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On fuzzy normed algebras

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Abstract

In this paper, a characterization for continuous product in a fuzzy normed algebra is established and it is proved that any fuzzy normed algebra is with continuous product. Another type of continuity for the product in a fuzzy normed algebras is introduced and studied. These concepts are illustrated by some examples. Also, the Cartesian product of fuzzy normed algebras is analyzed.

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1. Introduction

Fuzzy logic and fuzzy sets introduced by Zadeh in his famous paper [19] have quickly found their applicability in a wide variety of domains: control engineering, artificial intelligence, computer science, robotics and many more. At the same time, many mathematicians have tried to translate the classical results of mathematics in fuzzy context. This approach is motivated by the fact that fuzzy theory has proved a useful tool to describe situations in which data are imprecise.

An important issue is finding a suitable definition for the fuzzy norm. In the study of fuzzy topological vector spaces, Katsaras [10] was the one who first introduced the notion of fuzzy norm. In 1992, Felbin [8] introduced another concept of fuzzy norm by assigning a fuzzy real number to each element of the linear space. In 1994, Cheng and Mordenson [6] presented another idea of fuzzy norm on a linear space and in this situation the corresponding fuzzy metric is of Kramosil and Michálek type. Following Cheng and Mordenson, in 2003, Bag and Samanta [3] introduced a new concept of fuzzy norm and they studied the properties of finite dimensional fuzzy normed linear space. In the paper [4], Bag and Samanta introduced

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different types of continuities and boundedness for linear operators and established the principles of fuzzy functional analysis. A comparative study on fuzzy norms introduced by Katsaras, by Felbin and by Bag and Samanta was made in paper [5]. A new concept of fuzzy normed space was introduced by Saadati and Vaezpour in [15]. Other approaches for fuzzy normed linear spaces can be found in [1, 2, 9, 11, 12, 14, 17] etc. In this paper we will use the definition of the fuzzy norm introduced by Nădăban and Dzitac [14].

From the notion of fuzzy normed linear space to the concept of fuzzy normed algebra it is one step that had to be done. Thus in paper [16], Sadeqi and Amiripour gave a definition of fuzzy Banach algebra and established some results in this context. We also note that Dinda et al. introduced and studied some important properties of intuitionistic fuzzy Banach algebra in the paper [7]. In this paper, we will use the concept of fuzzy normed algebra recently introduced by Mirmostafaee [11] in 2012.

A characterization for continuous product in a fuzzy normed algebra is established and it is proved that any fuzzy normed algebra is with continuous product. Another type of continuity for the product in a fuzzy normed algebras is introduced and studied. These concepts are illustrated by some examples. Also, the Cartesian product of fuzzy normed algebras is analyzed. The results obtained in this paper constitute a foundation for the development of some spectral properties in fuzzy normed algebras.

2. Preliminaries

Definition 2.1 ([18]). A binary operation

$$* : [0,1] \times [0,1] \to [0,1]$$

is called triangular norm (t-norm) if it satisfies the following condition:

1. $a * b = b * a, \ \forall \ a, b \in [0,1]$;
2. $a * 1 = a, \ \forall \ a \in [0,1]$;
3. $(a * b) * c = a * (b * c), \ \forall \ a, b, c \in [0,1]$;
4. If $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$, then $a * b \leq c * d$.

Example 2.2. Three basic examples of continuous t-norms are $\wedge, \cdot, *_L$, which are defined by $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$ (usual multiplication in $[0,1]$) and $a *_L b = \max\{a + b - 1, 0\}$ (the Lukasiewicz t-norm).

Definition 2.3 ([13]). Let $*, *'$ be two t-norms. We say that $*'$ dominates $*$ and we denote $*' \gg *$ if $(x_1 *' x_2) * (y_1 *' y_2) \leq (x_1 * y_1) *' (x_2 * y_2)$ for all $x_1, x_2, y_1, y_2 \in [0,1]$.

Definition 2.4 ([14]). Let X be a vector space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and $*$ be a continuous t-norm. A fuzzy set N in $X \times [0, \infty)$ is called a fuzzy norm on X if it satisfies:

(N1) $N(x, 0) = 0, \ \forall \ x \in X$;

(N2) $[N(x,t) = 1, \ \forall \ t > 0]$ if and only if $x = 0$;

(N3) $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right), \ \forall \ x \in X, \ \forall \ t \geq 0, \forall \ \lambda \in \mathbb{K}^*$;

(N4) $N(x + y, t + s) \geq N(x,t) * N(y,s), \ \forall \ x, y \in X, \ \forall \ t, s \geq 0$;

(N5) $\forall \ x \in X$, $N(x, \cdot)$ is left continuous and $\lim\limits_{t \to \infty} N(x,t) = 1$.

The triple $(X, N, *)$ will be called fuzzy normed linear space.

Remark 2.5. $N(x, \cdot)$ is nondecreasing for all $x \in X$.

Theorem 2.6 ([14]). *Let $(X, N, *)$ be a fuzzy normed linear space. For $x \in X$, $r \in (0,1), t > 0$ we define the open ball*

$$B(x, r, t) := \{y \in X \ : \ N(x - y, t) > 1 - r\}.$$

Then

1. $\mathcal{T}_N := \{T \subset X \,:\, x \in T$ *iff* $\exists\, t > 0, r \in (0,1) \,:\, B(x,r,t) \subseteq T\}$ *is a topology on* X*;*
2. *If the t-norm* $*$ *satisfies* $\sup\limits_{x\in(0,1)} x * x = 1$*, then* $(X, \mathcal{T}_N)$ *is Hausdorff;*
3. $(X, \mathcal{T}_N)$ *is a metrizable topological vector space.*

Definition 2.7 ([4])**.** Let $(X, N_1, *_1)$, $(Y, N_2, *_2)$ be fuzzy normed linear spaces. A mapping $T : X \to Y$ is said to be fuzzy continuous at $x_0 \in X$, if

$$\forall \varepsilon > 0,\ \forall \alpha \in (0,1),\ \exists\, \delta = \delta(\varepsilon, \alpha, x_0) > 0,\ \exists\, \beta = \beta(\varepsilon, \alpha, x_0) \in (0,1)$$

such that

$$\forall x \in X : N_1(x - x_0, \delta) > \beta \text{ we have that } N_2(T(x) - T(x_0), \varepsilon) > \alpha.$$

If T is fuzzy continuous at each point of X, then T is called fuzzy continuous on X.

Definition 2.8 ([3])**.** Let $(X, N, *)$ be a fuzzy normed linear space and (x_n) be a sequence in X.

1. The sequence (x_n) is said to be convergent if there exists $x \in X$ such that
$$\lim_{n\to\infty} N(x_n - x, t) = 1, \ \ \forall t > 0.$$
In this case, x is called the limit of the sequence (x_n) and we denote $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$.
2. The sequence (x_n) is called Cauchy sequence if
$$\lim_{n\to\infty} N(x_{n+p} - x_n, t) = 1, \ \ \forall\, t > 0, \forall p \in \mathbb{N}^*.$$
3. $(X, N, *)$ is said to be complete if any Cauchy sequence in X converges to a point in X. A complete fuzzy normed linear space will be called a fuzzy Banach space.

3. Fuzzy normed algebras

Definition 3.1 ([11])**.** It is called fuzzy normed algebra the quadruplet $(X, N, *, \circ)$ if we have

(A1) $*, \circ$ are continuous t-norms;

(A2) X is an algebra;

(A3) $(X, N, *)$ is a fuzzy normed linear space;

(A4) $N(xy, ts) \geq N(x,t) \circ N(y,s)\, \forall x, y \in X,\ \forall t, s \geq 0.$

If $(X, N, *)$ is a fuzzy Banach space, then $(X, N, *, \circ)$ will be called fuzzy Banach algebra.

Example 3.2. Let $(X, ||\cdot||)$ be a normed algebra, $*, \circ$ be continuous t-norms and

$$N : X \times [0,\infty) \to [0,1] \text{ defined by } N(x,t) = \begin{cases} 0 & , \quad t \leq ||x|| \\ 1 & , \quad t > ||x|| \end{cases}.$$

Then $(X, N, *, \circ)$ is a fuzzy normed algebra.

Proof. It is easy to check (N1)-(N3) and (N5). We verify the condition (N4). Let $x, y \in X, t, s \in [0, \infty)$. If $||x+y|| \geq t+s$, then $t \leq ||x||$ or $s \leq ||y||$ (contrarily $t > ||x||$ and $s > ||y||$, thus $t+s > ||x||+||y|| \geq ||x+y||$, contradiction). If $t \leq ||x||$, then $N(x,t) = 0$. If $s \leq ||y||$, then $N(y,s) = 0$. Thus $N(x,t) * N(y,s) = 0$. Therefore the inequality $N(x+y, t+s) \geq N(x,t) * N(y,s)$ holds. If $||x+y|| < t+s$, then $N(x+y, t+s) = 1$ and the inequality $N(x+y, t+s) \geq N(x,t) * N(y,s)$ holds.

It remains to verify (A4). Let $x, y \in X, t, s \in [0,\infty)$. If $||xy|| \geq ts$, then $t \leq ||x||$ or $s \leq ||y||$ (contrarily $t > ||x||$ and $s > ||y||$, thus $ts > ||x|| \cdot ||y|| \geq ||xy||$, contradiction). If $t \leq ||x||$, then $N(x,t) = 0$. If $s \leq ||y||$, then $N(y,s) = 0$. Thus $N(x,t) \circ N(y,s) = 0$. Therefore the inequality $N(xy, ts) \geq N(x,t) \circ N(y,s)$ holds. If $||xy|| < ts$, then $N(xy, ts) = 1$ and the inequality $N(xy, ts) \geq N(x,t) \circ N(y,s)$ holds. □

Example 3.3. Let $(X, ||\cdot||)$ be a normed algebra and $N : X \times [0,\infty) \to [0,1]$ defined by

$$N(x,t) := \begin{cases} \dfrac{t}{t+||x||} & , \quad t > 0 \\ 0 & , \quad t = 0 \end{cases} .$$

Then $(X, N, \wedge, \cdot)$ is a fuzzy normed algebra.

Proof. By [4], $(X, N, \wedge)$ is a fuzzy normed linear space. It remains to verify (A4), that is

$$N(xy, ts) \geq N(x,t) \cdot N(y,s), \ \forall x, y \in X, \forall t, s \in [0,\infty).$$

For $t = 0$ or $s = 0$ the inequality is obvious. For $t \neq 0$ and $s \neq 0$, the inequality is equivalent to

$$\frac{ts}{ts + ||xy||} \geq \frac{t}{t+||x||} \cdot \frac{s}{s+||y||} ,$$

namely $ts + t||y|| + s||x|| + ||x|| \cdot ||y|| \geq ts + ||xy||$, which is evidently true. □

Example 3.4. Let $(X, ||\cdot||)$ be a normed algebra and $N : X \times [0,\infty) \to [0,1]$ defined by

$$N(x,t) := \begin{cases} \dfrac{t}{t+||x||} & , \quad t > 0 \\ 0 & , \quad t = 0 \end{cases} .$$

Then $(X, N, \cdot, \cdot)$ is a fuzzy normed algebra.

Proof. We will prove that $(X, N, \cdot)$ is a fuzzy normed linear space. According to [4], conditions (N1)-(N3) and (N5) are verified. It remains to prove (N4), that is,

$$N(x+y, t+s) \geq N(x,t) \cdot N(y,s), \ \ \forall x, y \in X, \ \forall t, s \geq 0.$$

Indeed, for $t = 0$ or $s = 0$ the inequality is obviously true. For $t \neq 0$ and $s \neq 0$, the inequality is equivalent to

$$\frac{t+s}{t+s+||x+y||} \geq \frac{t}{t+||x||} \cdot \frac{s}{s+||y||},$$

namely $(t+s)(t+||x||)(s+||y||) \geq ts(t+s+||x+y||)$, which is equivalent to

$$ts(||x||+||y||) + s^2||x|| + t^2||y|| + (t+s)||x||||y|| \geq ts||x+y||.$$

Because $ts(||x||+||y||) \geq ts||x+y||$ and all the other terms from the left member are positive, the inequality follows. Therefore $(X, N, \cdot)$ is a fuzzy normed linear space. Moreover, conditions (A1)-(A4) are satisfied similar to the proof from the previous example. It follows $(X, N, \cdot, \cdot)$ is a fuzzy normed algebra. □

Theorem 3.5. *A fuzzy normed algebra $(X, N, *, \circ)$ is with continuous product if and only if*

$$\forall \alpha \in (0,1), \exists \beta = \beta(\alpha) \in (0,1), \exists M = M(\alpha) > 0 \textit{ such that}$$

$$\forall x, y \in X, \ \forall s, t > 0 \ : \ N(x,s) > \beta, N(y,t) > \beta \Rightarrow N(xy, Mst) > \alpha.$$

Proof. ($\Rightarrow$) Let $\alpha \in (0,1)$ and $V := \{u \in X \ : \ N(u,1) > \alpha\}$ be an open neighbourhood of zero. As $X \times X \ni (x,y) \mapsto x \cdot y \in X$ is continuous at $(0,0)$, there exist $\epsilon_1 = \epsilon_1(\alpha) > 0, \ \epsilon_2 = \epsilon_2(\alpha) > 0, \ \gamma_1 = \gamma_1(\alpha) \in (0,1)$, $\gamma_2 = \gamma_2(\alpha) \in (0,1)$ such that

$$\forall u_1, u_2 \in X \ : \ N(u_1, \epsilon_1) > \gamma_1, N(u_2, \epsilon_2) > \gamma_2 \text{ we have that } N(u_1 u_2, 1) > \alpha.$$

Let $\beta = \max\{\gamma_1, \gamma_2\} \in (0,1), M = \frac{1}{\epsilon_1\epsilon_2} > 0$. Let $x, y \in X, s, t > 0$ such that $N(x,s) > \beta, N(y,t) > \beta$. Then $N(x/s, 1) > \beta \geq \gamma_1$ and $N(y/t, 1) > \beta \geq \gamma_2$. Let $u_1 = \frac{\epsilon_1 x}{s}, u_2 = \frac{\epsilon_2 y}{t}$. We note that

$$N(u_1, \epsilon_1) = N(u_1/\epsilon_1, 1) = N(x/s, 1) > \gamma_1,$$
$$N(u_2, \epsilon_2) = N(u_2/\epsilon_2, 1) = N(u/t, 1) > \gamma_2.$$

Hence $N(u_1u_2, 1) > \alpha$, i.e., $N\left(\frac{\epsilon_1 x}{s} \cdot \frac{\epsilon_2 y}{t}, 1\right) > \alpha$, namely $N\left(xy, \frac{st}{\epsilon_1\epsilon_2}\right) > \alpha$. Thus $N(xy, Mst) > \alpha$.

($\Leftarrow$) First we will prove that for each $y_0 \in X$, the mapping

$$X \ni x \mapsto xy_o \in X$$

is continuous.

Let $\epsilon > 0, \alpha \in (0,1)$. Thus there exist $\beta = \beta(\alpha) \in (0,1), M = M(\alpha) > 0$ such that

$$N(x,s) > \beta, N(y,t) > \beta \Rightarrow N(xy, Mst) > \alpha.$$

As $\lim\limits_{t\to\infty} N(y_0, t) = 1$, there exists $t_0 > 0$ such that $N(y_0, t_0) > \beta$. Let $\delta = \delta(\alpha, \epsilon) = \frac{\epsilon}{t_0 M}$ and $\beta(\alpha, \epsilon) = \beta$. Let $x \in X$ such that $N(x, \delta) > \beta$. As $N(y_0, t_0) > \beta$, we obtain that $N(xy_0, Mt_0\delta) > \alpha$, namely $N(xy_0, \epsilon) > \alpha$.

Similarly, we can establish that, for each $x_0 \in X$, the mapping

$$X \ni y \mapsto x_0 y \in X$$

is continuous.

Now, we will prove that $(X, N, *, \circ)$ is with continuous product. Let $x_n \to x_0, y_n \to y_0$. Thus $x_n y_0 \to x_0 y_0$ and $x_0 y_n \to x_0 y_0$. Hence $\lim\limits_{n\to\infty} N(x_n y_0 - x_0 y_0, s) = 1, \lim\limits_{n\to\infty} N(x_0 y_n - x_0 y_0, t) = 1$ for all $s, t > 0$. Therefore

$$\begin{aligned}
&N(x_n y_n - x_0 y_0, t)\\
&\quad = N((x_n - x_0)(y_n - y_0) + (x_n - x_0)y_0 + x_0(y_n - y_0), t)\\
&\quad \geq N\left((x_n - x_0)(y_n - y_0), \frac{t}{3}\right) * N\left((x_n - x_0)y_0, \frac{t}{3}\right) * N\left(x_0(y_n - y_0), \frac{t}{3}\right)\\
&\quad \geq \left(N\left(x_n - x_0, \sqrt{\frac{t}{3}}\right) \circ N\left(y_n - y_0, \sqrt{\frac{t}{3}}\right)\right) * N\left((x_n - x_0)y_0, \frac{t}{3}\right) * N\left(x_0(y_n - y_0), \frac{t}{3}\right)\\
&\quad \to 1.
\end{aligned}$$

Hence $x_n y_n \to x_0 y_0$. □

Lemma 3.6. *Any continuous t-norm $*$ satisfies:*

$$\forall \gamma \in (0,1), \exists\ \alpha, \beta \in (0,1) \text{ such that } \alpha * \beta = \gamma.$$

Proof. Let $\gamma \in (0,1)$. Choose $\alpha > \gamma$. Let $g : [0,1] \to [0,1]$ defined by $g(y) = \alpha * y$. As $*$ is continuous, we have that g is continuous. As $g(0) = \alpha * 0 = 0$ and $g(1) = \alpha * 1 = \alpha$, for $\gamma \in (0, \alpha)$ there exists $\beta \in (0,1)$ such that $g(\beta) = \gamma$, namely $\alpha * \beta = \gamma$. □

Theorem 3.7. *Any fuzzy normed algebra $(X, N, *, \circ)$ is with continuous product.*

Proof. Let $\alpha \in (0,1)$. Then there exists $\varepsilon > 0$ such that $\alpha + \varepsilon \in (0,1)$. As $\circ$ is a continuous t-norm, by the previous lemma, we obtain that there exist $\beta_\alpha, \gamma_\alpha \in (0,1)$ such that $\alpha + \varepsilon = \beta_\alpha \circ \gamma_\alpha$. We suppose that $\beta_\alpha \geq \gamma_\alpha$ (the case $\beta_\alpha \leq \gamma_\alpha$ is similar). We choose $M = M(\alpha) = 1$. Let $x, y \in X, s, t > 0$ such that $N(x,s) > \beta_\alpha, N(y,t) > \beta_\alpha$. Then

$$N(xy, Mst) \geq N(x,s) \circ N(y,t) \geq \beta_\alpha \circ \beta_\alpha \geq \beta_\alpha \circ \gamma_\alpha = \alpha + \varepsilon > \alpha.$$

□

Definition 3.8. The fuzzy normed algebra $(X, N, *, \circ)$ is called with multiplicatively continuous product if

$$\forall \alpha \in (0,1),\ \forall x, y \in X,\ \forall s, t > 0 : N(x,s) > \alpha, N(y,t) > \alpha \Rightarrow N(xy, st) \geq \alpha.$$

Example 3.9 (Fuzzy normed algebra with multiplicatively continuous product)**.** The fuzzy normed algebra $(X, N, *, \circ)$ from Example 3.2 is with multiplicatively continuous product.

Proof. Indeed, let $\alpha \in (0,1), x, y \in X, s, t > 0$ such that $N(x,s) > \alpha, N(y,t) > \alpha$. Then $N(x,s) = 1, N(y,t) = 1$. Thus $||x|| < s, ||y|| < t$. Therefore $||xy|| \leq ||x|| \cdot ||y|| < st$. Hence $N(xy, st) = 1 > \alpha$. □

Example 3.10 (Fuzzy normed algebra which is not with multiplicatively continuous product)**.** We consider the fuzzy normed algebra from Example 3.3, where $X = \mathbb{R}$ and the norm on X is the absolute value $|\cdot|$. Then $(\mathbb{R}, N, \wedge, \cdot)$ is not with multiplicatively continuous product.

Proof. Indeed, for $\alpha = \frac{1}{5}, x = \frac{5}{2}s, y = \frac{5}{2}t, s, t > 0$ we have that $N(x,s) = \frac{s}{s+|x|} = \frac{s}{s+\frac{5}{2}s} = \frac{2}{7} > \frac{1}{5}$ and $N(y,t) > \frac{1}{5}$. But $N(xy, st) = \frac{st}{st+|xy|} = \frac{st}{st+\frac{25}{4}st} = \frac{4}{29} < \frac{1}{5}$. Thus $(\mathbb{R}, N, \wedge, \cdot)$ is not with multiplicatively continuous product. □

Proposition 3.11. *Let $(X, N, *, \circ)$ be a fuzzy normed algebra such that $\alpha \circ \alpha \geq \alpha$ for all $\alpha \in (0,1)$. Then $(X, N, *, \circ)$ is with multiplicatively continuous product.*

Proof. Let $\alpha \in (0,1), x, y \in X, s, t > 0$ such that $N(x,s) > \alpha, N(y,t) > \alpha$. Then

$$N(xy, st) \geq N(x,s) \circ N(y,t) \geq \alpha \circ \alpha \geq \alpha.$$

Remark 3.12. The condition $\alpha \circ \alpha \geq \alpha$ for all $\alpha \in (0,1)$ from the previous proposition is sufficient but not necessary.

Indeed, the algebra $(X, N, *, \cdot)$ from Example 3.2 is with multiplicatively continuous product, although $\circ = \cdot$ does not verify $\alpha \circ \alpha \geq \alpha$ for all $\alpha \in (0,1)$.

Proposition 3.13. *Let $(X_1, N_1, *, \circ)$ and $(X_2, N_2, *, \circ)$ be two fuzzy normed algebras. If t-norm $*'$ dominates both $*$ and $\circ$, then $((X_1 \times X_2), N, *', \circ)$ is a fuzzy normed algebra, where $N((x_1, x_2), t) = N_1(x_1, t) *' N_2(x_2, t)$.*

Proof. According to [13], it remains to be proved that:

$$N((x_1y_1, x_2y_2), st) \geq N((x_1, x_2), s) \circ N((y_1, y_2), t),\ \ \forall x_1, x_2 \in X_1, y_1, y_2 \in Y_2,\ \forall s, t \in (0, \infty).$$

We have

$$\begin{aligned} N((x_1y_1, x_2y_2), st) &= N_1(x_1y_1, st) *' N_2(x_2y_2, st) \\ &\geq [N_1(x_1, s) \circ N_1(y_1, t)] *' [N_2(x_2, s) \circ N_2(y_2, t)] \\ &\geq [N_1(x_1, s) *' N_2(x_2, s)] \circ [N_1(y_1, t) *' N_2(y_2, t)] \\ &= N((x_1, x_2), s) \circ N((y_1, y_2), t)\ \ \forall x_1, x_2 \in X_1, y_1, y_2 \in Y_2,\ \forall s, t \in (0, \infty). \end{aligned}$$

□

Proposition 3.14. *Let $*$ be a t-norm satisfying $\alpha * \alpha \geq \alpha$ for all $\alpha \in (0;1)$ and let $(X_1, N_1, *, \circ)$ and $(X_2, N_2, *, \circ)$ be two fuzzy normed algebras with multiplicatively continuous product. If $*'$ is a t-norm that dominates both $*$ and $\circ$ then $((X_1 \times X_2), N, *', \circ)$ is a fuzzy normed algebra with multiplicatively continuous product.*

Proof. Let $\alpha \in (0,1), (x_1,x_2) \in X_1 \times X_2$, and $(y_1,y_2) \in X_1 \times X_2, s,t > 0$ such that $N((x_1,x_2),s) > \alpha$ and $N((y_1,y_2),t) > \alpha$. Then we have successively:

$$\begin{aligned} N((x_1y_1,x_2y_2),st) &= N_1(x_1y_1,st) *' N_2(x_2y_2,st) \\ &\geq [N_1(x_1,s) \circ N_1(y_1,t)] *' [N_2(x_2,s) \circ N_2(y_2,t)] \\ &\geq [N_1(x_1,s) *' N_2(x_2,s)] \circ [N_1(y_1,t) *' N_2(y_2,t)] \\ &= N((x_1,x_2),s) \circ N((y_1,y_2),t) \\ &\geq \alpha \circ \alpha \\ &\geq \alpha. \end{aligned}$$

□

Example 3.15. Let $(X,N,*,\circ)$ be a fuzzy normed algebra with multiplicatively continuous product and let $S \subset X$ be a linear closed subalgebra of X. Then $(S,N,*,\circ)$ is a fuzzy normed algebra with multiplicatively continuous product.

Example 3.16 (Cartesian product of fuzzy normed algebras with multiplicatively continuous product that is not with multiplicatively continuous product)**.** Let $(X,N,\cdot,\cdot)$ be the fuzzy normed algebra from Example 3.2, where $X = \mathbb{R}$. The fuzzy normed algebra $(X \times X, N', \cdot, \cdot)$, where

$$N'((x_1,x_2),t) = N(x_1,t) \cdot N(x_2,t),\ \ \forall (x_1,x_2) \in X \times X,\ \forall\, t > 0$$

is not with multiplicatively continuous product.

Proof. Taking into account that

$$\begin{aligned} N'((x_1,x_2),t) &= N(x_1,t) \cdot N(x_2,t) \\ &= \begin{cases} 1, & t \geq |x_1|, \\ 0, & \text{for the rest,} \end{cases} \cdot \begin{cases} 1, & t \geq |x_2|, \\ 0, & \text{for the rest,} \end{cases} \\ &= \begin{cases} 1, & t \geq \max\{|x_1|,|x_2|\}, \\ 0, & \text{for the rest} \end{cases} \end{aligned}$$

for $\alpha = \dfrac{1}{2}, x_1 = x_2 = y_1 = y_2 = \dfrac{1}{2}, s = t = \dfrac{3}{5}$, we obtain

$$N'((x_1,x_2),t) = 1 > \frac{1}{2},\ N'((y_1,y_2),s) = 1 > \frac{1}{2},\ N'((x_1y_1,x_2y_2),st) = 0 < \frac{1}{2}\ .$$

Therefore $(X \times X, N', \cdot, \cdot)$ is not with multiplicatively continuous product. □

Proposition 3.17. *Let $(X,N,*,\circ)$ be a fuzzy normed algebra and let $I \subset X$ be a bilateral closed ideal. Then $(X/I,N',*,\circ)$ is a fuzzy normed algebra, where*

$$N'(\hat{x},s) := \inf_{x \in \hat{x}} N(x,s),\ \ \forall \hat{x} \in X/I,\ \forall s > 0.$$

Proof. Prove first that $(X/I,N',*)$ is a fuzzy normed space.

(N1) $N'(\hat{x},0) = \inf_{x\in\hat{x}} N(x,0) = 0,\ \forall \hat{x} \in X/I.$

(N2) One has to show that $[N'(\hat{x},t) = 1\ \ \forall t > 0]$ iff $\hat{x} = \hat{0}$. Indeed,

$$\begin{aligned} [\inf_{x\in\hat{x}} N(x,t) = 1\ \ \forall t > 0] &\Leftrightarrow [N(x,t) = 1\ \ \forall t > 0,\ \forall x \in \hat{x}] \\ &\Leftrightarrow x = 0\ \ \forall x \in \hat{x} \\ &\Leftrightarrow \hat{x} = \hat{0}. \end{aligned}$$

(N3) Let $t > 0, \lambda \in \mathbb{K}^*, \hat{x} \in X/I$. Then

$$N'(\lambda\hat{x}, t) = \inf_{x\in\hat{x}} N(\lambda x, t) = \inf_{x\in\hat{x}} N\left(x, \frac{t}{|\lambda|}\right) = N'\left(\hat{x}, \frac{t}{|\lambda|}\right).$$

(N4)

$$\begin{aligned} N'(\hat{x}+\hat{y}, t+s) = N'(\widehat{x+y}, t+s) &= \inf_{x+y\in\widehat{x+y}} N(x+y, t+s) \\ &\geq \inf_{x+y\in\widehat{x+y}} N(x,t) * N(y,s) = \inf_{x+y\in\widehat{x+y}} N(x,t) * \inf_{x+y\in\widehat{x+y}} N(y,s) \\ &\geq \inf_{x\in\hat{x}} N(x,t) * \inf_{y\in\hat{y}} N(y,s) = N'(\hat{x},t) * N'(\hat{y},s) \quad \forall s,t>0,\ \forall \hat{x},\hat{y} \in X/I. \end{aligned}$$

(N5) Let $x \in X$ and $t > 0$ fixed. Consider an arbitrary sequence $(t_n)_{n\geq 0}$ such that $t_n \to t$, $t_n \leq t$. Then, we have $N(x, t_n) \to N(x,t)$. Passing to infimum, it results $\inf\limits_{x\in\hat{x}} N(x,t_n) \to \inf\limits_{x\in\hat{x}} N(x,t)$. Hence $N'(\hat{x}, t_n) \to N'(\hat{x}, t), \quad \forall \hat{x} \in X/I$.

Moreover, $\lim\limits_{t\to\infty} N'(\hat{x},t) = \lim\limits_{t\to\infty} \inf\limits_{x\in\hat{x}} N(x,t) = \inf\limits_{x\in\hat{x}} \lim\limits_{t\to\infty} N(x,t) = 1$. It follows $(X/I, N', *)$ is a fuzzy normed linear space. It is clear that conditions (A1), (A2) are satisfied. To verify (A4), fix $\hat{x}, \hat{y} \in X/I, s, t > 0$. Then

$$\begin{aligned} N'(\hat{x}\hat{y}, ts) = N'(\widehat{xy}, ts) &= \inf_{xy\in\widehat{xy}} N(xy, ts) \geq \inf_{xy\in\widehat{xy}} N(x,t) \circ N(y,s) \\ &= \inf_{xy\in\widehat{xy}} N(x,t) \circ \inf_{xy\in\widehat{xy}} N(y,s) \geq \inf_{x\in\hat{x}} N(x,t) \circ \inf_{y\in\hat{y}} N(y,s) = N'(\hat{x},t) \circ N'(\hat{y},s). \end{aligned}$$

That concludes the proof. □

Proposition 3.18. *If $(X, N, *, \circ)$ is a fuzzy normed algebra with multiplicatively continuous product then $(X/I, N', *, \circ)$ is a fuzzy normed algebra with multiplicatively continuous product, where*

$$N'(\hat{x}, s) := \inf_{x\in\hat{x}} N(x,s), \ \ \forall \hat{x}, \hat{y} \in X/I, \ \forall s > 0.$$

Proof. Let $\alpha \in (0,1), \hat{x}, \hat{y} \in X/I, s, t > 0, N'(\hat{x}, s) > \alpha$ and $N'(\hat{y}, t) > \alpha$. It follows $\inf\limits_{x\in\hat{x}} N(x,s) > \alpha$ and $\inf\limits_{y\in\hat{y}} N(y,t) > \alpha$. This implies $N(x,s) > \alpha, N(y,t) > \alpha$ for all $x \in \hat{x}, y \in \hat{y}$. Since $(X, N, *, \circ)$ is a fuzzy normed algebra with multiplicatively continuous product, it results $N(xy, st) \geq \alpha$, for all $x \in \hat{x}, \ y \in \hat{y}, \ xy \in \widehat{xy}$. Therefore

$$\inf_{xy\in\widehat{xy}} N(xy, st) \geq \alpha.$$

□

4. Conclusion and future work

In this paper, we initiated the study of fuzzy normed algebras. We have built a fertile ground for studying in the coming papers, some spectral properties in fuzzy normed algebras. Also, fuzzy normed algebras will be used for applications in the theory of dynamical systems, particle physics, etc.

References

[1] C. Alegre, S. Romaguera, *Characterizations of fuzzy metrizable topological vector spaces and their asymmetric generalization in terms of fuzzy (quasi-)norms*, Fuzzy Sets and Systems, **161** (2010), 2182–2192. 1

[2] R. Ameri, *Fuzzy inner product and fuzzy norm of hyperspaces*, Iran. J. Fuzzy Syst., **11** (2014), 125–135. 1

[3] T. Bag, S. K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math., **11** (2003), 687–705. 1, 2.8

[4] T. Bag, S. K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets and Systems, **151** (2005), 513–547. 1, 2.7, 3, 3

[5] T. Bag, S. K. Samanta, *A comparative study of fuzzy norms on a linear space*, Fuzzy Sets and Systems, **159** (2008), 670–684. 1

[6] S. C. Cheng, J. N. Mordeson, *Fuzzy linear operator and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc., **86** (1994), 429–436. 1

[7] B. Dinda, T. K. Samanta, U. K. Bera, *Intuitionistic fuzzy Banach algebra*, Bull. Math. Anal. Appl., **3** (2010), 273–281. 1

[8] C. Felbin, *Finite-dimensional fuzzy normed linear space*, Fuzzy Sets and Systems, **48** (1992), 239–248. 1

[9] I. Goleţ, *On generalized fuzzy normed spaces and coincidence point theorems*, Fuzzy Sets and Systems, **161** (2010), 1138–1144. 1

[10] A. K. Katsaras, *Fuzzy topological vector spaces*, II, Fuzzy Sets and Systems, **12** (1984), 143–154. 1

[11] A. K. Mirmostafaee, *Perturbation of generalized derivations in fuzzy Menger normed algebras*, Fuzzy Sets and Systems, **195** (2012), 109–117. 1, 3.1

[12] A. K. Mirmostafaee, M. Mirzavaziri, *Uniquely remotal sets in c_0-sums and ℓ^∞-sums of fuzzy normed spaces*, Iran. J. Fuzzy Syst., **9** (2012), 113–122. 1

[13] S. Nădăban, *Fuzzy euclidean normed spaces for data mining applications*, Int. J. Comput. Commun. Control, **10** (2014), 70–77. 2.3, 3

[14] S. Nădăban, I. Dzitac, *Atomic decompositions of fuzzy normed linear spaces for wavelet applications*, Informatica (Vilnius), **25** (2014), 643–662. 1, 2.4, 2.6

[15] R. Saadati, S. M. Vaezpour, *Some results on fuzzy Banach spaces*, J. Appl. Math. Comput., **17** (2005), 475–484. 1

[16] I. Sadeqi, A. Amiripour, *Fuzzy Banach algebra*, First joint congress on fuzzy and intelligent systems, Ferdorwsi university of mashhad, Iran, (2007). 1

[17] I. Sadeqi, F. Moradlou, M. Salehi, *On approximate Cauchy equation in Felbin's type fuzzy normed linear spaces*, Iran. J. Fuzzy Syst., **10** (2013), 51–63. 1

[18] B. Schweizer, A. Sklar, *Statistical metric spaces*, Pacific J. Math., **10** (1960), 314–334. 2.1

[19] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338–353. 1