-1} x_0, TT^{j-1} x_0)\} \leq q.\delta[O(x_0, n)].
\]

This proves the lemma.

\[\square\]
Remark 3.4. It follows from the above lemma that if $T$ is a generalized $\alpha$-quasi contraction and $x_0 \in X$, then for every positive integer $n$, there exists a positive integer $k \leq n$ such that

$$d(x_0, T^k x_0) = \delta[O(x_0, n)].$$

Theorem 3.5. Let $X$ be a $T$-orbitally complete $b$-metric space (with constant $s \geq 1$) and $T : X \to X$ a generalized $\alpha$-quasi contraction satisfying conditions (A) and (B) of Lemma 3.3. Then $T$ has a fixed point in $X$.

Proof. By assumption, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = T^nx_0$ for all $n \in \mathbb{N}$. We show that the sequence $\{T^nx_0\}$ is a Cauchy sequence. By the triangle inequality and Lemma 3.3 and Remark 3.4, we have

$$d(x_0, T^kx_0) \leq s[d(x_0, Tx_0) + d(Tx_0, T^kx_0)]$$

$$\leq s[d(x_0, Tx_0) + q.\delta[O(x_0, n)]]$$

$$= s[d(x_0, Tx_0) + q.d(x_0, T^kx_0)].$$

Therefore,

$$\delta[O(x_0, n)] = d(x_0, T^kx_0) \leq \frac{s}{1-q.s}d(x_0, Tx_0).$$

Let $n$ and $m$ be positive integers with $n < m$. Since $T$ is a generalized $\alpha$-quasi contraction, it follows from Lemma 3.3 that

$$d(T^nx_0, T^mx_0) = d(TT^{n-1}x_0, TT^{m-1}x_0)$$

$$\leq \alpha(T^n x_0, T^m x_0)d(TT^{n-1}x_0, TT^{m-1}x_0)$$

$$\leq q.\max\{d(T^{n-1}x_0, T^{m-1}x_0), d(T^{n-1}x_0, TT^{n-1}x_0), d(T^{m-1}x_0, TT^{m-1}x_0), d(T^{n-1}x_0, T^{m-1}x_0), d(T^{m-1}x_0, TT^{m-1}x_0), d(T^{n-1}x_0, T^{m-1}x_0), d(T^{m-1}x_0, TT^{m-1}x_0)\}$$

$$= q.\max\{d(T^{n-1}x_0, T^{m-n}T^{n-1}x_0), d(T^{n-1}x_0, TT^{n-1}x_0),$$

$$d(T^{m-n}T^{n-1}x_0, T^{m-n+1}T^{n-1}x_0), d(T^{n-1}x_0, T^{m-n+1}T^{n-1}x_0),$$

$$d(T^{m-n}T^{n-1}x_0, TT^{m-n+1}T^{n-1}x_0), d(T^{m-n}T^{n-1}x_0, TT^{m-n+1}T^{n-1}x_0),$$

$$d(T^{n-1}x_0, T^{m-n}T^{n-1}x_0), d(T^{n-1}x_0, T^{m-n+1}T^{n-1}x_0)\}.$$

Since

$$O(T^{n-1}x_0, m-n+1) = \{T^{n-1}x_0, TT^{n-1}x_0, T^2T^{n-1}x_0, \ldots, T^{m-n}T^{n-1}x_0, T^{m-n+1}T^{n-1}x_0\},$$

the above inequality reduces to

$$d(T^nx_0, T^mx_0) \leq q.\delta[O(T^{n-1}x_0, m-n+1)]. \tag{3.1}$$

By Remark 3.4, there exists an integer $k_1, 1 \leq k_1 \leq m-n+1$ such that

$$\delta[O(T^{n-1}x_0, m-n+1)] = d(T^{n-1}x_0, T^{k_1}T^{n-1}x_0). \tag{3.2}$$

Again, by Lemma 3.3 we have

$$d(T^{n-1}x_0, T^{k_1}T^{n-1}x_0) = d(TT^{n-2}x_0, T^{k_1+1}T^{n-2}x_0)$$

$$\leq q.\delta[O(T^{n-2}x_0, k_1+1)]$$

$$\leq q.\delta[O(T^{n-2}x_0, m-n+2)].$$
Then (3.2) becomes

\[ \delta[O(T^{n-1}x_0, m - n + 1)] \leq q.\delta[O(T^{n-2}x_0, m - n + 2)]. \tag{3.3} \]

Therefore, from (3.1) and (3.3), we get

\[ d(T^n x_0, T^m x_0) \leq q.\delta[O(T^{n-1}x_0, m - n + 1)] \]
\[ \leq q^2.\delta[O(T^{n-2}x_0, m - n + 2)] \]
\[ \vdots \]
\[ \leq q^n.\delta[O(x_0, m)] \]
\[ \leq \frac{q^n s}{1 - qs} d(x_0, Tx_0). \]

Since \( \lim_{n \to \infty} q^n = 0 \), the sequence \( \{T^n x_0\} \) is Cauchy in \( X \). Since \( X \) is \( T \)-orbitally complete, there exists \( u \in X \) such that

\[ \lim_{n \to \infty} T^n x_0 = u. \]

By the triangular inequality, we get

\[ d(u, Tu) \leq s[d(u, T^{n+1}x_0) + d(Tu, T^{n+1}x_0)] \]
\[ = s[d(u, T^{n+1}x_0) + d(Tu, TT^n x_0)] \]
\[ \leq s[d(u, T^{n+1}x_0) + \alpha(u, T^n x_0)d(Tu, TT^n x_0)] \]
\[ \leq s[d(u, T^{n+1}x_0) + q \max\{d(T^n x_0, u), d(T^n x_0, TT^n x_0), d(u, Tu), d(T^n x_0, Tu), \]
\[ d(u, TT^n x_0), d(T^2T^n x_0, T^n x_0), d(T^2T^n x_0, TT^n x_0), d(T^2T^n x_0, Tu), d(T^2T^n x_0, Tu)\}] \]
\[ = s[d(u, T^{n+1}x_0) + q \max\{d(T^n x_0, u), d(T^n x_0, T^{n+1} x_0), d(u, Tu), d(T^n x_0, Tu), \]
\[ d(u, T^{n+1}x_0), d(T^{n+2} x_0, T^n x_0), d(T^{n+2} x_0, T^{n+1} x_0), d(T^{n+2} x_0, u), d(T^{n+2} x_0, Tu)\}] \]
\[ \leq s[d(u, T^{n+1}x_0) + q \max\{d(T^n x_0, u), s[d(T^n x_0, u) + d(u, T^{n+1} x_0)], d(u, Tu), \]
\[ s[d(T^n x_0, u) + d(u, Tu)], d(u, T^{n+1} x_0), s[d(T^{n+2} x_0, u) + d(u, T^n x_0)], \]
\[ s[d(T^{n+2} x_0, u) + d(u, T^{n+1} x_0)], d(T^{n+2} x_0, u), s[d(T^{n+2} x_0, u) + d(u, Tu)]\}. \]

By letting \( n \to \infty \), we get

\[ d(u, Tu) \leq qs \max\{d(u, Tu), sd(u, Tu)\} \]
\[ = qs^2 d(u, Tu). \]

Since \( q < \frac{1}{s^2} \), we get \( d(u, Tu) = 0 \). Hence \( u \) is a fixed point of \( T \).

\[ \square \]

**Corollary 3.6 (21).** Let \( (X, d) \) be a complete \( b \)-metric space (with constant \( s \geq 1 \)), \( \alpha : X \times X \to [0, \infty) \) a functional and \( T : X \to X \) be an \( \alpha \)-quasi-contraction, that is,

\[ \alpha(x, y)d(Tx, Ty) \leq qm(x, y) \]

for all \( x, y \in X \), where \( 0 \leq q < 1 \) and

\[ m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \]

Suppose that the following conditions hold:
(i) $T$ is $\alpha$-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.

If we set $q < \frac{1}{s^2 + s}$, then $T$ has a fixed point in $X$.

When $\alpha(x, y) = 1$ for all $x, y \in X$, we obtain the following results:

**Corollary 3.7.** *Theorem 1.4.*

**Corollary 3.8.** *Theorem 1.2.*

The following example shows the generality of Theorem 3.5 over 1.4.

**Example 3.9.** Let $X = [0, 4]$ be endowed with the $b$-metric $d : X \times X \to [0, \infty)$ defined by $d(x, y) = |x - y|^2$. Define $T : X \to X$ by

$$Tx = \begin{cases} 
\frac{x}{4} & \text{if } x \in [0, 1], \\
4 & \text{if } x \in (1, 4],
\end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 
2 & \text{if } (x, y) \in [0, 1], \\
0 & \text{otherwise}.
\end{cases}$$

Then $(X, d)$ is a $T$-orbitally complete $b$-metric space with $s = 2$.

If $x, y \in [0, 1]$, then

$$\alpha(x, y)d(Tx, Ty) = 2 \frac{|x/4 - y/4|^2}{4} = \frac{1}{8}|x - y|^2 = qd(x, y) \leq qM(x, y),$$

where $q = \frac{1}{8} < \frac{1}{4} = \frac{1}{s^2}$. If $x \in [0, 1]$ and $y \in (1, 4]$, then $\alpha(x, y)d(Tx, Ty) = 0 \leq qM(x, y)$. Now, if $x = 0$ and $y = 4$, then $d(T0, T4) = 16 = M(0, 4)$. Hence $d(T0, T4) > qM(0, 4)$ for any $q < 1$. Therefore, the contractive condition of Theorem 1.4 is not satisfied. Since $\alpha(x, y)d(Tx, Ty) = 0 \leq qM(x, y)$, the mapping $T$ is a generalized $\alpha$-quasi-contraction. Further, it is easy to check that $T$ is triangular $\alpha$-admissible. Therefore, the mapping $T$ satisfies all the conditions of Theorem 3.5 and $x = 0$ and $x = 4$ are the fixed points of $T$.

### 4. Geraghty type contractive mapping

In this section, we present some Geraghty type results for admissible mappings.

**Definition 4.1.** Let $X$ be a $b$-metric space, $T : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$. The mapping $T$ is said to be an $(\alpha, \beta)$-admissible mapping, if $\alpha(x, y) \geq 1$ and $\beta(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ and $\beta(Tx, Ty) \geq 1$ for all $x, y \in X$.

**Definition 4.2.** Let $\alpha, \beta : X \times X \to [0, \infty)$. A $b$-metric space $X$ is $(\alpha, \beta)$-regular, if $\{x_n\}$ is a sequence in $X$ such that $x_n \to x \in X$, $\alpha(x_n, x_{n+1}) \geq 1$ and $\beta(x_n, x_{n+1}) \geq 1$ for all $n$ and there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that $\alpha(x_{nk}, x_{nk+1}) \geq 1$, $\beta(x_{nk}, x_{nk+1}) \geq 1$ for all $k \in \mathbb{N}$. Also $\alpha(x, Tx) \geq 1$, $\beta(x, Tx) \geq 1$.

We need the following class of functions to prove certain results of this section:

1. $\Theta$ is a family of functions $\theta : [0, \infty) \to [0, 1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\theta(t_n) \to 1$ implies $t_n \to 0$;
2. $\Psi$ is a family of functions $\psi : [0, \infty) \to [0, \infty)$ such that $\psi$ is continuous, strictly increasing and $\psi(0) = 0$. 

**Definition 4.3.** Let $X$ be a $b$-metric space, $T : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$. A mapping $T$ is said to be $(\alpha, \beta)$-Geraghty type contractive mapping, if there exists $\theta \in \Theta$ such that for all $x, y \in X$, the following condition holds:

$$
\alpha(x, Tx)\beta(y, Ty)\psi(s^3 d(Tx, Ty)) \leq \theta(\psi(N(x, y)))\psi(N(x, y)),
$$

(4.1)

where $N(x, y) = \max\left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s} \right\}$ and $\psi \in \Psi$.

**Theorem 4.4.** Let $(X, d)$ be a complete $b$-metric space, $T : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$. Suppose the following conditions hold:

(A) $T$ is an $(\alpha, \beta)$-admissible mapping;

(B) $T$ is an $(\alpha, \beta)$-Geraghty type contractive mapping;

(C) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;

(D) either $T$ is continuous or $X$ is $(\alpha, \beta)$-regular.

Then $T$ has a unique fixed point.

**Proof.** By assumption, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in $X$ by $x_n = T^n x_0 = Tx_{n-1}$ for $n \in \mathbb{N}$. It is obvious that if $x_{n_k} = x_{n_k+1}$ for some $n_k \in \mathbb{N}$, then $x_{n_k}$ is a fixed point of $T$ and we are done. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since $T$ is $(\alpha, \beta)$-admissible, so

$$
\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1 \Rightarrow \alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1.
$$

By continuing this manner, we get $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$. Similarly $\beta(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$. From (4.1), we have

$$
\psi(d(x_{n+1}, x_{n+2})) = \psi(d(Tx_n, Tx_{n+1})) \leq \psi(s^3 d(Tx_n, Tx_{n+1})) \\
\leq \alpha(x_n, Tx_n)\beta(x_{n+1}, Tx_{n+1})\psi(s^3 d(Tx_n, Tx_{n+1})) \\
\leq \theta(\psi(N(x_n, x_{n+1})))\psi(N(x_n, x_{n+1})),
$$

where

$$
N(x_n, x_{n+1}) = \max\left\{ d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \frac{d(x_{n+1}, Tx_{n+1}) + d(x_{n+1}, Tx_{n})}{2s} \right\} \\
= \max\left\{ d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_{n+1}, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2s} \right\} \\
= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}.
$$

Now, if $N(x_n, x_{n+1}) = d(x_n, x_{n+1})$, then

$$
\psi(d(x_{n+1}, x_{n+2})) \leq \theta(\psi(N(x_n, x_{n+1})))\psi(N(x_n, x_{n+1})) \\
= \theta(\psi(N(x_n, x_{n+1})))\psi(d(x_{n+1}, x_{n+2})) \\
< \psi(d(x_{n+1}, x_{n+2})),
$$

a contradiction. Therefore $N(x_n, x_{n+1}) = d(x_n, x_{n+1})$. Now

$$
\psi(d(x_{n+1}, x_{n+2})) \leq \theta(\psi(N(x_n, x_{n+1})))\psi(N(x_n, x_{n+1})) \\
= \theta(\psi(N(x_n, x_{n+1})))\psi(d(x_n, x_{n+1})) \\
< \psi(d(x_n, x_{n+1})).
$$

(4.2)

Since $\psi$ is a strictly increasing mapping, the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and bounded from below. Thus, there exists $r \geq 0$ such that

$$
\lim_{n \to \infty} d(x_n, x_{n+1}) = r.
$$
From (4.2), we get
\[
\frac{\psi(d(x_{n+1},x_{n+2}))}{\psi(N(x_n,x_{n+1}))} \leq \theta(\psi(N(x_n,x_{n+1}))) < 1.
\] (4.3)

By letting \( n \to \infty \) in (4.3), we have \( 1 \leq \lim_{n \to \infty} \theta(\psi(N(x_n,x_{n+1}))) < 1. \)

That is, \( \lim_{n \to \infty} \theta(\psi(N(x_n,x_{n+1}))) = 1 \) and \( \theta \in \Theta \) implies \( \lim_{n \to \infty} \psi(N(x_n,x_{n+1})) = 0 \) which yields that
\[
r = \lim_{n \to \infty} d(x_n,x_{n+1}) = 0.
\] (4.4)

We show that \( \{x_n\} \) is a Cauchy sequence in \( X \). Suppose \( \{x_n\} \) is not Cauchy. Then there exists \( \epsilon > 0 \) and the subsequences \( \{x_{m_k}\} \) and \( \{x_{n_k}\} \) of \( \{x_n\} \) with \( n_k > m_k > k \) such that
\[
d(x_{n_k},x_{m_k}) \geq \epsilon, \tag{4.5}
\]
and \( n_k \) is the smallest number such that (4.5) holds. From (4.5) we get
\[
d(x_{n_k-1},x_{m_k}) < \epsilon. \tag{4.6}
\]

By using triangle inequality, (4.5) and (4.6) we have
\[
\epsilon \leq d(x_{n_k},x_{m_k}) \\
\leq s[d(x_{n_k},x_{n_k-1}) + d(x_{n_k-1},x_{m_k})] \\
< s[d(x_{n_k},x_{n_k-1}) + \epsilon]. \tag{4.7}
\]

By taking the upper limit as \( k \to \infty \) in (4.7) and using (4.4), we get
\[
\epsilon \leq \limsup_{k \to \infty} d(x_{n_k},x_{m_k}) < s\epsilon. \tag{4.8}
\]

From the triangle inequality, we have
\[
d(x_{n_k},x_{m_k}) \leq s[d(x_{n_k},x_{n_k+1}) + d(x_{n_k+1},x_{m_k})], \tag{4.9}
\]
and
\[
d(x_{n_k+1},x_{m_k}) \leq s[d(x_{n_k+1},x_{n_k}) + d(x_{n_k},x_{m_k})]. \tag{4.10}
\]

By taking the upper limit as \( k \to \infty \) in (4.9) and applying (4.4), (4.8) becomes
\[
\epsilon \leq s \left( \limsup_{k \to \infty} d(x_{n_k+1},x_{m_k}) \right),
\]
and taking the upper limit as \( k \to \infty \) in (4.10) gives
\[
\limsup_{k \to \infty} d(x_{n_k+1},x_{m_k}) \leq s.s\epsilon = s^2\epsilon.
\]

Thus
\[
\frac{\epsilon}{s} \leq \limsup_{k \to \infty} d(x_{n_k+1},x_{m_k}) \leq s^2\epsilon. \tag{4.11}
\]

Similarly, we get
\[
\frac{\epsilon}{s} \leq \limsup_{k \to \infty} d(x_{n_k},x_{m_k+1}) \leq s^2\epsilon. \tag{4.12}
\]

By triangular inequality, we have
\[
d(x_{n_k+1},x_{m_k}) \leq s[d(x_{n_k+1},x_{m_k+1}) + d(x_{m_k+1},x_{m_k})]. \tag{4.13}
\]
By taking the upper limit as \( k \to \infty \) in (4.13), from (4.4) and (4.11) we obtain that
\[
\frac{\epsilon}{s} \leq s \limsup_{k \to \infty} d(x_{nk+1}, x_{mk+1}).
\]
That is,
\[
\frac{\epsilon}{s^2} \leq \limsup_{k \to \infty} d(x_{nk+1}, x_{mk+1}).
\]
Again, by following the above process, we get
\[
\limsup_{k \to \infty} d(x_{nk+1}, x_{mk+1}) \leq s^3 \epsilon.
\]
From (4.14) and (4.15), we get
\[
\frac{\epsilon}{s^2} \leq \limsup_{k \to \infty} d(x_{nk+1}, x_{mk+1}) \leq s^3 \epsilon.
\]
Since \( X \) is \((\alpha, \beta)\)-regular, by (4.4) we have
\[
\psi(s^3 d(x_{nk+1}, x_{mk+1})) = \psi(s^3 d(Tx_{nk}, Tx_{mk})) \\
\leq \alpha(x_{nk}, Tx_{nk}) \beta(x_{mk}, Tx_{mk}) \psi(s^3 d(Tx_{nk}, Tx_{mk})) \\
\leq \theta(\psi(N(x_{nk}, x_{mk}))) \psi(N(x_{nk}, x_{mk})),
\]
where
\[
N(x_{nk}, x_{mk}) = \max \left\{ d(x_{nk}, x_{mk}), d(x_{nk}, Tx_{nk}), d(x_{mk}, Tx_{mk}), \frac{d(x_{nk}, Tx_{mk}) + d(x_{mk}, Tx_{nk})}{2s} \right\} \\
= \max \left\{ d(x_{nk}, x_{mk}), d(x_{nk}, x_{mk+1}), d(x_{mk}, x_{mk+1}), \frac{d(x_{nk}, x_{mk+1}) + d(x_{mk}, x_{mk+1})}{2s} \right\}.
\]
By taking limit supremum as \( k \to \infty \) in the above equation and using (4.4), (4.8), (4.11) and (4.12), we obtain
\[
\epsilon = \max \left\{ \epsilon, \frac{\epsilon}{s} + \frac{s^2 \epsilon}{2s} \right\} \leq \limsup_{k \to \infty} N(x_{nk}, x_{mk}) \leq \max \left\{ s\epsilon, \frac{s^2 \epsilon + s^2 \epsilon}{2s} \right\} = s\epsilon.
\]
Similarly, we can show that
\[
\epsilon = \max \left\{ \epsilon, \frac{\epsilon}{s} + \frac{s^2 \epsilon}{2s} \right\} \leq \liminf_{k \to \infty} N(x_{nk}, x_{mk}) \leq \max \left\{ s\epsilon, \frac{s^2 \epsilon + s^2 \epsilon}{2s} \right\} = s\epsilon.
\]
Hence, it follows from (4.14) that
\[
\psi(s\epsilon) = \psi(s^3 \left(\frac{\epsilon}{s^2}\right)) \\
\leq \psi(s^3 \limsup_{k \to \infty} d(x_{nk+1}, x_{mk+1})) \\
\leq \alpha(x_{nk}, x_{nk+1}) \beta(x_{mk}, x_{mk+1}) \psi(s^3 \limsup_{k \to \infty} d(x_{nk+1}, x_{mk+1})) \\
\leq \theta(\psi(\limsup_{k \to \infty} N(x_{nk}, x_{mk}))) \psi(\limsup_{k \to \infty} N(x_{nk}, x_{mk})) \\
\leq \theta(\psi(s\epsilon)) \psi(s\epsilon) \\
< \psi(s\epsilon),
\]
which is a contradiction. Therefore \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is complete, there exists \( x^* \in X \) such that \( x_n \to x^* \). First, suppose that \( T \) is continuous. Then we have

\[
x^* = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} Tx_n = T \lim_{n \to \infty} x_n = Tx^*.
\]

Now, suppose that \( X \) is \((\alpha, \beta)\)-regular. Then, there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \alpha(x_{n_k+1}, x_{n_k}) \geq 1 \) and \( \beta(x_{n_k+1}, x_{n_k}) \geq 1 \) for all \( k \in \mathbb{N} \) and \( \alpha(x^*, Tx^*) \geq 1 \) and \( \beta(x^*, Tx^*) \geq 1 \). Now from (4.1), with \( x = x_{n_k} \) and \( y = x^* \), we obtain

\[
\psi(d(x_{n_k+1}, Tx^*)) = \psi(d(Tx_{n_k}, Tx^*)) \\
\leq \psi(s^3 d(Tx_{n_k}, Tx^*)) \\
\leq \alpha(x_{n_k}, Tx_{n_k}) \beta(x^*, Tx^*) \psi(s^3 d(Tx_{n_k}, Tx^*)) \\
\leq \theta(\psi(N(x_{n_k}, x^*)) \psi(N(x_{n_k}, x^*))
\]

where

\[
N(x_{n_k}, x^*) = \max \left\{ d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), \frac{d(x_{n_k}, Tx^*) + d(x^*, Tx_{n_k})}{2s} \right\}
\]

\[
= \max \left\{ d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, Tx^*), \frac{d(x_{n_k}, Tx^*) + d(x^*, x_{n_k+1})}{2s} \right\}
\]

\[
\leq \max \left\{ d(x_{n_k}, x^*), s[d(x_{n_k}, x^*) + d(x_{n_k+1}, x^*)], d(x^*, Tx^*), \frac{s[d(x_{n_k}, x^*) + d(x^*, Tx^*)] + d(x^*, x_{n_k+1})}{2s} \right\}
\]

By letting \( k \to \infty \), we get

\[
\lim_{k \to \infty} N(x_{n_k}, x^*) \leq \max \left\{ d(x^*, Tx^*), \frac{d(x^*, Tx^*)}{2s} \right\} = d(x^*, Tx^*).
\]

Therefore, by taking the limit as \( k \to \infty \) in (4.16), we get

\[
\psi(d(x^*, Tx^*)) \leq \lim_{k \to \infty} \theta(\psi(N(x_{n_k}, x^*)) \psi(d(x^*, Tx^*)).
\]

That is, \( 1 \leq \lim_{k \to \infty} \theta(\psi(N(x_{n_k}, x^*))) \), which implies that \( \lim_{k \to \infty} \theta(\psi(N(x_{n_k}, x^*))) = 1 \). Consequently, we obtain \( \lim_{k \to \infty} N(x_{n_k}, x^*) = 0 \). Hence \( d(x^*, Tx^*) = 0 \), that is, \( x^* = Tx^* \).

Further, suppose that \( x^* \) and \( y^* \) are two fixed points of \( T \) such that \( x^* \neq y^* \) and \( \alpha(x^*, Tx^*) \geq 1 \), \( \alpha(y^*, Ty^*) \geq 1 \) and \( \beta(x^*, Tx^*) \geq 1 \), \( \beta(y^*, Ty^*) \geq 1 \). Now by applying (4.1), we have

\[
\psi(d(x^*, y^*)) = \psi(d(Tx^*, Ty^*)) \\
\leq \psi(s^3 d(Tx^*, Ty^*)) \\
\leq \alpha(x^*, Tx^*) \beta(y^*, Ty^*) \psi(s^3 d(Tx^*, Ty^*)) \\
\leq \theta(\psi(N(x^*, y^*)) \psi(N(x^*, y^*))
\]

where

\[
N(x^*, y^*) = \max \left\{ d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*), \frac{d(x^*, Ty^*) + d(y^*, Tx^*)}{2s} \right\}
\]

\[
= d(x^*, y^*).
\]

Hence, \( \psi(d(x^*, y^*)) \leq \theta(\psi(N(x^*, y^*)) \psi(d(x^*, y^*)) < \psi(d(x^*, y^*)) \), which is a contradiction unless \( d(x^*, y^*) = 0 \) and \( T \) has a unique fixed point.
Corollary 4.5. Let \((X, d)\) be a complete \(b\)-metric space, \(T : X \to X\) and \(\alpha, \beta : X \times X \to [0, \infty)\). Suppose the following conditions hold:

(a) \(T\) is an \(\alpha\)-admissible mapping;

(b) \(T\) is an \(\alpha\)-Geraghty type contractive mapping;

(c) there exists \(x_0 \in X\) such that \(\alpha(x_0, Tx_0) \geq 1\);

(d) either \(T\) is continuous or \(X\) is \(\alpha\)-regular.

Then \(T\) has a unique fixed point.

Example 4.6. Let \(X = [0, \infty)\) be endowed with the \(b\)-metric \(d : X \times X \to [0, \infty)\) defined by \(d(x, y) = |x - y|^2\). Then \((X, d)\) is a complete \(b\)-metric space with \(s = 2\). Let \(T : X \to X\) be defined by

\[
T_x = \begin{cases} 
  \frac{1 - x^2}{8} & \text{if } x \in [0, 1], \\
  \frac{8}{2x} & \text{otherwise.}
\end{cases}
\]

Define \(\alpha, \beta : X \times X \to [0, \infty), \theta : [0, \infty) \to [0, 1)\) and \(\psi : [0, \infty) \to [0, \infty)\) as

\[
\alpha(x, y) = \begin{cases} 
  \frac{3}{2} & \text{if } (x, y) \in [0, 1], \\
  1 & \text{otherwise.}
\end{cases}
\]

\[
\beta(x, y) = \begin{cases} 
  1 & \text{if } (x, y) \in [0, 1], \\
  0 & \text{otherwise.}
\end{cases}
\]

\[
\theta(t) = \frac{3}{4} \quad \text{and} \quad \psi(t) = t.
\]

First we show that \(T\) is an \((\alpha, \beta)\)-admissible mapping.

If \(x, y \in [0, 1]\), then \(\alpha(x, y) > 1\) and \(\beta(x, y) \geq 1\), \(T_x \leq 1\) and \(T_y \leq 1\). By the definition of \(\alpha\) and \(\beta\), it follows that \(\alpha(Tx, Ty) > 1\) and \(\beta(Tx, Ty) \geq 1\). Further, if \(\{x_n\}\) is a sequence in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) and \(\beta(x_n, x_{n+1}) \geq 1\) for all \(n \in \mathbb{N} \cup \{0\}\) and \(x_n \to x \in X\) as \(n \to \infty\), then \(x_n \subseteq [0, 1]\) and hence \(x \in [0, 1]\). This implies that \(\alpha(x, Tx) \geq 1\) and \(\beta(x, Tx) \geq 1\).

For \(x, y \in [0, 1]\), we have

\[
\alpha(x, Tx)\beta(y, Ty)\psi(s^3d(Tx, Ty)) = 12|Tx - Ty|^2
\]

\[
= \frac{3}{16} |x^2 - y^2|^2 = \frac{3}{16} |x - y|^2 |x + y|^2 \leq \frac{3}{4} |x - y|^2
\]

\[
= \theta(\psi(d(x, y)))\psi(d(x, y)) \leq \theta(\psi(M(x, y)))\psi(M(x, y)).
\]

Hence the contractive condition of Theorem 4.4 is satisfied. If \(x, y \in (1, \infty)\), then \(Tx > 1\) and \(\alpha(x, Tx) \geq 1\). Then we have

\[
\alpha(x, Tx)\psi(s^3d(Tx, Ty)) = \frac{8}{16} |2x - 2y|^2
\]

\[
= 32|x - y|^2 \geq \theta(\psi(M(x, y)))\psi(M(x, y)).
\]

Hence the contractive condition of Corollary 4.5 is not satisfied by \(T\). However,

\[
\alpha(x, Tx)\beta(y, Ty)\psi(s^3d(Tx, Ty)) = 0 \leq \theta(\psi(M(x, y)))\psi(M(x, y)).
\]

Again, if \(x \in [0, 1]\) and \(y > 1\), \(\alpha(x, Tx)\beta(y, Ty)\psi(s^3d(Tx, Ty)) = 0 \leq \theta(\psi(M(x, y)))\psi(M(x, y))\). Therefore, all the conditions of Theorem 4.4 are satisfied and \(T\) has a fixed point \(x^* = \sqrt{17} - 4\).

5. Applications to nonlinear integral equations

In this section, we discuss an application to nonlinear quadratic integral equation.

Consider the integral equation

\[
x(t) = h(t) + \lambda \int_0^1 k(t, s)f(s, x(s))ds, \quad t \in I = [0, 1], \quad \lambda \geq 0.
\]

(5.1)
Proof. Define an operator $C$ solution in Theorem 5.1. Under assumptions (a)-(e) the space $X = C(I)$ of continuous functions defined on $I = [0, 1]$ with the standard metric given by

$$
\rho(x, y) = \sup_{t \in I} |x(t) - y(t)| \quad \text{for } x, y \in C(I).
$$

Now, for $p \geq 1$, we define

$$
d(x, y) = (\rho(x, y))^p = \left(\sup_{t \in I} |x(t) - y(t)|\right)^p = \sup_{t \in I} |x(t) - y(t)|^p, \quad \text{for } x, y \in C(I).
$$

Then $(X, d)$ is a complete $b$-metric space with $s = 2^{p-1}$ (cf. [1], [2]).

**Theorem 5.1.** Under assumptions (a)-(e) the nonlinear quadratic integral equation (5.1) has a unique solution in $C(I)$.

Proof. Define an operator $T : X \to X$ by

$$
Tx(t) = h(t) + \lambda \int_0^1 k(t, s)f(s, x(s))ds, \quad t \in I = [0, 1], \quad \lambda \geq 0.
$$

Now, for $x, y \in X$, we have

$$
|Tx(t) - Ty(t)| = \left| h(t) + \lambda \int_0^1 k(t, s)f(s, x(s))ds - h(t) - \lambda \int_0^1 k(t, s)f(s, y(s))ds \right|
$$

$$
\leq \lambda \int_0^1 k(t, s)|f(s, x(s)) - f(s, y(s))|ds
$$

$$
\leq \lambda \int_0^1 k(t, s)L|x(s) - y(s)|ds.
$$

Since $|x(s) - y(s)| \leq \sup_{s \in I} |x(s) - y(s)| = \rho(x, y)$, we get

$$
|Tx(t) - Ty(t)| \leq \lambda KL \rho(x, y).
$$

Now, we can write

$$
d(Tx, Ty) = \sup_{t \in I} |Tx(t) - Ty(t)|^p
$$

$$
\leq (\lambda KL(p(x, y)))^p
$$

$$
\leq \lambda^p K^p L^p d(x, y)
$$

$$
\leq \frac{1}{2^{3p-3}} M(x, y).
$$

Therefore, all the assumptions of Corollary 3.7 are satisfied by the operator $T$ and (5.1) has a unique solution in $C(I)$. \qed
Example 5.2. Consider the following functional integral equation:

\[
x(t) = \frac{t}{1+t^2} + \frac{1}{18} \int_0^t \frac{s}{9e^t(1+t)} \frac{|x(s)|}{1+|x(s)|} \, ds, \quad t \in I = [0, 1].
\]

It is observed that the above equation is a special case of (5.1) with

\[
h(t) = \frac{t}{1+t^2},
\]

\[
k(t, s) = \frac{s}{1+t},
\]

\[
f(t, x) = \frac{|x|}{9e^t(1+|x|)}.
\]

Now, for arbitrary \(x, y \in \mathbb{R}\) such that \(x \geq y\) and for \(t \in [0, 1]\), we obtain

\[
|f(t, x) - f(t, y)| = \left| \frac{|x|}{9e^t(1+|x|)} - \frac{|y|}{9e^t(1+|y|)} \right|
\]

\[
= \frac{1}{9e^t} \left| \frac{|x|}{1+|x|} - \frac{|y|}{1+|y|} \right|
\]

\[
\leq \frac{1}{9} |x - y|.
\]

Thus, \(f\) satisfies condition (b) of the integral equation (5.1) with \(L = \frac{1}{9}\). It can be easily seen that \(h\) is a continuous function and \(k\) satisfies condition (c) with

\[
\int_0^1 k(t, s) \, ds = \int_0^t \frac{s}{1+t} \, ds = \frac{1}{2(1+t)} \leq \frac{1}{2} = K.
\]

By substituting \(L = \frac{1}{9}\), \(K = \frac{1}{2}\) and \(\lambda = \frac{1}{18}\) in condition (d), we obtain

\[
\frac{1}{9^p} \times \frac{1}{18^p} \times \frac{1}{2^p} \leq \frac{1}{23^{p-3}}.
\]

The above inequality is true for each \(p \geq 1\). Consequently, all the conditions of Theorem 5.1 are satisfied and hence the integral equation (5.1) has a unique solution in \(C(I)\).

References


