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New Hermite-Hadamard inequalities for twice differentiable ϕ-MT-preinvex functions

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Abstract

New Hermite-Hadamard-type integral inequalities for ϕ-MT-preinvex functions are obtained. Our results in special cases yield some of those results proved in recent articles concerning with the differentiable MT-convex functions. Some applications to special means and the trapezoidal formula are also considered, respectively.

Keywords: ϕ-MT-preinvex functions, Hermite-Hadamard's integral inequality, Hölder's inequality.
2010 MSC: 26D15, 26A51, 26B12.

1. Introduction and preliminaries

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2},$$

referred to as Hermite-Hadamard inequality, is one of the most famous results for convex functions.

Over the last decade, this has been extended in diverse approaches. Some recent results on generalizations, refinements and new inequalities are involved with the Hermite-Hadamard inequality, please see [2, 4, 5, 9, 11, 24, 25] and the references therein.

Let us recall some necessary definitions and preliminary results which are used for further discussion.

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Definition 1.1 ([1])**.** A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \to \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true. For more details please refer to [1, 32].

Definition 1.2 ([17])**.** Let $x \in K \subseteq \mathbb{R}^n$ and let $\phi : K \to \mathbb{R}$ be a continuous function. Then the set K is said to be ϕ-convex at x with respect to ϕ, if

$$x + \lambda e^{i\phi}(y - x) \in K, \ \forall x, y \in K, \ \lambda \in [0, 1].$$

For every $x, y \in K$, $e^{i\phi}(y - x) = y - x$, if and only if $\phi = 0$ and consequently ϕ-convexity reduces to convexity. It is true that every convex set is also a ϕ-convex, but the converse is not necessarily true.

Definition 1.3 ([16])**.** The set $K_{\phi\eta} \subseteq \mathbb{R}^n$ is said to be ϕ-invex at u with respect to $\phi(\cdot)$, if there exists $\phi(\cdot) : K \to \mathbb{R}$ and a bifunction $\eta(\cdot, \cdot) : K_{\phi\eta} \times K_{\phi\eta} \to \mathbb{R}^n$, such that

$$u + te^{i\phi}\eta(v, u) \in K_{\phi\eta}, \quad \forall u, v \in K_{\phi\eta}, \ t \in [0, 1].$$

The ϕ-invex set $K_{\phi\eta}$ is also described as $\phi\eta$-connected set. Note that the convex set with $\phi = 0$ and $\eta(v, u) = v - u$ is a ϕ-invex set, but the converse is not true (see [16]).

Definition 1.4 ([23])**.** The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to η, if for every $x, y \in K$ and $t \in [0, 1]$ we have that

$$f\big(x + t\eta(y, x)\big) \le (1 - t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity, since every convex function is preinvex with respect to the mapping $\eta(x, y) = y - x$. Further, there exist preinvex functions which are not convex.

Definition 1.5 ([15])**.** The function f on the ϕ-convex set K is said to be ϕ-convex with respect to ϕ, if

$$f\big(x + \lambda e^{i\phi}(y - x)\big) \le (1 - \lambda)f(x) + \lambda f(y), \quad \forall x, y \in K, \ \lambda \in [0, 1].$$

The function f is said to be ϕ-concave, if and only if $-f$ is ϕ-convex.

In [29] (see also [28]), Tunç and Yidirim defined the notion of MT-convex as follows.

Definition 1.6. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class of $\mathrm{MT}(I)$, if it is nonnegative and satisfies the following inequality

$$f\big(tx + (1 - t)y\big) \le \frac{\sqrt{t}}{2\sqrt{1 - t}}f(x) + \frac{\sqrt{1 - t}}{2\sqrt{t}}f(y)$$

for all $x, y \in I$ and $t \in (0, 1)$. For some interesting and significant integral inequalities concerning with the MT-convex functions, one can see in the recent papers [14, 21, 27–29]. It can be easily observed that convexity means just Jensen-convex when $t = \frac{1}{2}$.

In [18], Özdemir et al. established inequalities for twice differentiable m-convex functions which are connected with Hermite-Hadamard inequality, and they used the following lemma to prove their results.

Lemma 1.7. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I$ with $a < b$ and $f'' \in L[a, b]$. Then, the following equality holds:*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b - a}\int_a^b f(x)\mathrm{d}x = \frac{(b - a)^2}{2}\int_0^1 t(1 - t)f''\big(ta + (1 - t)b\big)\mathrm{d}t.$$

Note that under the conditions of Lemma 1.7, let $t = 1 - u$, then $\mathrm{d}t = -\mathrm{d}u$ and we also have

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = \frac{(b-a)^2}{2}\int_0^1 u(1-u)f''\big(ub+(1-u)a\big)\mathrm{d}u.$$

Currently, integral inequalities concerning with different kinds of convex functions remain attractive topics for many scholars in the field of convex analysis. For further information on the topic, the reader may refer to [3, 6–8, 10, 12, 13, 22, 26, 30, 31] and plenty of references cited therein.

The aim of this article is to establish new Hermite-Hadamard-type inequalities that are associated with the right-side of Hermite-Hadamard inequality for twice differentiable ϕ-MT-preinvex functions which generalize those results provided for twice differentiable MT-convex functions presented in [20].

2. Main results

As one can see, the definitions of the ϕ-convex, preinvex, and MT-convex functions have similar forms. This observation leads us to generalize these varieties of convexity. Firstly, the so-called 'ϕ-MT-convex', may be introduced as follows.

Definition 2.1. The function f defined on the ϕ-convex set $K_\phi \subseteq \mathbb{R}^n$ is said to be ϕ-MT-convex, if it is nonnegative and for all $x, y \in K_\phi$ and $t \in (0,1)$ satisfies the following inequality

$$f\big(x+te^{i\phi}(y-x)\big) \le \frac{\sqrt{1-t}}{2\sqrt{t}}f(x) + \frac{\sqrt{t}}{2\sqrt{1-t}}f(y).$$

The concept of the ϕ-MT-convex function may be further generalized as in the definition below.

Definition 2.2. The function f defined on the ϕ-invex set $K_{\phi\eta} \subseteq \mathbb{R}^n$ is said to be ϕ-MT-preinvex, if it is nonnegative and for all $x, y \in K_{\phi\eta}$ and $t \in (0,1)$ satisfies the following inequality

$$f\big(x+te^{i\phi}\eta(y,x)\big) \le \frac{\sqrt{1-t}}{2\sqrt{t}}f(x) + \frac{\sqrt{t}}{2\sqrt{1-t}}f(y).$$

To derive main results in this section, we prove the following lemma.

Lemma 2.3. *Let $K_{\phi\eta} \subseteq \mathbb{R}$ be a ϕ-invex subset with respect to $\phi(\cdot)$ and $\eta : K_{\phi\eta} \times K_{\phi\eta} \subseteq \mathbb{R}$, $a, b \in K_{\phi\eta}$ with $a < a+e^{i\phi}\eta(b,a)$ and $0 \le \phi \le \frac{\pi}{2}$. If $f : K_{\phi\eta} \to \mathbb{R}$ is a twice differentiable function and $f'' \in L[a, a+e^{i\phi}\eta(b,a)]$ we have that*

$$\begin{aligned}\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2} &- \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x \\ &= \frac{[e^{i\phi}\eta(b,a)]^2}{2}\int_0^1 t(1-t)f''\big(a+te^{i\phi}\eta(b,a)\big)\mathrm{d}t.\end{aligned}$$

Proof. Set

$$J = \frac{[e^{i\phi}\eta(b,a)]^2}{2}\int_0^1 t(1-t)f''\big(a+te^{i\phi}\eta(b,a)\big)\mathrm{d}t.$$

Since $a, b \in K_{\phi\eta}$ and $K_{\phi\eta}$ is ϕ-invex subset with respect to ϕ and η, for every $t \in [0,1]$, we have $a+te^{i\phi}\eta(b,a) \in K_{\phi\eta}$. By integrating by part, it yields that

$$\begin{aligned}J =& \frac{[e^{i\phi}\eta(b,a)]^2}{2}\left[\frac{t(1-t)}{e^{i\phi}\eta(b,a)}f'\big(a+te^{i\phi}\eta(b,a)\big)\Big|_0^1 + \int_0^1 \frac{2t-1}{e^{i\phi}\eta(b,a)}f'\big(a+te^{i\phi}\eta(b,a)\big)\mathrm{d}t\right] \\ =& \frac{[e^{i\phi}\eta(b,a)]^2}{2}\left[\frac{2t-1}{[e^{i\phi}\eta(b,a)]^2}f\big(a+te^{i\phi}\eta(b,a)\big)\Big|_0^1 - \frac{2}{[e^{i\phi}\eta(b,a)]^2}\int_0^1 f\big(a+te^{i\phi}\eta(b,a)\big)\mathrm{d}t\right].\end{aligned}$$

Let $x = a + te^{i\phi}\eta(b,a)$, then $\mathrm{d}x = e^{i\phi}\eta(b,a)\,\mathrm{d}t$ and we have

$$\begin{aligned} J &= \frac{[e^{i\phi}\eta(b,a)]^2}{2}\left[\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{[e^{i\phi}\eta(b,a)]^2} - \frac{2}{[e^{i\phi}\eta(b,a)]^3}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right] \\ &= \frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2} - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x, \end{aligned}$$

which is required. □

Remark 2.4. By applying Lemma 2.3 with $\phi = 0$ and $\eta(b,a) = b-a$, we can obtain [18, Lemma 1] which may be discovered also in [19].

With the help of Lemma 2.3, let us begin with the following result involving differentiable ϕ-MT-preinvex function.

Theorem 2.5. *Let $A_{\phi\eta} \subseteq \mathbb{R}_0$ be an open ϕ-invex subset with respect to $\phi(\cdot)$ and $\eta : A_{\phi\eta} \times A_{\phi\eta} \to \mathbb{R}_0$, $a, b \in A_{\phi\eta}$ with $a < a+e^{i\phi}\eta(b,a)$ and $0 \le \phi \le \frac{\pi}{2}$. Suppose that $f : A_{\phi\eta} \to \mathbb{R}_0$ is a twice differentiable function and $f'' \in L[a, a+e^{i\phi}\eta(b,a)]$. If $|f''|$ is ϕ-MT-preinvex on $A_{\phi\eta}$ and $|f''(x)| \le M$, $x \in [a, a+e^{i\phi}\eta(b,a)]$, then the following inequality holds:*

$$\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2} - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \le \frac{M\pi[e^{i\phi}\eta(b,a)]^2}{32}.$$

Proof. Since $a + te^{i\phi}\eta(b,a) \in A_{\phi\eta}$, by using Lemma 2.3 and ϕ-MT-preinvex of $|f''|$, we can obtain that

$$\begin{aligned} &\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2} - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \\ &\le \frac{[e^{i\phi}\eta(b,a)]^2}{2}\int_0^1 t(1-t)\big|f''\big(a+te^{i\phi}\eta(b,a)\big)\big|\mathrm{d}t \\ &\le \frac{[e^{i\phi}\eta(b,a)]^2}{2}\int_0^1 t(1-t)\left[\frac{\sqrt{1-t}}{2\sqrt{t}}|f''(a)| + \frac{\sqrt{t}}{2\sqrt{1-t}}|f''(b)|\right]\mathrm{d}t \\ &\le \frac{M[e^{i\phi}\eta(b,a)]^2}{4}\int_0^1 \left[t^{\frac{1}{2}}(1-t)^{\frac{3}{2}} + t^{\frac{3}{2}}(1-t)^{\frac{1}{2}}\right]\mathrm{d}t \\ &= \frac{M\pi[e^{i\phi}\eta(b,a)]^2}{32}, \end{aligned}$$

where Euler Beta function is defined by

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \forall x, y > 0.$$

The proof is completed. □

Corollary 2.6. *Under the conditions of Theorem 2.5, if $\phi = 0$ and $\eta(b,a) = b-a$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{M\pi(b-a)^2}{32}.$$

This is the result given in [20, Theorem 2.1].

Next, by utilizing Lemma 2.3 again, we prove the following result.

Theorem 2.7. *Let $A_{\phi\eta} \subseteq \mathbb{R}_0$ be an open ϕ-invex subset with respect to $\phi(\cdot)$ and $\eta : A_{\phi\eta} \times A_{\phi\eta} \to \mathbb{R}_0$, $a, b \in A_{\phi\eta}$ with $a < a+e^{i\phi}\eta(b,a)$ and $0 \le \phi \le \frac{\pi}{2}$. Suppose that $f : A_{\phi\eta} \to \mathbb{R}_0$ is a twice differentiable* *function and* $f'' \in L[a, a + e^{i\phi}\eta(b,a)]$. *If* $|f''|^q$ *is* ϕ*-MT-preinvex on* $A_{\phi\eta}$, $q > 1$, $p^{-1} + q^{-1} = 1$ *and* $|f''(x)| \leq M$, $x \in [a, a + e^{i\phi}\eta(b,a)]$, *then we have:*

$$\left|\frac{f(a) + f\big(a + e^{i\phi}\eta(b,a)\big)}{2} - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \leq \frac{M[e^{i\phi}\eta(b,a)]^2}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\beta^{\frac{1}{p}}(p+1,p+1).$$

Proof. Since $a + te^{i\phi}\eta(b,a) \in A_{\phi\eta}$, by applying Lemma 2.3 and the Hölder's inequality, it follows that

$$\begin{aligned}
&\left|\frac{f(a) + f\big(a + e^{i\phi}\eta(b,a)\big)}{2} - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \\
&\leq \frac{[e^{i\phi}\eta(b,a)]^2}{2}\int_0^1 t(1-t)\Big|f''\big(a + te^{i\phi}\eta(b,a)\big)\Big|\mathrm{d}t \\
&\leq \frac{[e^{i\phi}\eta(b,a)]^2}{2}\left[\int_0^1 t^p(1-t)^p\mathrm{d}t\right]^{\frac{1}{p}}\left[\int_0^1 \Big|f''\big(a + te^{i\phi}\eta(b,a)\big)\Big|^q\mathrm{d}t\right]^{\frac{1}{q}}.
\end{aligned}$$

Since $|f''|^q$ is ϕ-MT-preinvex function on $A_{\phi\eta}$ and $|f''(x)| \leq M$, we have that

$$\begin{aligned}
\int_0^1 \Big|f''\big(a + te^{i\phi}\eta(b,a)\big)\Big|^q\mathrm{d}t &\leq \int_0^1 \left[\frac{\sqrt{1-t}}{2\sqrt{t}}|f''(a)|^q + \frac{\sqrt{t}}{2\sqrt{1-t}}|f''(b)|^q\right]\mathrm{d}t \\
&= \frac{\pi}{4}\Big[|f''(a)|^q + |f''(b)|^q\Big] \\
&\leq \frac{\pi}{2}M^q.
\end{aligned}$$

Therefore, we have

$$\left|\frac{f(a) + f\big(a + e^{i\phi}\eta(b,a)\big)}{2} - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \leq \frac{M[e^{i\phi}\eta(b,a)]^2}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\beta^{\frac{1}{p}}(p+1,p+1),$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Hence the theorem is proved. □

Corollary 2.8. *Under the conditions of Theorem* 2.7, *if* $\phi = 0$ *and* $\eta(b,a) = b - a$, *then*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{M(b-a)^2}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\beta^{\frac{1}{p}}(p+1,p+1).$$

This is the result given in [20, Theorem 2.2].

A different approach leads us to the following result.

Theorem 2.9. *Suppose that all the assumptions of Theorem* 2.7 *are satisfied. Then the following inequality holds:*

$$\left|\frac{f(a) + f\big(a + e^{i\phi}\eta(b,a)\big)}{2} - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \leq \frac{M[e^{i\phi}\eta(b,a)]^2}{2}\left[\frac{1}{(p+1)(p+2)}\right]^{\frac{1}{p}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}.$$

Proof. Since $a + te^{i\phi}\eta(b,a) \in A_{\phi\eta}$, by using Lemma 2.3 and properties of modulus, it yields that

$$\begin{aligned}
&\left|\frac{f(a) + f\big(a + e^{i\phi}\eta(b,a)\big)}{2} - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \\
&\leq \frac{[e^{i\phi}\eta(b,a)]^2}{2}\int_0^1 t(1-t)\Big|f''\big(a + te^{i\phi}\eta(b,a)\big)\Big|\mathrm{d}t.
\end{aligned} \tag{2.1}$$

Now, if we use the following weighted version of Hölder's inequality

$$\left|\int_I f(s)g(s)h(s)\mathrm{d}s\right| \le \left(\int_I |f(s)|^p h(s)\mathrm{d}s\right)^{\frac{1}{p}} \left(\int_I |g(s)|^q h(s)\mathrm{d}s\right)^{\frac{1}{q}}, \tag{2.2}$$

for $p > 1$, $p^{-1} + q^{-1} = 1$, h is nonnegative on I and provided all the other integrals exist and are finite. If we rewrite the (2.1) regarding (2.2) with $|f''|^q$ is a ϕ-MT-preinvex function on $A_{\phi\eta}$ for some fixed $q > 1$ and $|f''(x)| \le M$, then we have

$$\begin{aligned}
&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2} - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \\
&\le \frac{[e^{i\phi}\eta(b,a)]^2}{2}\int_0^1 t(1-t)\Big|f''\big(a+te^{i\phi}\eta(b,a)\big)\Big|\mathrm{d}t \\
&\le \frac{[e^{i\phi}\eta(b,a)]^2}{2}\left[\int_0^1 (1-t)^p t\mathrm{d}t\right]^{\frac{1}{p}}\left[\int_0^1 \Big|f''\big(a+te^{i\phi}\eta(b,a)\big)\Big|^q t\mathrm{d}t\right]^{\frac{1}{q}} \\
&\le \frac{[e^{i\phi}\eta(b,a)]^2}{2}\left[\int_0^1 (1-t)^p t\mathrm{d}t\right]^{\frac{1}{p}} \\
&\quad \times \left\{\int_0^1 t\left[\frac{\sqrt{1-t}}{2\sqrt{t}}|f''(a)|^q + \frac{\sqrt{t}}{2\sqrt{1-t}}|f''(b)|^q\right]\mathrm{d}t\right\}^{\frac{1}{q}} \\
&= \frac{[e^{i\phi}\eta(b,a)]^2}{2}\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}} \\
&\quad \times \left\{\frac{1}{2}\int_0^1 \left[t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}|f''(a)|^q + t^{\frac{3}{2}}(1-t)^{-\frac{1}{2}}|f''(b)|^q\right]\mathrm{d}t\right\}^{\frac{1}{q}} \\
&\le \frac{M[e^{i\phi}\eta(b,a)]^2}{2}\left[\frac{1}{(p+1)(p+2)}\right]^{\frac{1}{p}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, which completes the proof. □

Corollary 2.10. *Under the conditions of Theorem 2.9, if $\phi = 0$ and $\eta(b,a) = b - a$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{M(b-a)^2}{2}\left[\frac{1}{(p+1)(p+2)}\right]^{\frac{1}{p}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}.$$

By using a similar way of Theorem 2.7, we can prove the following theorem.

Theorem 2.11. *Suppose that all the assumptions of Theorem 2.7 are satisfied. Then the following inequality holds:*

$$\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2} - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \le \frac{M\big[e^{i\phi}\eta(b,a)\big]^2}{2^{1+\frac{1}{q}}}\Big(\frac{1}{1+p}\Big)^{\frac{1}{p}}\beta^{\frac{1}{q}}\Big(\frac{1}{2}, q+\frac{1}{2}\Big).$$

Proof. Since $a + te^{i\phi}\eta(b,a) \in A_{\phi\eta}$, by using Lemma 2.3 and the Hölder's inequality, it follows that

$$\begin{aligned}
&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2} - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \\
&\le \frac{[e^{i\phi}\eta(b,a)]^2}{2}\int_0^1 t(1-t)\Big|f''\big(a+te^{i\phi}\eta(b,a)\big)\Big|\mathrm{d}t \\
&\le \frac{[e^{i\phi}\eta(b,a)]^2}{2}\left(\int_0^1 t^p\mathrm{d}t\right)^{\frac{1}{p}}\left[\int_0^1 (1-t)^q\Big|f''\big(a+te^{i\phi}\eta(b,a)\big)\Big|^q\mathrm{d}t\right]^{\frac{1}{q}}.
\end{aligned}$$

Since $|f''|^q$ is a ϕ-MT-preinvex function on $A_{\phi\eta}$ for some fixed $q>1$ and $|f''(x)|\le M$, we have that

$$\begin{aligned}\int_0^1 (1-t)^q\Big|f''\big(a+te^{i\phi}\eta(b,a)\big)\Big|^q\mathrm{d}t &\le \int_0^1 (1-t)^q\Big[\frac{\sqrt{1-t}}{2\sqrt{t}}|f''(a)|^q+\frac{\sqrt{t}}{2\sqrt{1-t}}|f''(b)|^q\Big]\mathrm{d}t\\ &= \frac{\beta(\frac{1}{2},q+\frac{3}{2})|f''(a)|^q+\beta(\frac{3}{2},q+\frac{1}{2})|f''(b)|^q}{2}\\ &\le M^q\beta(\frac{1}{2},q+\frac{1}{2}).\end{aligned}$$

Therefore, we deduce that

$$\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2}-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right|\le \frac{M\big[e^{i\phi}\eta(b,a)\big]^2}{2^{1+\frac{1}{q}}}\Big(\frac{1}{1+p}\Big)^{\frac{1}{p}}\beta^{\frac{1}{q}}\Big(\frac{1}{2},q+\frac{1}{2}\Big),$$

where $\frac{1}{p}+\frac{1}{q}=1$, which completes the proof. □

Corollary 2.12. *Under the conditions of Theorem 2.11, if $\phi=0$ and $\eta(b,a)=b-a$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\frac{M\big(b-a\big)^2}{2^{1+\frac{1}{q}}}(\frac{1}{1+p})^{\frac{1}{p}}\beta^{\frac{1}{q}}(\frac{1}{2},q+\frac{1}{2}).$$

This is the result given in [20, Theorem 2.3].

Theorem 2.13. *Under the conditions of Theorem 2.7, we have*

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2}-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right|\\ &\le \frac{M[e^{i\phi}\eta(b,a)]^2}{2}\Big(\frac{q-1}{2q-p-1}\Big)^{\frac{q-1}{q}}\Big(\frac{1}{2}\Big)^{\frac{1}{q}}\Big[\frac{\Gamma\Big(p+\frac{1}{2}\Big)\Gamma\Big(q+\frac{3}{2}\Big)}{\Gamma(p+q+2)}+\frac{\Gamma\Big(p+\frac{3}{2}\Big)\Gamma\Big(q+\frac{1}{2}\Big)}{\Gamma(p+q+2)}\Big]^{\frac{1}{q}}.\end{aligned}$$

Proof. By using Lemma 1.7 and the Hölder's inequality for $q>1$, it follows that

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2}-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right|\\ &\le \frac{[e^{i\phi}\eta(b,a)]^2}{2}\int_0^1 t(1-t)\Big|f''\big(a+te^{i\phi}\eta(b,a)\big)\Big|\mathrm{d}t\\ &\le \frac{[e^{i\phi}\eta(b,a)]^2}{2}\Big[\int_0^1 t^{\frac{q-p}{q-1}}\mathrm{d}t\Big]^{\frac{q-1}{q}}\Big[\int_0^1 t^p(1-t)^q\Big|f''\big(a+te^{i\phi}\eta(b,a)\big)\Big|^q\mathrm{d}t\Big]^{\frac{1}{q}}.\end{aligned}$$

On the other hand, since $|f''|^q$ is a ϕ-MT-preinvex function on $A_{\phi\eta}$ and $|f''(x)|\le M$, we know that

$$\begin{aligned}\int_0^1 t^p(1-t)^q\Big|f''\big(a+te^{i\phi}\eta(b,a)\big)\Big|^q\mathrm{d}t &\le \int_0^1 t^p(1-t)^q\Big[\frac{\sqrt{1-t}}{2\sqrt{t}}|f''(a)|^q+\frac{\sqrt{t}}{2\sqrt{1-t}}|f''(b)|^q\Big]\mathrm{d}t\\ &= \frac{M^q}{2}\Big[\frac{\Gamma\Big(p+\frac{1}{2}\Big)\Gamma\Big(q+\frac{3}{2}\Big)}{\Gamma(p+q+2)}+\frac{\Gamma\Big(p+\frac{3}{2}\Big)\Gamma\Big(q+\frac{1}{2}\Big)}{\Gamma(p+q+2)}\Big].\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2}-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right|\\ &\le \frac{M[e^{i\phi}\eta(b,a)]^2}{2}\Big(\frac{q-1}{2q-p-1}\Big)^{\frac{q-1}{q}}\Big(\frac{1}{2}\Big)^{\frac{1}{q}}\\ &\quad\times\Big[\frac{\Gamma\Big(p+\frac{1}{2}\Big)\Gamma\Big(q+\frac{3}{2}\Big)}{\Gamma(p+q+2)}+\frac{\Gamma\Big(p+\frac{3}{2}\Big)\Gamma\Big(q+\frac{1}{2}\Big)}{\Gamma(p+q+2)}\Big]^{\frac{1}{q}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, which is required. □

Corollary 2.14. *Under the conditions of Theorem 2.7, if $\phi=0$ and $\eta(b,a)=b-a$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\frac{M(b-a)^2}{2}\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\times\left[\frac{\Gamma\left(p+\frac{3}{2}\right)\Gamma\left(q+\frac{1}{2}\right)}{\Gamma(p+q+2)}+\frac{\Gamma\left(p+\frac{1}{2}\right)\Gamma\left(q+\frac{3}{2}\right)}{\Gamma(p+q+2)}\right]^{\frac{1}{q}}.$$

Theorem 2.15. *Let $A_{\phi\eta}\subseteq\mathbb{R}_0$ be an open ϕ-invex subset with respect to $\phi(\cdot)$ and $\eta: A_{\phi\eta}\times A_{\phi\eta}\to\mathbb{R}_0$, $a,b\in A_{\phi\eta}$ with $a<a+e^{i\phi}\eta(b,a)$ and $0\le\phi\le\frac{\pi}{2}$. Suppose that $f:A_{\phi\eta}\to\mathbb{R}_0$ is a twice differentiable function and $f''\in L[a,a+e^{i\phi}\eta(b,a)]$. If $|f''|^q$ is ϕ-MT-preinvex on $A_{\phi\eta}$ with $q\ge1$ and $|f''(x)|\le M$, $x\in[a,a+e^{i\phi}\eta(b,a)]$, then we have*

$$\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2}-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|\le\frac{M\left[e^{i\phi}\eta(b,a)\right]^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{\pi}{16}\right)^{\frac{1}{q}}.$$

Proof. Since $a+te^{i\phi}\eta(b,a)\in A_{\phi\eta}$, from Lemma 2.3 and by using the Hölder's inequality, we have that

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2}-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|\\&\le\frac{\left[e^{i\phi}\eta(b,a)\right]^2}{2}\int_0^1(t-t^2)\left|f''\big(a+te^{i\phi}\eta(b,a)\big)\right|\mathrm{d}t\\&\le\frac{\left[e^{i\phi}\eta(b,a)\right]^2}{2}\left[\int_0^1(t-t^2)\mathrm{d}t\right]^{1-\frac{1}{q}}\left[\int_0^1(t-t^2)\left|f''\big(a+te^{i\phi}\eta(b,a)\big)\right|^q\mathrm{d}t\right]^{\frac{1}{q}}.\end{aligned}$$

Since $|f''|$ is a ϕ-MT-preinvex function on $A_{\phi\eta}$ and $|f''(x)|\le M$, we have

$$\begin{aligned}\int_0^1(t-t^2)\left|f''\big(a+te^{i\phi}\eta(b,a)\big)\right|^q\mathrm{d}t&\le\int_0^1\left[\frac{t(1-t)\sqrt{1-t}}{2\sqrt{t}}|f''(a)|^q+\frac{t(1-t)\sqrt{t}}{2\sqrt{1-t}}|f''(b)|^q\right]\mathrm{d}t\\&=\frac{\pi M^q}{16}.\end{aligned}$$

Therefore, we obtain

$$\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2}-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|\le\frac{M\left[e^{i\phi}\eta(b,a)\right]^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{\pi}{16}\right)^{\frac{1}{q}},$$

which is required. □

Corollary 2.16. *Under the conditions of Theorem 2.15, if $\phi=0$ and $\eta(b,a)=b-a$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\frac{M(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{\pi}{16}\right)^{\frac{1}{q}}.$$

This is the result given in [20, Theorem 2.4]. We continue with the following result.

Theorem 2.17. *Suppose that all the assumptions of Theorem 2.15 are satisfied. Then the following inequality holds:*

$$\left|\frac{f(a)+f(a+e^{i\phi}\eta(b,a))}{2}-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|\le\frac{M\left[e^{i\phi}\eta(b,a)\right]^2}{2}\left[\frac{\Gamma(q+\frac{1}{2})\Gamma(q+\frac{3}{2})}{\Gamma(2q+2)}\right]^{\frac{1}{q}}.$$

Proof. Since $a + te^{i\phi}\eta(b,a) \in A_{\phi\eta}$, from Lemma 2.3 and by using the Hölder's inequality, we have that

$$\begin{aligned}
&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2} - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \\
&\leq \frac{\big[e^{i\phi}\eta(b,a)\big]^2}{2}\int_0^1 (t-t^2)\big|f''\big(a+te^{i\phi}\eta(b,a)\big)\big|\mathrm{d}t \\
&\leq \frac{\big[e^{i\phi}\eta(b,a)\big]^2}{2}\left(\int_0^1 1\mathrm{d}t\right)^{1-\frac{1}{q}}\left[\int_0^1 (t-t^2)^q\big|f''\big(a+te^{i\phi}\eta(b,a)\big)\big|^q\mathrm{d}t\right]^{\frac{1}{q}}.
\end{aligned}$$

Since $|f''|^q$ is a ϕ-MT-preinvex function on $A_{\phi\eta}$ and $|f''| \leq M$, it yields that

$$\begin{aligned}
\int_0^1 \big(t-t^2\big)^q\big|f''\big(a+te^{i\phi}\eta(b,a)\big)\big|^q\mathrm{d}t &\leq \int_0^1 [t(1-t)]^q\Big[\frac{\sqrt{1-t}}{2\sqrt{t}}|f''(a)|^q + \frac{\sqrt{t}}{2\sqrt{1-t}}|f''(b)|^q\Big]\mathrm{d}t \\
&= \frac{1}{2}\int_0^1 \big[t^{q-\frac{1}{2}}(1-t)^{q+\frac{1}{2}}|f''(a)|^q + t^{q+\frac{1}{2}}(1-t)^{q-\frac{1}{2}}|f''(b)|^q\big]\mathrm{d}t \\
&\leq \frac{1}{2}M^q\Big[\beta\big(q+\frac{1}{2}, q+\frac{3}{2}\big) + \beta\big(q+\frac{3}{2}, q+\frac{1}{2}\big)\Big] \\
&= \frac{M^q\Gamma(q+\frac{1}{2})\Gamma(q+\frac{3}{2})}{\Gamma(2q+2)}.
\end{aligned}$$

Therefore, we have

$$\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2} - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \leq \frac{M\big[e^{i\phi}\eta(b,a)\big]^2}{2}\left[\frac{\Gamma(q+\frac{1}{2})\Gamma(q+\frac{3}{2})}{\Gamma(2q+2)}\right]^{\frac{1}{q}},$$

which is required. □

Corollary 2.18. *Under the conditions of Theorem* 2.17*, if* $\phi = 0$ *and* $\eta(b,a) = b-a$*, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{M(b-a)^2}{2}\left[\frac{\Gamma(q+\frac{1}{2})\Gamma(q+\frac{3}{2})}{\Gamma(2q+2)}\right]^{\frac{1}{q}}.$$

Finally we prove the following result.

Theorem 2.19. *Suppose that all the assumptions of Theorem* 2.15 *are satisfied, then we have:*

$$\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2} - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \leq \frac{M\big[e^{i\phi}\eta(b,a)\big]^2}{2}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\beta^{\frac{1}{q}}\Big(\frac{3}{2}, q+\frac{1}{2}\Big).$$

Proof. By using Lemma 2.3 and well-known power-mean inequality, it follows that

$$\begin{aligned}
&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2} - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \\
&\leq \frac{[e^{i\phi}\eta(b,a)]^2}{2}\int_0^1 t(1-t)\Big|f''\big(a+te^{i\phi}\eta(b,a)\big)\Big|\mathrm{d}t \\
&\leq \frac{[e^{i\phi}\eta(b,a)]^2}{2}\left(\int_0^1 t\mathrm{d}t\right)^{1-\frac{1}{q}}\left[\int_0^1 t(1-t)^q\Big|f''\big(a+te^{i\phi}\eta(b,a)\big)\Big|^q\mathrm{d}t\right]^{\frac{1}{q}}.
\end{aligned}$$

Since $|f''|^q$ is ϕ-MT-preinvex on $A_{\phi\eta}$ and $|f''(x)| \leq M$, we have

$$\begin{aligned}
\int_0^1 t(1-t)^q\Big|f''\big(a+te^{i\phi}\eta(b,a)\big)\Big|^q\mathrm{d}t &\leq \int_0^1 t(1-t)^q\left[\frac{\sqrt{1-t}}{2\sqrt{t}}|f''(a)|^q + \frac{\sqrt{t}}{2\sqrt{1-t}}|f''(b)|^q\right]\mathrm{d}t \\
&= \frac{\beta(\frac{3}{2}, q+\frac{3}{2})|f''(a)|^q + \beta(\frac{5}{2}, q+\frac{1}{2})|f''(b)|^q}{2} \\
&\leq M^q\beta\Big(\frac{3}{2}, q+\frac{1}{2}\Big).
\end{aligned}$$

Therefore, we deduce that

$$\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2}-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|\leq\frac{M\big[e^{i\phi}\eta(b,a)\big]^2}{2}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\beta^{\frac{1}{q}}\Big(\frac{3}{2},q+\frac{1}{2}\Big),$$

where $\frac{1}{p}+\frac{1}{q}=1$, which completes the proof. □

Corollary 2.20. *Under the conditions of Theorem* 2.19*, if* $\phi=0$ *and* $\eta(b,a)=b-a$*, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\leq\frac{M\big(b-a\big)^2}{2}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\beta^{\frac{1}{q}}\Big(\frac{3}{2},q+\frac{1}{2}\Big).$$

This is the result given in [20, Theorem 2.5].

3. Applications to special means

We consider the means for arbitrary positive numbers a,b $(a\neq b)$ as below:

(1) The arithmetic mean: $A=A(a,b)=\frac{a+b}{2}; a,b\in\mathbb{R}$;

(2) The logarithmic mean: $L(a,b)=\frac{b-a}{\ln|b|-\ln|a|}; |a|\neq|b|, ab\neq 0, a,b\in\mathbb{R}$;

(3) The generalized logarithmic mean:

$$L_n(a,b)=\Big[\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\Big]^{\frac{1}{n}}; n\in\mathbb{Z}\backslash\{-1,0\}, a,b\in\mathbb{R}, a\neq b.$$

Now by using the results of Section 2, we give some applications to special means of real numbers.

Proposition 3.1. *Let* $a,b\in\mathbb{R}$*,* $0<a<b$ *and* $n\in\mathbb{Z}$*,* $|n|\geq 2$*. Then, for all* $q\geq 1$

$$|A(a^n,b^n)-L_n^n(a,b)|\leq\frac{M(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{\pi}{16}\right)^{\frac{1}{q}},$$

$$|A(a^n,b^n)-L_n^n(a,b)|\leq\frac{M(b-a)^2}{2}\left[\frac{\Gamma(q+\frac{1}{2})\Gamma(q+\frac{3}{2})}{\Gamma(2q+2)}\right]^{\frac{1}{q}},$$

and

$$|A(a^n,b^n)-L_n^n(a,b)|\leq\frac{M\big(b-a\big)^2}{2}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\beta^{\frac{1}{q}}\Big(\frac{3}{2},q+\frac{1}{2}\Big).$$

Proof. The assertion follows from Corollaries 2.16, 2.18 and 2.20, for $f(x)=x^n$, $x\in\mathbb{R}$, $n\in\mathbb{Z}$, $|n|\geq 2$. □

Proposition 3.2. *Let* $a,b\in\mathbb{R}$*,* $0<a<b$*. Then, for all* $q\geq 1$

$$|A(a^1,b^{-1})-L^{-1}(a,b)|\leq\frac{M(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{\pi}{16}\right)^{\frac{1}{q}},$$

$$|A(a^1,b^{-1})-L^{-1}(a,b)|\leq\frac{M(b-a)^2}{2}\left[\frac{\Gamma(q+\frac{1}{2})\Gamma(q+\frac{3}{2})}{\Gamma(2q+2)}\right]^{\frac{1}{q}},$$

and

$$|A(a^1,b^{-1})-L^{-1}(a,b)|\leq\frac{M\big(b-a\big)^2}{2}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\beta^{\frac{1}{q}}\Big(\frac{3}{2},q+\frac{1}{2}\Big).$$

Proof. The assertion follows from Corollaries 2.16, 2.18 and 2.20, for $f(x)=\frac{1}{x}$. □

4. Some error estimates for the Trapezoidal formula

Let d be a division $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of the interval $[a, b]$ and consider the quadrature formula

$$\int_a^b f(x)\mathrm{d}x = T(f,d) + E(f,d), \tag{4.1}$$

where

$$T(f,d) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)$$

for the trapezoidal version and $E(f, d)$ stands for the associated approximation error.

Proposition 4.1. *Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a,b]$, where $a, b \in I$ with $a < b$ and $|f''|^q$ is MT-convex on $[a, b]$, where $p > 1, p^{-1} + q^{-1} = 1$. Then in (4.1), for every division d of $[a, b]$ and $|f''(x)| \le M$, $x \in [a, b]$, the trapezoidal error estimate satisfies*

$$|E(f,d)| \le \frac{M}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{q}} \beta^{\frac{1}{p}}(p+1, p+1) \sum_{i=0}^{n-1}(x_{i+1} - x_i)^3.$$

Proof. By applying Corollary 2.8 to the subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, 2, \cdots, n-1)$ of the division, we have that

$$\left|\frac{f(x_i) - f(x_{i+1})}{2} - \frac{1}{x_{i+1} - x_i}\int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x\right| \le \frac{M}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{q}} \beta^{\frac{1}{p}}(p+1, p+1) \sum_{i=0}^{n-1}(x_{i+1} - x_i)^2.$$

Hence in (4.1), we deduce that

$$\begin{aligned}\left|\int_a^b f(x)\mathrm{d}x - T(f,d)\right| &= \left|\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x - \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)\right\}\right| \\ &\le \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x - \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)\right| \\ &\le \frac{M}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{q}} \beta^{\frac{1}{p}}(p+1, p+1) \sum_{i=0}^{n-1}(x_{i+1} - x_i)^3,\end{aligned}$$

which completes the proof. □

Proposition 4.2. *Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a,b]$, where $a, b \in I$ with $a < b$ and $|f''|^q$ is MT-convex on $[a, b]$, where $q \ge 1$. Then in (4.1), for every division d of $[a, b]$ and $|f''(x)| \le M$, $x \in [a, b]$, the trapezoidal error estimate satisfies*

$$|E(f,d)| \le \frac{M}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{\pi}{16}\right)^{\frac{1}{q}} \sum_{i=0}^{n-1}(x_{i+1} - x_i)^3.$$

Proof. The proof is similar to that of Proposition 4.1 and by using Corollary 2.16. □

Proposition 4.3. *Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a,b]$, where $a, b \in I$ with $a < b$ and $|f''|^q$ is MT-convex on $[a, b]$, where $q \ge 1$. Then in (4.1), for every division d of $[a, b]$ and $|f''(x)| \le M$, $x \in [a, b]$, the trapezoidal error estimate satisfies*

$$|E(f,d)| \le \frac{M}{2}\left[\frac{\Gamma(q+\frac{1}{2})\Gamma(q+\frac{3}{2})}{\Gamma(2q+2)}\right]^{\frac{1}{q}} \sum_{i=0}^{n-1}(x_{i+1} - x_i)^3.$$

Proof. The proof is similar to that of Proposition 4.1 and by using Corollary 2.18. □

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