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Strong convergence for a common solution of variational inequalities, fixed point problems and zeros of finite maximal monotone mappings

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Abstract

In this paper, by the strongly positive linear bounded operator technique, a new generalized Mann-type hybrid composite extragradient CQ iterative algorithm is first constructed. Then by using the algorithm, we find a common element of the set of solutions of the variational inequality problem for a monotone, Lipschitz continuous mapping, the set of zeros of two families of finite maximal monotone mappings and the set of fixed points of an asymptotically κ-strict pseudocontractive mappings in the intermediate sense in a real Hilbert space. Finally, we prove the strong convergence of the iterative sequences, which extends and improves the corresponding previous works.

Keywords: Hybrid method, extragradient method, proximal method, zeros, strong convergence.
2010 MSC: 47H05, 47H10, 49J40, 49J53.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, I be the identity mapping on H, C be a nonempty closed and convex subset of H and P_C be the metric projection of H onto C. Let $F(T)$ be the set of fixed points of T and $T^{-1}0$ be the set of zeros of T.

On the one hand, very recently, Sahu et al. [15] introduced a new class of mappings, asymptotically κ-strict pseudocontractive mappings in the intermediate sense, which are not necessarily Lipschitzian. Pseudocontractions [18] and asymptotically pseudocontractions [19] are its special cases.

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Definition 1.1 ([15])**.** Let C be a nonempty subset of a Hilbert space H. A mapping $T : C \to C$ is said to be an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ if there exist a constant $\kappa \in [0, 1)$ and a sequence $\{\gamma_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|T^n x - T^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 - \kappa\|x - T^n x - (y - T^n y)\|^2\right) \le 0.$$

Throughout this paper, we assume that

$$c_n := \max \left\{0, \sup_{x,y\in C} \left(\|T^n x - T^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 - \kappa\|x - T^n x - (y - T^n y)\|^2\right)\right\}. \tag{1.1}$$

Then $c_n \ge 0$ for all $n \in \mathbb{N}$, $c_n \to 0$ as $n \to \infty$ and Definition 1.1 reduces to the relation

$$\|T^n x - T^n y\|^2 \le (1 + \gamma_n)\|x - y\|^2 + \kappa\|x - T^n x - (y - T^n y)\|^2 + c_n, \quad \forall x, y \in C,\ n \in \mathbb{N}. \tag{1.2}$$

On the other hand, a strongly positive operator is defined as follows.

Definition 1.2. An operator $V : H \to H$ is called $\overline{\gamma}$-strongly positive, if there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Vx, x\rangle \ge \overline{\gamma}\|x\|^2, \qquad \forall x \in H.$$

By using the strongly positive linear bounded operator technique, Marino and Xu [8] and Qin et al. [10] approximated the fixed point of a nonexpansive mapping and a non-self strictly pseudo-contraction, respectively.

In addition, to approximate the zeros of a maximal monotone mapping, Rockafellar [14] introduced the proximal point method. And then Wei and Tan [16] extended it to the case of two families of finite maximal monotone mappings.

Furthermore, To find the solution of the classic variational inequality problem in Euclidean space R^n, Korpelevich [7] introduced the extragradient method. Yao et al. [17] proposed a modified Korpelevich's method.

Finally, by combining the ideas of the projection method and the outer-approximation method (see [1]), Iiduka and Takahashi [6] introduced the CQ algorithm. Recently, Nadezhkina and Takahashi [9] introduced and studied the combined hybrid-extragradient method.

In this paper, motivated and inspired by the above work, by the strongly positive linear bounded operator technique, we first construct a new generalized Mann-type hybrid composite extragradient CQ iterative algorithm. Then by using the new algorithm, we find a common element of the set of solutions of the variational inequality problem for a monotone, Lipschitz continuous mapping, the set of zeros of two families of finite maximal monotone mappings and the set of fixed points of an asymptotically κ-strict pseudocontractive mappings in the intermediate sense in a real Hilbert space. Finally, we prove the strong convergence of the iterative sequences, which extends and improves the corresponding previous works, see [6, 7, 9, 16].

2. Preliminaries

Lemma 2.1 ([20])**.** *Let X be a real inner product space. Then we have the following inequality:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \qquad \forall x, y \in X.$$

Lemma 2.2 ([20])**.** *Let H be a real Hilbert space. For every $x, y, z \in H$* and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha+\beta+\gamma = 1$, *the following equality holds:*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Lemma 2.3 ([20])**.** *Let H be a real Hilbert space. Given a nonempty closed convex subset C of H, points $x, y, z \in H$ and a real number $a \in \mathbb{R}$, the set*

$$\{v \in C : \|y - v\|^2 \le \|x - v\|^2 + \langle z, v\rangle + a\}$$

is convex (and closed).

Lemma 2.4 ([5]). *(Demiclosedness Principle) Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demi-closed. That is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ (for short $x_n \rightharpoonup x \in C$), and the sequence $\{(I-T)x_n\}$ converges strongly to some y (for short $x_n - Tx_n \to y$), it implies that $(I-T)x = y$.*

Lemma 2.5 ([8]). *Assume that $V : H \to H$ is a $\overline{\gamma}$-strongly positive linear bounded operator with $0 < \mu < \|V\|^{-1}$. Then $\|I - \mu V\| \le 1 - \mu\overline{\gamma}$.*

Lemma 2.6 ([15]). *Let C be a nonempty subset of a Hilbert space H and $T : C \to C$ be an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then*

$$\|T^n x - T^n y\| \le \frac{1}{1-\kappa}\left(\kappa\|x-y\| + \sqrt{(1+(1-\kappa)\gamma_n)\|x-y\|^2 + (1-\kappa)c_n}\right)$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Lemma 2.7 ([15]). *Let C be a nonempty subset of a Hilbert space H and $T : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$. Then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.*

Lemma 2.8 ([15]). *(Demiclosedness principle) Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then $I - T$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{m\to\infty} \limsup_{n\to\infty}\|x_n - T^m x_n\| = 0$, then $(I-T)x = 0$.*

Lemma 2.9 ([15]). *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then the set of fixed points of T is closed and convex.*

Lemma 2.10. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, $A_i, B_j (i = 1, 2, \ldots, k; j = 1, 2, \ldots, l) : C \to C$ be two families of finite maximal monotone mappings such that $(\cap_{i=1}^k A_i^{-1}0) \cap (\cap_{j=1}^l B_j^{-1}0)$ is nonempty. Suppose that $S_{r_n}^{A_k A_{k-1}\cdots A_1} = J_{r_n}^{A_k} J_{r_n}^{A_{k-1}} \cdots J_{r_n}^{A_1}, G_{r_n}^{B_l B_{l-1}\cdots B_1} = a_0 I + a_1 J_{r_n}^{B_1} + a_2 J_{r_n}^{B_2} + \cdots + a_l J_{r_n}^{B_l}$ with $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$, $J_{r_n}^{B_j} = (I + r_n B_j)^{-1}$, $a_m \in (0,1)$, $\Sigma_{m=0}^l a_m = 1$, $m = 0, 1, 2, \ldots, l$ and $r_n > 0$. Then*

(i) $S_{r_n}^{A_k A_{k-1}\cdots A_1}$, $G_{r_n}^{B_l B_{l-1}\cdots B_1}$ *and* $S_{r_n}^{A_k A_{k-1}\cdots A_1} G_{r_n}^{B_l B_{l-1}\cdots B_1} : C \to C$ *are all nonexpansive.*

(ii) $F(S_{r_n}^{A_k A_{k-1}\cdots A_1}) = \cap_{i=1}^k A_i^{-1}0$ *and* $F(G_{r_n}^{B_l B_{l-1}\cdots B_1}) = \cap_{j=1}^l B_j^{-1}0$.

(iii) $F(S_{r_n}^{A_k A_{k-1}\cdots A_1} G_{r_n}^{B_l B_{l-1}\cdots B_1}) = F(S_{r_n}^{A_k A_{k-1}\cdots A_1}) \cap F(G_{r_n}^{B_l B_{l-1}\cdots B_1})$.

Proof. It is easy to get the result. □

3. Iterative algorithm and strong convergence theorem

We construct the following new iterative algorithm and then get the main result.

Theorem 3.1. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, $\mathcal{A} : C \to H$ be a monotone and ρ-Lipschitz continuous mapping, $f : H \to H$ be a contraction with the coefficient $\eta \in (0,1)$, $T : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ and c_n be defined as in (1.1). Let $V : H \to H$ be a $\overline{\gamma}$-strongly positive linear bounded operator, $A_i, B_j (i = 1, 2, \ldots, k; j = 1, 2, \ldots, l) : C \to C$ be two families of finite maximal monotone mappings such that $\Omega = (\cap_{i=1}^k A_i^{-1}0) \cap (\cap_{j=1}^l B_j^{-1}0) \cap VI(C, \mathcal{A}) \cap F(T)$ is nonempty and bounded. Suppose that*

$S_{r_n}^{A_kA_{k-1}\cdots A_1} = J_{r_n}^{A_k} J_{r_n}^{A_{k-1}} \cdots J_{r_n}^{A_1}$, $G_{r_n}^{B_lB_{l-1}\cdots B_1} = a_0 I + a_1 J_{r_n}^{B_1} + a_2 J_{r_n}^{B_2} + \cdots + a_l J_{r_n}^{B_l}$ *with* $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$, $J_{r_n}^{B_j} = (I + r_n B_j)^{-1}$, $a_m \in (0,1)$, $\Sigma_{m=0}^{l} a_m = 1$, $m = 0, 1, 2, \ldots, l$ *and* $r_n > 0$. *The sequences* $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ *and* $\{z_n\}$ *are generated by:*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n \mathcal{A} x_n), \\ u_n = \alpha_n x_n + (1-\alpha_n) S_{r_n}^{A_kA_{k-1}\cdots A_1} G_{r_n}^{B_lB_{l-1}\cdots B_1} P_C(x_n - \lambda_n \mathcal{A} y_n), \\ z_n = \beta_n u_n + \sigma_n \mu f(u_n) + ((1-\beta_n) I - \sigma_n V) T^n u_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases} \tag{3.1}$$

where $\theta_n = (1 - \sigma_n(1 + \mu - \overline{\gamma}))^{-1}(c_n + (\sigma_n(1 + \mu - \overline{\gamma}) + \gamma_n)\Delta_n)$,

$$\Delta_n = \sup\left\{ \|x_n - p\|^2 + \frac{1}{1-\eta}\|f(p) - p\|^2 + \|\mu p - V p\|^2 : p \in \Omega \right\} < \infty$$

for every $n = 0, 1, 2, \cdots$. *If* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\sigma_n\}$, $\{\lambda_n\}$ *and* μ *satisfy the following conditions:*

(i) $0 \le \alpha_n \le a < 1$;

(ii) $0 \le \beta_n \le 1$;

(iii) $\sigma_n \in [0,1]$ and $\lim_{n\to\infty} \sigma_n = 0$;

(iv) $\beta_n + \sigma_n \mu \le 1$;

(v) $0 \le \kappa \le b < \beta_n(1 - \beta_n - \sigma_n \mu)$;

(vi) $\overline{\gamma} < \mu < \frac{\overline{\gamma}}{\eta}$;

(vii) $0 < p \le \lambda_n \le q < \frac{1}{\rho}$;

where a, b, p *and* q *are constants, then the following statements hold:*

(1) $\{x_n\}$ *converges strongly to* $p_0 = P_\Omega x$;

(2) $\{x_n\}$ *converges strongly to* $p_0 = P_\Omega x$, *which is the unique solution in* Ω *of the following variational inequality:*

$$\langle (V - \mu f) p_0, p - p_0 \rangle \ge 0, \qquad \forall p \in \Omega,$$

provided that $\|x_n - z_n\| = \circ(\sigma_n)$ *and* $\gamma_n + c_n + \|u_n - T^n u_n\|^2 = \circ(\sigma_n)$. *Equivalently,*

$$p_0 = P_\Omega(I - V + \mu f) p_0.$$

Proof. We shall split the proof into six steps.

Step 1. $\Omega \subset C_n \cap Q_n$ and $\{x_n\}$ is well-defined.

First, we will show $\Omega \subset C_n$. Put $t_n = P_C(x_n - \lambda_n \mathcal{A} y_n)$, for all $p \in \Omega$, we have

$$\begin{aligned} \|t_n - p\|^2 &\le \|x_n - \lambda_n \mathcal{A} y_n - p\|^2 - \|x_n - \lambda_n \mathcal{A} y_n - t_n\|^2 \\ &= \|x_n - p\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle \mathcal{A} y_n, p - t_n \rangle \\ &= \|x_n - p\|^2 - \|x_n - t_n\|^2 + 2\lambda_n (\langle \mathcal{A} y_n - \mathcal{A} p, p - y_n \rangle \end{aligned}$$

$$
\begin{aligned}
&+ \langle \mathcal{A}p, p - y_n\rangle + \langle \mathcal{A}y_n, y_n - t_n\rangle) \\
&\le \|x_n - p\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle \mathcal{A}y_n, y_n - t_n\rangle \\
&= \|x_n - p\|^2 - \big(\|x_n - y_n\|^2 + 2\langle x_n - y_n, y_n - t_n\rangle + \|y_n - t_n\|^2\big) + 2\lambda_n\langle \mathcal{A}y_n, y_n - t_n\rangle \\
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n\mathcal{A}y_n - y_n, t_n - y_n\rangle.
\end{aligned} \tag{3.2}
$$

Since $y_n = P_C(x_n - \lambda_n\mathcal{A}x_n) \in C$ and $\mathcal{A}$ is ρ-Lipschitz continuous, we have

$$
\begin{aligned}
\langle x_n - \lambda_n\mathcal{A}y_n - y_n, t_n - y_n\rangle &= \langle x_n - \lambda_n\mathcal{A}x_n - y_n, t_n - y_n\rangle + \langle \lambda_n\mathcal{A}x_n - \lambda_n\mathcal{A}y_n, t_n - y_n\rangle \\
&\le \langle \lambda_n\mathcal{A}x_n - \lambda_n\mathcal{A}y_n, t_n - y_n\rangle \\
&\le \lambda_n\rho\|x_n - y_n\|\|t_n - y_n\|.
\end{aligned} \tag{3.3}
$$

Submitting (3.3) into (3.2) and by condition (vii) we have

$$
\begin{aligned}
\|t_n - p\|^2 &\le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n\rho\|x_n - y_n\|\|t_n - y_n\| \\
&\le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n\rho\left(\|x_n - y_n\|^2 + \|t_n - y_n\|^2\right) \\
&= \|x_n - p\|^2 + (\lambda_n\rho - 1)\left(\|x_n - y_n\|^2 + \|t_n - y_n\|^2\right) \\
&\le \|x_n - p\|^2.
\end{aligned} \tag{3.4}
$$

From (3.1), (3.4), Lemma 2.10 and the convexity of $\|\cdot\|^2$, and since $p \in (\cap_{i=1}^k A_i^{-1}0) \cap (\cap_{j=1}^l B_j^{-1}0)$, we have

$$
\begin{aligned}
\|u_n - p\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)S_{r_n}^{A_kA_{k-1}\cdots A_1}G_{r_n}^{B_lB_{l-1}\cdots B_1}t_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|S_{r_n}^{A_kA_{k-1}\cdots A_1}G_{r_n}^{B_lB_{l-1}\cdots B_1}t_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|t_n - p\|^2 \\
&\le \|x_n - p\|^2 + (1 - \alpha_n)(\lambda_n\rho - 1)\left(\|x_n - y_n\|^2 + \|t_n - y_n\|^2\right) \\
&\le \|x_n - p\|^2.
\end{aligned} \tag{3.5}
$$

By virtue of (1.2), (3.1), Lemma 2.1, conditions iv–vii and the definition of T, and since $p = Tp$, we obtain

$$
\begin{aligned}
\|z_n - p\|^2 &= \|\beta_n u_n + \sigma_n\mu f(u_n) + ((1 - \beta_n)I - \sigma_n V)T^n u_n - p\|^2 \\
&= \|\beta_n u_n + \sigma_n\mu f(u_n) + (1 - \beta_n - \sigma_n\mu)T^n u_n + ((1 - \beta_n)I - \sigma_n V)T^n u_n \\
&\quad - (1 - \beta_n - \sigma_n\mu)T^n u_n - p\|^2 \\
&\le \|\beta_n u_n + \sigma_n\mu f(u_n) + (1 - \beta_n - \sigma_n\mu)T^n u_n - p\|^2 \\
&\quad + 2\langle((1 - \beta_n)I - \sigma_n V)T^n u_n - (1 - \beta_n - \sigma_n\mu)T^n u_n, z_n - p\rangle \\
&\le \beta_n\|u_n - p\|^2 + \sigma_n\mu\|f(u_n) - p\|^2 + (1 - \beta_n - \sigma_n\mu)\|T^n u_n - p\|^2 \\
&\quad - \beta_n(1 - \beta_n - \sigma_n\mu)\|u_n - T^n u_n\|^2 + 2\sigma_n\langle(\mu I - V)(T^n u_n - p) + (\mu I - V)p, z_n - p\rangle \\
&\le \beta_n\|u_n - p\|^2 + \sigma_n\mu(\|f(u_n) - f(p)\| + \|f(p) - p\|)^2 + (1 - \beta_n - \sigma_n\mu)\|T^n u_n - p\|^2 \\
&\quad - \beta_n(1 - \beta_n - \sigma_n\mu)\|u_n - T^n u_n\|^2 + 2\sigma_n[\|\mu I - V\|\|T^n u_n - p\|\|z_n - p\| \\
&\quad + \|\mu p - Vp\|\|z_n - p\|] \\
&\le \beta_n\|u_n - p\|^2 + \sigma_n\mu\Big(\eta\|u_n - p\| + (1 - \eta)\frac{1}{1 - \eta}\|f(p) - p\|\Big)^2 \\
&\quad + (1 - \beta_n - \sigma_n\mu)\|T^n u_n - p\|^2 - \beta_n(1 - \beta_n - \sigma_n\mu)\|u_n - T^n u_n\|^2 \\
&\quad + \sigma_n[(\mu - \overline{\gamma})\left(\|T^n u_n - p\|^2 + \|z_n - p\|^2\right) + \|\mu p - Vp\|^2 + \|z_n - p\|^2] \\
&\le \beta_n\|u_n - p\|^2 + \sigma_n\mu\Big(\eta\|u_n - p\|^2 + (1 - \eta)\frac{1}{(1 - \eta)^2}\|f(p) - p\|^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+(1-\beta_n-\sigma_n\mu)\|T^n u_n-p\|^2-\beta_n(1-\beta_n-\sigma_n\mu)\|u_n-T^n u_n\|^2\\
&+\sigma_n[(\mu-\overline{\gamma})\left(\|T^n u_n-p\|^2+\|z_n-p\|^2\right)+\|\mu p-Vp\|^2+\|z_n-p\|^2]\\
=\,&(\beta_n+\sigma_n\mu\eta)\|u_n-p\|^2+\frac{\sigma_n\mu}{1-\eta}\|f(p)-p\|^2+[1-\beta_n-\sigma_n\mu+\sigma_n(\mu-\overline{\gamma})]\|T^n u_n-p\|^2\\
&-\beta_n(1-\beta_n-\sigma_n\mu)\|u_n-T^n u_n\|^2+\sigma_n(1+\mu-\overline{\gamma})\|z_n-p\|^2+\sigma_n\|\mu p-Vp\|^2\\
\le\,&(\beta_n+\sigma_n\mu\eta)\|u_n-p\|^2+\frac{\sigma_n\mu}{1-\eta}\|f(p)-p\|^2-\beta_n(1-\beta_n-\sigma_n\mu)\|u_n-T^n u_n\|^2\\
&+[1-\beta_n-\sigma_n\mu+\sigma_n(\mu-\overline{\gamma})]((1+\gamma_n)\|u_n-p\|^2+\kappa\|u_n-T^n u_n\|^2+c_n)\\
&+\sigma_n(1+\mu-\overline{\gamma})\|z_n-p\|^2+\sigma_n\|\mu p-Vp\|^2\\
\le\,&[\beta_n+\sigma_n\overline{\gamma}+(1-\beta_n-\sigma_n\overline{\gamma})(1+\gamma_n)]\|u_n-p\|^2+\frac{\sigma_n\mu}{1-\eta}\|f(p)-p\|^2\\
&+(1-\beta_n-\sigma_n\overline{\gamma})\big(\kappa-\beta_n\frac{1-\beta_n-\sigma_n\mu}{1-\beta_n-\sigma_n\overline{\gamma}}\big)\|u_n-T^n u_n\|^2\\
&+(1-\beta_n-\sigma_n\overline{\gamma})c_n+\sigma_n(1+\mu-\overline{\gamma})\|z_n-p\|^2+\sigma_n\|\mu p-Vp\|^2\\
\le\,&(1+\gamma_n)\|u_n-p\|^2+\frac{\sigma_n\mu}{1-\eta}\|f(p)-p\|^2\\
&+(1-\beta_n-\sigma_n\overline{\gamma})(\kappa-\beta_n(1-\beta_n-\sigma_n\mu))\|u_n-T^n u_n\|^2\\
&+(1-\beta_n-\sigma_n\overline{\gamma})c_n+\sigma_n(1+\mu-\overline{\gamma})\|z_n-p\|^2+\sigma_n\|\mu p-Vp\|^2\\
\le\,&(1+\gamma_n)\|u_n-p\|^2+\frac{\sigma_n\mu}{1-\eta}\|f(p)-p\|^2\\
&+(1-\beta_n-\sigma_n\overline{\gamma})(\kappa-\beta_n(1-\beta_n-\sigma_n\mu))\|u_n-T^n u_n\|^2\\
&+c_n+\sigma_n(1+\mu-\overline{\gamma})\|z_n-p\|^2+\sigma_n\|\mu p-Vp\|^2\\
\le\,&(1+\gamma_n)\|u_n-p\|^2+\frac{\sigma_n\mu}{1-\eta}\|f(p)-p\|^2\\
&+(1-\beta_n-\sigma_n\overline{\gamma})(\kappa-\beta_n(1-\beta_n-\sigma_n\mu))\|u_n-T^n u_n\|^2\\
&+c_n+\sigma_n(1+\mu-\overline{\gamma})\|z_n-p\|^2+\sigma_n\|\mu p-Vp\|^2\\
\le\,&(1+\gamma_n)(\|x_n-p\|^2+(1-\alpha_n)(\lambda_n\rho-1)(\|x_n-y_n\|^2+\|t_n-y_n\|^2))\\
&+\frac{\sigma_n\mu}{1-\eta}\|f(p)-p\|^2+(1-\beta_n-\sigma_n\overline{\gamma})(\kappa-\beta_n(1-\beta_n-\sigma_n\mu))\|u_n-T^n u_n\|^2\\
&+c_n+\sigma_n(1+\mu-\overline{\gamma})\|z_n-p\|^2+\sigma_n\|\mu p-Vp\|^2\\
\le\,&(1+\gamma_n)\|x_n-p\|^2+\frac{\sigma_n\mu}{1-\eta}\|f(p)-p\|^2+c_n+\sigma_n(1+\mu-\overline{\gamma})\|z_n-p\|^2+\sigma_n\|\mu p-Vp\|^2.
\end{aligned}
\tag{3.6}
$$

This implies

$$
(1-\sigma_n(1+\mu-\overline{\gamma}))\|z_n-p\|^2\le(1+\gamma_n)\|x_n-p\|^2+\frac{\sigma_n\mu}{1-\eta}\|f(p)-p\|^2+c_n+\sigma_n\|\mu p-Vp\|^2.
$$

By virtue of conditions (iii) and (vi), we conclude

$$
\begin{aligned}
\|z_n-p\|^2\le\,&\|x_n-p\|^2+\frac{1}{1-\sigma_n(1+\mu-\overline{\gamma})}\big[(\sigma_n(1+\mu-\overline{\gamma})+\gamma_n)\|x_n-p\|^2\\
&+\frac{\sigma_n\mu}{1-\eta}\|f(p)-p\|^2+c_n+\sigma_n\|\mu p-Vp\|^2\big]\\
\le\,&\|x_n-p\|^2+\frac{1}{1-\sigma_n(1+\mu-\overline{\gamma})}\big[(\sigma_n(1+\mu-\overline{\gamma})+\gamma_n)\big(\|x_n-p\|^2\\
&+\frac{1}{1-\eta}\|f(p)-p\|^2+\|\mu p-Vp\|^2\big)+c_n\big]
\end{aligned}
\tag{3.7}
$$

$$\leq \|x_n - p\|^2 + \frac{1}{1 - \sigma_n(1 + \mu - \overline{\gamma})}[(\sigma_n(1 + \mu - \overline{\gamma}) + \gamma_n)\Delta_n + c_n]$$
$$= \|x_n - p\|^2 + \theta_n,$$

where $\theta_n = (1 - \sigma_n(1 + \mu - \overline{\gamma}))^{-1}(c_n + (\sigma_n(1 + \mu - \overline{\gamma}) + \gamma_n)\Delta_n)$,

$$\Delta_n = \sup\left\{\|x_n - p\|^2 + \frac{1}{1-\eta}\|f(p) - p\|^2 + \|\mu p - Vp\|^2 : p \in \Omega\right\} < \infty.$$

By virtue of (3.7) and the definition of C_n, we have $p \in C_n$. So, $\Omega \subset C_n$, for every $n = 0, 1, 2, \cdots$.

Next, through the mathematical induction method, we shall prove that $\{x_n\}$ is well-defined and $\Omega \subset C_n \cap Q_n$, for all $n = 0, 1, 2, \cdots$. For $n = 0$, $Q_0 = \{z \in C : \langle x_0 - z, x - x_0\rangle \geq 0\} = C$, hence $\Omega \subset C_0 \cap Q_0$. Suppose that x_k is given and $\Omega \subset C_k \cap Q_k$ for some $k \in N$. Because Ω is nonempty, we have $C_k \cap Q_k$ is nonempty. It is obvious that C_n is closed and Q_n is closed and convex. Since

$$C_n = \{z \in C : \|z_n - x_n\|^2 + 2\langle z_n - x_n, x_n - z\rangle + \theta_n \leq 0\} = \{z \in C : 2\langle x_n - z_n, z\rangle \leq \|x_n\|^2 - \|z_n\|^2 + \theta_n\},$$

by Lemma 2.3, we also have C_n is convex. Thus, $C_k \cap Q_k$ is a nonempty closed convex subset of C, so there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k} x$. It is obvious that

$$\langle x_{k+1} - z, x - x_{k+1}\rangle \geq 0, \quad \forall z \in C_k \cap Q_k.$$

Since $\Omega \subset C_k \cap Q_k$, we have

$$\langle x_{k+1} - z, x - x_{k+1}\rangle \geq 0, \quad \forall z \in \Omega.$$

That is, $z \in Q_{k+1}$. Hence $\Omega \subset Q_{k+1}$. Therefore, we get $\Omega \subset C_{k+1} \cap Q_{k+1}$., and then $\Omega \subset C_n \cap Q_n$, for every $n = 0, 1, 2, \cdots$. Therefore, $\{x_n\}$ is well-defined.

Step 2. $\{x_n\}, \{y_n\}, \{u_n\}, \{z_n\}$, $\{t_n\}$, $\{f(u_n)\}$ and $\{T^n u_n\}$ are all bounded.

Let $p_0 = P_\Omega x$, from Step 1, we have $p_0 \in \Omega \subset C_n \cap Q_n$. From $x_{n+1} = P_{C_n \cap Q_n} x$ and the definition of the metric projection, we have

$$\|x_{n+1} - x\| \leq \|p_0 - x\| \tag{3.8}$$

for every $n = 0, 1, 2, \cdots$. Therefore, $\{x_n\}$ is bounded. By virtue of (3.4), (3.5) and (3.7), we also obtain $\{t_n\}$, $\{u_n\}$ and $\{z_n\}$ are also bounded, respectively.

Again from (3.4), condition (vii) and the boundedness of $\{x_n\}$ and $\{t_n\}$, we have

$$\begin{aligned}
\|x_n - y_n\|^2 &\leq \|x_n - y_n\|^2 + \|t_n - y_n\|^2 \\
&\leq \frac{1}{1 - \lambda_n\rho}\left(\|x_n - p\|^2 - \|t_n - p\|^2\right) \\
&\leq \frac{1}{1 - \lambda_n\rho}(\|x_n - p\| + \|t_n - p\|)(\|x_n - p\| - \|t_n - p\|) \\
&\leq \frac{1}{1 - \lambda_n\rho}(\|x_n - p\| + \|t_n - p\|)\|x_n - t_n\| \\
&\leq \frac{1}{1 - q\rho}(\|x_n - p\| + \|t_n - p\|)\|x_n - t_n\|.
\end{aligned}$$

So, $\{y_n\}$ is bounded. Because

$$\|f(u_n)\| \leq \|f(u_n) - f(p)\| + \|f(p)\| \leq \eta\|u_n - p\| + \|f(p)\|,$$

we have $\{f(u_n)\}$ is bounded. By (3.1), we have

$$\|z_n - p\|^2 = \|\beta_n u_n + \sigma_n \mu f(u_n) + ((1 - \beta_n)I - \sigma_n V)T^n u_n - p\|^2.$$

Combining with the boundedness of $\{u_n\}$, $\{z_n\}$ and $\{f(u_n)\}$, we obtain that $\{T^n u_n\}$ is bounded.

Step 3. $\lim_{n\to\infty}\|x_n - y_n\| = 0$, $\lim_{n\to\infty}\|x_n - u_n\| = 0$, $\lim_{n\to\infty}\|x_n - z_n\| = 0$ and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

As $Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}$, we have $\langle x_n - z, x - x_n\rangle \geq 0$ for all $z \in Q_n$. And by the definition of the metric projection, we obtain $x_n = P_{Q_n}x$. Because $x_{n+1} = P_{C_n\cap Q_n}x \in C_n \cap Q_n \subset Q_n$ and by (3.8), we have

$$\|x_n - x\| \leq \|x_{n+1} - x\| \leq \|p_0 - x\|$$

for every $n = 0, 1, 2, \cdots$. Therefore, $\lim_{n\to\infty}\|x_n - x\|$ exists. We have

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x\|^2 - \|x_n - x\|^2,$$

which implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.9}$$

By $x_{n+1} \in C_n$ and the definition of C_n, we have

$$\|z_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \theta_n \leq \left(\|x_n - x_{n+1}\| + \sqrt{\theta_n}\right)^2,$$

which yields that

$$\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| + \sqrt{\theta_n}.$$

Hence

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_n - x_{n+1}\| + \sqrt{\theta_n}. \tag{3.10}$$

By virtue of condition (iii), we have $\theta_n \to 0$. Again from (3.9) and (3.10), we conclude

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.11}$$

From (3.6), we obtain

$$\begin{aligned}\|z_n - p\|^2 \leq &(1+\gamma_n)(\|x_n - p\|^2 + (1-\alpha_n)(\lambda_n\rho - 1)(\|x_n - y_n\|^2 + \|t_n - y_n\|^2))\\ &+ \frac{\sigma_n\mu}{1-\eta}\|f(p) - p\|^2 + c_n + \sigma_n(1+\mu-\overline{\gamma})\|z_n - p\|^2 + \sigma_n\|\mu p - Vp\|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|x_n - y_n\|^2 + \|t_n - y_n\|^2 \leq &\frac{1}{(1+\gamma_n)(1-\alpha_n)(1-\lambda_n\rho)}\big[(1+\gamma_n)\|x_n - p\|^2\\ &- (1-\sigma_n(1+\mu-\overline{\gamma}))\|z_n - p\|^2\\ &+ \frac{\sigma_n\mu}{1-\eta}\|f(p) - p\|^2 + c_n + \sigma_n\|\mu p - Vp\|^2\big]\\ = &\frac{1}{(1+\gamma_n)(1-\alpha_n)(1-\lambda_n\rho)}\big[(1+\gamma_n)(\|x_n - p\|^2 - \|z_n - p\|^2)\\ &+ (1+\sigma_n\mu+\gamma_n - (1-\sigma_n(1+\mu-\overline{\gamma})))\|z_n - p\|^2\\ &+ \frac{\sigma_n\mu}{1-\eta}\|f(p) - p\|^2 + c_n + \sigma_n\|\mu p - Vp\|^2\big]\\ = &\frac{1}{(1+\gamma_n)(1-\alpha_n)(1-\lambda_n\rho)}\big[(1+\gamma_n)(\|x_n - p\|\\ &+ \|z_n - p\|)(\|x_n - p\| - \|z_n - p\|)\\ &+ (\sigma_n(1+2\mu-\overline{\gamma}) + \gamma_n)\|z_n - p\|^2\\ &+ \frac{\sigma_n\mu}{1-\eta}\|f(p) - p\|^2 + c_n + \sigma_n\|\mu p - Vp\|^2\big]\\ \leq &\frac{1}{(1+\gamma_n)(1-\alpha_n)(1-\lambda_n\rho)}\big[(1+\gamma_n)(\|x_n - p\|\end{aligned} \tag{3.12}$$

$$+ \|z_n - p\|)\|x_n - z_n\| + (\sigma_n(1 + 2\mu - \overline{\gamma}) + \gamma_n)\|z_n - p\|^2 + \frac{\sigma_n\mu}{1-\eta}\|f(p) - p\|^2 + c_n + \sigma_n\|\mu p - Vp\|^2\big].$$

From (3.11), (3.12), condition (i), (iii), (vii) and the boundedness of $\{x_n\}$ and $\{z_n\}$, we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0, \tag{3.13}$$

and

$$\lim_{n\to\infty} \|y_n - t_n\| = 0. \tag{3.14}$$

By $\|x_n - t_n\| \le \|x_n - y_n\| + \|y_n - t_n\|$, (3.13) and (3.14), we get

$$\lim_{n\to\infty} \|x_n - t_n\| = 0. \tag{3.15}$$

In view of (3.6), we obtain

$$\begin{aligned}\|z_n - p\|^2 \le (1+\gamma_n)\|x_n - p\|^2 &+ \frac{\sigma_n\mu}{1-\eta}\|f(p) - p\|^2 \\ &+ (1 - \beta_n - \sigma_n\overline{\gamma})(\kappa - \beta_n(1 - \beta_n - \sigma_n\mu))\|u_n - T^n u_n\|^2 \\ &+ c_n + \sigma_n(1 + \mu - \overline{\gamma})\|z_n - p\|^2 + \sigma_n\|\mu p - Vp\|^2.\end{aligned}$$

By conditions (iv) and (vi), we have $\beta_n + \sigma_n\overline{\gamma} < \beta_n + \sigma_n\mu \le 1$. Therefore, repeating the similar method in (3.12), we have

$$\begin{aligned}\|u_n - T^n u_n\|^2 \le& \frac{1}{(1 - \beta_n - \sigma_n\overline{\gamma})(\beta_n(1 - \beta_n - \sigma_n\mu) - \kappa)}\big[(1+\gamma_n)\|x_n - p\|^2 \\ &- (1 - \sigma_n(1 + \mu - \overline{\gamma}))\|z_n - p\|^2 + \frac{\sigma_n\mu}{1-\eta}\|f(p) - p\|^2 + c_n + \sigma_n\|\mu p - Vp\|^2\big] \\ \le& \frac{1}{(1 - \beta_n - \sigma_n\overline{\gamma})(\beta_n(1 - \beta_n - \sigma_n\mu) - \kappa)}\big[(1+\gamma_n)(\|x_n - p\| + \|z_n - p\|)\|x_n - z_n\| \\ &+ (\sigma_n(1 + \mu - \overline{\gamma}) + \gamma_n)\|z_n - p\|^2 + \frac{\sigma_n\mu}{1-\eta}\|f(p) - p\|^2 + c_n + \sigma_n\|\mu p - Vp\|^2\big].\end{aligned} \tag{3.16}$$

Again from (3.11), (3.16), condition (iii) and the boundedness of $\{x_n\}$ and $\{z_n\}$, we have

$$\lim_{n\to\infty} \|u_n - T^n u_n\| = 0. \tag{3.17}$$

By (3.1), we have

$$\begin{aligned}\|z_n - u_n\| =& \|\beta_n u_n + \sigma_n\mu f(u_n) + ((1-\beta_n)I - \sigma_n V)T^n u_n - u_n\| \\ =& \|(1-\beta_n)(T^n u_n - u_n) + \sigma_n(\mu f(u_n) - VT^n u_n)\| \\ \le& (1-\beta_n)\|T^n u_n - u_n\| + \sigma_n\|\mu f(u_n) - VT^n u_n\|.\end{aligned} \tag{3.18}$$

From (3.17), (3.18), condition (iii) and the boundedness of V, $\{f(u_n)\}$ and $\{T^n u_n\}$, we have

$$\lim_{n\to\infty} \|z_n - u_n\| = 0. \tag{3.19}$$

By $\|x_n - u_n\| \le \|x_n - z_n\| + \|z_n - u_n\|$, (3.11) and (3.19), we get

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.20}$$

Furthermore, by Lemma 2.6, we have

$$\begin{aligned}\|x_n - T^n x_n\| \le& \|x_n - u_n\| + \|u_n - T^n u_n\| + \|T^n u_n - T^n x_n\| \\ \le& \|x_n - u_n\| + \|u_n - T^n u_n\| + \frac{1}{1-\kappa}\big(\kappa\|x_n - u_n\| \\ &+ \sqrt{(1+(1-\kappa)\gamma_n)\|x_n - u_n\|^2 + (1-\kappa)c_n}\big).\end{aligned} \tag{3.21}$$

From (3.17), (3.20) and (3.21), we get

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0.$$

Since $\|x_{n+1} - x_n\| \to 0$, $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$ and T is uniformly continuous, by Lemma 2.7, we obtain

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0.$$

Step 4. $W(x_n) \subset \Omega$, where $W(x_n)$ denotes the set of all the weak limit points of $\{x_n\}$.

Indeed, since $\{x_n\}$ is bounded, we know that $W(x_n)$ is nonempty. Take $u \in W(x_n)$ arbitrarily. Then there exists a subsequence of $\{x_n\}$, for simplicity, we still denote it by $\{x_n\}$, such that $x_n \rightharpoonup u$ as $n \to \infty$. In the following, we shall prove $u \in \Omega$. First, we show $u \in (\cap_{i=1}^k A_i^{-1}0) \cap (\cap_{j=1}^l B_j^{-1}0)$. From (3.4), for all $p \in \Omega$, we have

$$\begin{aligned}\|u_n - x_n\| =& \|\alpha_n x_n + (1-\alpha_n) S_{r_n}^{A_k A_{k-1}\cdots A_1} G_{r_n}^{B_l B_{l-1}\cdots B_1} t_n - x_n\| \\ =& (1-\alpha_n)\|S_{r_n}^{A_k A_{k-1}\cdots A_1} G_{r_n}^{B_l B_{l-1}\cdots B_1} t_n - x_n\|.\end{aligned}$$

This implies that

$$\|S_{r_n}^{A_k A_{k-1}\cdots A_1} G_{r_n}^{B_l B_{l-1}\cdots B_1} t_n - x_n\| = \frac{1}{1-\alpha_n}\|x_n - u_n\|. \tag{3.22}$$

From (3.20), (3.22) and condition (i), we have

$$S_{r_n}^{A_k A_{k-1}\cdots A_1} G_{r_n}^{B_l B_{l-1}\cdots B_1} t_n - x_n \to 0, \quad (n \to \infty). \tag{3.23}$$

By $\|S_{r_n}^{A_k A_{k-1}\cdots A_1} G_{r_n}^{B_l B_{l-1}\cdots B_1} t_n - t_n\| \le \|S_{r_n}^{A_k A_{k-1}\cdots A_1} G_{r_n}^{B_l B_{l-1}\cdots B_1} t_n - x_n\| + \|t_n - x_n\|$, (3.15) and (3.23), we get $S_{r_n}^{A_k A_{k-1}\cdots A_1} G_{r_n}^{B_l B_{l-1}\cdots B_1} t_n - t_n \to 0$. Since $t_n \rightharpoonup u$, $S_{r_n}^{A_k A_{k-1}\cdots A_1} G_{r_n}^{B_l B_{l-1}\cdots B_1}$ is nonexpansive and by Lemma 2.4, we have $S_{r_n}^{A_k A_{k-1}\cdots A_1} G_{r_n}^{B_l B_{l-1}\cdots B_1}$ is demiclosed at zero, i.e., $u \in F(S_{r_n}^{A_k A_{k-1}\cdots A_1} G_{r_n}^{B_l B_{l-1}\cdots B_1})$.

Again By Lemma 2.10, we get $u \in (\cap_{i=1}^k A_i^{-1}0) \cap (\cap_{j=1}^l B_j^{-1}0)$. Secondly, we show $u \in VI(C, \mathcal{A})$. Let

$$Tv = \begin{cases} \mathcal{A}v + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C, \end{cases}$$

where $N_C v$ is the normal cone to C at $v \in C$, $G(T)$ is the graph of T and $(v, \omega) \in G(T)$. So, we have $\omega \in Tv = \mathcal{A}v + N_C v$ and hence $\omega - \mathcal{A}v \in N_C v$ (see [13]). From the definition of the normal cone and since $t_n = P_C(x_n - \lambda_n \mathcal{A} y_n) \in C$, we have

$$\langle v - t_n, \omega - \mathcal{A}v\rangle \ge 0. \tag{3.24}$$

From the property of P_C, we have

$$\langle x_n - \lambda_n \mathcal{A} y_n - t_n, t_n - v\rangle \ge 0, \quad \forall v \in C,$$

and hence

$$\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + \mathcal{A} y_n\rangle \ge 0. \tag{3.25}$$

For simplicity, we assume that $\{y_n\}$ and $\{t_n\}$ are also subsequences of $\{y_n\}$ and $\{t_n\}$, respectively. Because of $x_n - y_n \to 0$, $x_n - t_n \to 0$ and $x_n \rightharpoonup u$, we have $y_n \rightharpoonup u$ and $t_n \rightharpoonup u$. By the monotonicity of $\mathcal{A}$, (3.24) and (3.25), we obtain

$$\begin{aligned}\langle v - t_n, \omega\rangle &\geq \langle v - t_n, \mathcal{A}v\rangle \\ &\geq \langle v - t_n, \mathcal{A}v\rangle - \langle v - t_n, \frac{t_n - x_n}{\lambda_n} + \mathcal{A}y_n\rangle \\ &= \langle v - t_n, \mathcal{A}v - \mathcal{A}t_n\rangle + \langle v - t_n, \mathcal{A}t_n - \mathcal{A}y_n\rangle - \langle v - t_n, \frac{t_n - x_n}{\lambda_n}\rangle \\ &\geq \langle v - t_n, \mathcal{A}t_n - \mathcal{A}y_n\rangle - \langle v - t_n, \frac{t_n - x_n}{\lambda_n}\rangle.\end{aligned}$$

Hence, we obtain $\langle v - u, \omega - 0\rangle \geq 0$ as $n \to \infty$. Since T is maximal monotone, we have $0 \in Tu$ and hence $u \in VI(C, \mathcal{A})$.

Thirdly, we show $u \in F(T)$. We observe that T is uniformly continuous and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Hence it is easy to get that $\lim_{n\to\infty} \|x_n - T^m x_n\| = 0$ for all $m \geq 1$, $m \in \mathbb{N}$. From Lemma 2.8, we have $u \in F(T)$. So $u \in \Omega$ and we get $W(x_n) \subset \Omega$.

Step 5. $x_n \to P_\Omega x$ as $n \to \infty$.

Suppose $x_n \rightharpoonup u$ as $n \to \infty$, where $\{x_n\}$ is looked as a subsequence of $\{x_n\}$ for simplicity, from Step 4, we have $u \in \Omega$. Let $p_0 = P_\Omega x$, from the definition of the metric projection and the weak lower semi-continuity of $\|\cdot\|$, we have

$$\|p_0 - x\| \leq \|u - x\| \leq \liminf_{n\to\infty} \|x_n - x\| \leq \limsup_{n\to\infty} \|x_n - x\| \leq \|p_0 - x\|.$$

So, we obtain

$$\lim_{n\to\infty} \|x_n - x\| = \|u - x\|.$$

Again from $x_n - x \rightharpoonup u - x$ and the Kadec-Klee property (see [12]), we have $x_n - x \to u - x$ and hence $x_n \to u$. Since $x_n = P_{Q_n} x$ and $p_0 \in \Omega \subset C_n \cap Q_n \subset Q_n$, we have

$$-\|p_0 - x_n\|^2 = \langle p_0 - x_n, x_n - x\rangle + \langle p_0 - x_n, x - p_0\rangle \geq \langle p_0 - x_n, x - p_0\rangle.$$

Let $n \to \infty$ in both sides of the above inequality, we get $-\|p_0 - u\|^2 \geq \langle p_0 - u, x - p_0\rangle \geq 0$. Hence $u = p_0$. This implies that $x_n \to p_0 = P_\Omega x$ as $n \to \infty$.

Step 6. Let $p_0 = P_\Omega x$, assume additionally that $\|x_n - z_n\| = \circ(\sigma_n)$ and $\gamma_n + c_n + \|u_n - T^n u_n\|^2 = \circ(\sigma_n)$, we will show p_0 is the unique solution in Ω to solve the following variational inequality:

$$\langle (V - \mu f)p_0, p - p_0\rangle \geq 0, \qquad \forall p \in \Omega. \tag{3.26}$$

Equivalently, $p_0 = P_\Omega(I - V + \mu f)p_0$. Indeed, by the definition of V and f, we have

$$\langle (V - \mu f)x - (V - \mu f)y, x - y\rangle = \langle Vx - Vy, x - y\rangle - \mu\langle f(x) - f(y), x - y\rangle \geq (\overline{\gamma} - \mu\eta)\|x - y\|^2.$$

This implies that $V - \mu f$ is $(\overline{\gamma} - \mu\eta)$-strongly monotone. In the meantime, we obtain

$$\|(V - \mu f)x - (V - \mu f)y\| \leq \|Vx - Vy\| + \mu\|f(x) - f(y)\| \leq (\|V\| + \mu\eta)\|x - y\|.$$

That is, $V - \mu f$ is $(\|V\| + \mu\eta)$-Lipschitz continuous. Thus, there exists a unique solution $\widehat{p} \in \Omega$ to satisfy the following variational inequality:

$$\langle (V - \mu f)\widehat{p}, p - \widehat{p}\rangle \geq 0, \qquad \forall p \in \Omega.$$

Equivalently, $\widehat{p} = P_{\Omega}(I - V + \mu f)\widehat{p}$. Furthermore, by using another technique in (3.6), by virtue of (1.2), (3.1), (3.5), Lemma 2.1 and condition (vi), we obtain

$$\begin{aligned}
\|z_n - p\|^2 &= \|\beta_n(u_n - p) + \sigma_n\mu(f(u_n) - f(p)) \\
&\quad + ((1-\beta_n)I - \sigma_n V)(T^n u_n - p) + \sigma_n(\mu f - V)p\|^2 \\
&\le \|\beta_n(u_n - p) + \sigma_n\mu(f(u_n) - f(p)) \\
&\quad + ((1-\beta_n)I - \sigma_n V)(T^n u_n - p)\|^2 \\
&\quad + 2\sigma_n\langle(\mu f - V)p, z_n - p\rangle \\
&\le [\beta_n\|u_n - p\| + \sigma_n\mu\|f(u_n) - f(p)\| \\
&\quad + (1 - \beta_n - \sigma_n\overline{\gamma})\|T^n u_n - p\|]^2 + 2\sigma_n\langle(\mu f - V)p, z_n - p\rangle \\
&\le [\beta_n\|u_n - p\| + \sigma_n\mu\eta\|u_n - p\| + (1 - \beta_n - \sigma_n\overline{\gamma})\|T^n u_n - p\|]^2 \\
&\quad + 2\sigma_n\langle(\mu f - V)p, z_n - p\rangle \\
&\le [(\beta_n + \sigma_n\overline{\gamma})\|u_n - p\| + (1 - \beta_n - \sigma_n\overline{\gamma})\|T^n u_n - p\|]^2 + 2\sigma_n\langle(\mu f - V)p, z_n - p\rangle \\
&\le (\beta_n + \sigma_n\overline{\gamma})\|u_n - p\|^2 + (1 - \beta_n - \sigma_n\overline{\gamma})\|T^n u_n - p\|^2 + 2\sigma_n\langle(\mu f - V)p, z_n - p\rangle \\
&\le (\beta_n + \sigma_n\overline{\gamma})\|u_n - p\|^2 + (1 - \beta_n - \sigma_n\overline{\gamma})((1+\gamma_n)\|u_n - p\|^2 + \kappa\|u_n - T^n u_n\|^2 + c_n) \\
&\quad + 2\sigma_n\langle(\mu f - V)p, z_n - p\rangle \\
&\le \|u_n - p\|^2 + (1 - \beta_n - \sigma_n\overline{\gamma})(\gamma_n\|u_n - p\|^2 \\
&\quad + \kappa\|u_n - T^n u_n\|^2 + c_n) + 2\sigma_n\langle(\mu f - V)p, z_n - p\rangle \\
&\le \|x_n - p\|^2 + (1 - \beta_n - \sigma_n\overline{\gamma}) \\
&\quad \times (\gamma_n + \|u_n - T^n u_n\|^2 + c_n)(\|u_n - p\|^2 + \kappa + 1) \\
&\quad + 2\sigma_n\langle(\mu f - V)p, z_n - p\rangle,
\end{aligned} \tag{3.27}$$

which yields

$$\begin{aligned}
\langle(\mu f - V)p, p - z_n\rangle &\le \frac{1}{2\sigma_n}[\|x_n - p\|^2 - \|z_n - p\|^2 + (1 - \beta_n - \sigma_n\overline{\gamma}) \\
&\quad \times (\gamma_n + \|u_n - T^n u_n\|^2 + c_n)(\|u_n - p\|^2 + \kappa + 1)] \\
&= \frac{\|x_n - p\| - \|z_n - p\|}{2\sigma_n}(\|x_n - p\| + \|z_n - p\|) \\
&\quad + \frac{\gamma_n + \|u_n - T^n u_n\|^2 + c_n}{2\sigma_n}(1 - \beta_n - \sigma_n\overline{\gamma}) \\
&\quad \times (\|u_n - p\|^2 + \kappa + 1) \\
&\le \frac{\|x_n - z_n\|}{2\sigma_n}(\|x_n - p\| + \|z_n - p\|) \\
&\quad + \frac{\gamma_n + c_n + \|u_n - T^n u_n\|^2}{2\sigma_n}(1 - \beta_n - \sigma_n\overline{\gamma})(\|u_n - p\|^2 + \kappa + 1).
\end{aligned} \tag{3.28}$$

By means of (3.11) and (3.17), i.e., $\lim_{n\to\infty}\|x_n - z_n\| = 0$ and $\lim_{n\to\infty}\|u_n - T^n u_n\| = 0$, and by condition (iii), we can assume that $\|x_n - z_n\| = \circ(\sigma_n)$ and $\gamma_n + c_n + \|u_n - T^n u_n\|^2 = \circ(\sigma_n)$. And since $x_n \to p_0$ as $n \to \infty$, $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ are bounded, we conclude from (3.28) that

$$\langle(V - \mu f)p, p - p_0\rangle \ge 0, \qquad \forall p \in \Omega,$$

which together with Minty's lemma (also see [4]) implies that

$$\langle(V - \mu f)p_0, p - p_0\rangle \ge 0, \qquad \forall p \in \Omega.$$

This shows that p_0 is a solution in Ω to the variational inequality (3.26). By using the uniqueness of solutions in Ω of the variational inequality (3.26), we obtain that $p_0 = \widehat{p}$. This completes the proof. □

Remark 3.2.

(1) In Theorem 3.1, we consider the problem $(\cap_{i=1}^{k} A_i^{-1}0) \cap (\cap_{j=1}^{l} B_j^{-1}0) \cap VI(C, \mathcal{A}) \cap F(T)$, which is more general than Sahu et al. [15] and Qiu et al. [11].

(2) In Theorem 3.1, the strongly positive linear bounded operator is first put into the CQ algorithm, such that $C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}$ holds, and θ_n is different from that of other literatures, e.g. [2, 3, 15].

(3) The strongly positive linear bounded operator technique is first used to approximate the fixed point of asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$.

(4) The proof of Theorem 3.1 is interesting, e.g., in (3.6) and (3.27), by using two different techniques, we transfer the nonconvex style into the convex style to complete it, respectively. In (3.4) and (3.12) of the proof of Theorem 3.1, the method is different from that of Nadezhkina and Takahashi [9] and Qiu et al. [11]. The proof of Theorem 3.1 is easier than that of Theorem 3.4 in [8] and Theorem 2.1 in [10], respectively, which need the implicit iteration.

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