Essential norm of weighted composition operators from $H^\infty$ to the Zygmund space

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Abstract

Let $\varphi$ be an analytic self-map of the unit disk $\mathbb{D}$ and $u \in H(\mathbb{D})$, the space of analytic functions on $\mathbb{D}$. The weighted composition operator, denoted by $u C_\varphi$, is defined by $(u C_\varphi f)(z) = u(z)f(\varphi(z))$, $f \in H(\mathbb{D})$, $z \in \mathbb{D}$. In this paper, we give three different estimates for the essential norm of the operator $u C_\varphi$ from $H^\infty$ into the Zygmund space, denoted by $Z$. In particular, we show that $\| u C_\varphi \|_{e, H^\infty \to Z} \approx \limsup_{n \to \infty} \| u \varphi^n \|_Z$. ©2016 All rights reserved.

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1. Introduction and preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. Let $H^\infty$ denote the bounded analytic function space, i.e.,

$$
H^\infty = \{ f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} |f(z)| < \infty \}.
$$

The Bloch space, denoted by $\mathcal{B}$, is the space of all functions $f \in H(\mathbb{D})$ such that

$$
\| f \|_\mathcal{B} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.
$$

For more details of the Bloch space we refer the reader to [21].

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Let $Z$ denote the set of all functions $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that
\[
\|f\| = \sup_{n} \frac{|f(e^{i(\theta+h)})| + f(e^{i(-\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty,
\]
where the supremum is taken over all $\theta \in \mathbb{R}$ and $h > 0$. By Theorem 5.3 of [3], we see that $f \in Z$ if and only if $\sup_{z \in \mathbb{D}}(1 - |z|^2)|f''(z)| < \infty$. $Z$, called the Zygmund space, a Banach space with the norm defined by
\[
\|f\|_Z = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}}(1 - |z|^2)|f''(z)|.
\]
See [1, 3, 6] for more details on the space $Z$.

Let $\varphi$ be an analytic self-map of $\mathbb{D}$. The composition operator $C_{\varphi}$ is defined by
\[
(C_{\varphi}f)(z) = f(\varphi(z)), \quad f \in H(\mathbb{D}).
\]
Let $u \in H(\mathbb{D})$. The weighted composition operator, denoted by $uC_{\varphi}$, is defined by
\[
(uC_{\varphi}f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.
\]

Let $X, Y$ be Banach spaces and $\| \cdot \|_{X \rightarrow Y}$ denotes the operator norm. Recall that the essential norm of a bounded linear operator $T : X \rightarrow Y$ is its distance to the set of compact operators $K$ mapping $X$ into $Y$, that is,
\[
\|T\|_{e,X \rightarrow Y} = \inf \{\|T - K\|_{X \rightarrow Y} : K \text{ is a compact operator}\}.
\]
It is well-known that $\|T\|_{e,X \rightarrow Y} = 0$ if and only if $T : X \rightarrow Y$ is compact.

The composition operator $C_{\varphi} : B \rightarrow B$ is bounded for any $\varphi$ by the Schwarz-Pick Lemma. Madigan and Matheson studied the compactness of the operator $C_{\varphi} : B \rightarrow B$ in [11]. Montes-Rodrieguez [12] studied the essential norm of the operator $C_{\varphi} : B \rightarrow B$ and got the exact value for it, i.e.,
\[
\|C_{\varphi}\|_{e,B \rightarrow B} = \lim_{s \rightarrow 1} \sup_{|\varphi(z)| > s} \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2}.
\]
Tjani [16] proved that $C_{\varphi} : B \rightarrow B$ is compact if and only if $\lim_{|z| \rightarrow 1} \|C_{\varphi}\sigma_a\| = 0$, where $\sigma_a(z) = \frac{a-z}{1-ar{a}z}$. Wulan et al. [17] proved that $C_{\varphi} : B \rightarrow B$ is compact if and only if $\lim_{|z| \rightarrow 1} \|\varphi'(z)\| = 0$. In [20], Zhao obtained that
\[
\|C_{\varphi}\|_{e,B \rightarrow B} = \frac{e}{2} \lim_{n \rightarrow \infty} \sup \|\varphi^n\|.
\]
The boundedness and compactness of the operator $uC_{\varphi} : B \rightarrow B$ were studied in [13]. The essential norm of the operator $uC_{\varphi} : B \rightarrow B$ was studied in [3, 10].

The composition operators, weighted composition operators and related operators on the Zygmund space were studied in [1, 2, 4, 6, 9, 11, 15, 18, 19]. In [2], the authors studied the operator $uC_{\varphi} : H^{\infty} \rightarrow Z$. Among others, they showed that $uC_{\varphi} : H^{\infty} \rightarrow Z$ is compact if and only if $\lim_{n \rightarrow \infty} \|u\varphi^n\| = 0$. In fact, from the proof of Theorem 2 in [2], or [13, 19], we find that they obtained the following result.

**Theorem 1.1** ([2, 13, 19]). Let $u \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$ such that the operator $uC_{\varphi} : H^{\infty} \rightarrow Z$ is bounded. Then the following statements are equivalent:

(a) The operator $uC_{\varphi} : H^{\infty} \rightarrow Z$ is compact.

(b) $\lim_{n \rightarrow \infty} \|u\varphi^n\| = 0$.

(c) $\limsup_{|\varphi(w)| \rightarrow 1} \|uC_{\varphi}f_{\varphi(w)}\| = \limsup_{|\varphi(w)| \rightarrow 1} \|uC_{\varphi}g_{\varphi(w)}\| = \limsup_{|\varphi(w)| \rightarrow 1} \|uC_{\varphi}h_{\varphi(w)}\| = 0$,

where
\[
f_a(z) = \frac{1 - |a|^2}{1 - \overline{a}z}, \quad g_a(z) = \frac{(1 - |a|^2)^2}{(1 - \overline{a}z)^2}, \quad h_a(z) = \frac{(1 - |a|^2)^3}{(1 - \overline{a}z)^3}, \quad a \in \mathbb{D}.
\]
Hence $z_f$ and Proof. First, we prove that $\max$ and $Q$. Hu, X. Zhu, J. Nonlinear Sci. Appl. 9 (2016), 5082–5092 5084

where $\phi$ is a compact operator if and only if given a bounded sequence $(\{f_n\})$ in $X$ such that $f_n \to 0$ uniformly on compact subsets of $X$, then the sequence $\{Tf_n\}$ converges to zero in the norm of $X$.

Then, $T$ is a compact operator if and only if given a bounded sequence $\{f_n\}$ in $X$ such that $f_n \to 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of $X$.

**Theorem 2.2.** Let $u \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$ such that $\text{uC}_{\varphi} : H^\infty \to Z$ is bounded. Then

$$\|\text{uC}_{\varphi}\|_{E, H^\infty \to Z} \approx \max \{A, B, C\} \approx \max \{E, F, G\},$$

where

$$A := \limsup_{|a| \to 1} \|\text{uC}_{\varphi} f_a\|_Z,$$

$$B := \limsup_{|a| \to 1} \|\text{uC}_{\varphi} g_a\|_Z,$$

$$C := \limsup_{|a| \to 1} \|\text{uC}_{\varphi} h_a\|_Z,$$

$$E := \limsup_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{1 - |\varphi(z)|^2},$$

$$F := \limsup_{|\varphi(z)| \to 1} (1 - |z|^2)|u''(z)|,$$

and

$$G := \limsup_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)|u(z)||\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2}.$$

**Proof.** First, we prove that $\max \{A, B, C\} \lesssim \|\text{uC}_{\varphi}\|_{E, H^\infty \to Z}$. Let $a \in \mathbb{D}$. It is easy to see that $f_a, g_a, h_a \in H^\infty$ and $f_a, g_a, h_a$ converge to 0 uniformly on compact subsets of $\mathbb{D}$. Thus, for any compact operator $K : H^\infty \to Z$, by Lemma 2.1 we have

$$\lim_{|a| \to 1} \|Kf_a\|_Z = 0, \quad \lim_{|a| \to 1} \|Kg_a\|_Z = 0, \quad \lim_{|a| \to 1} \|Kh_a\|_Z = 0.$$

Hence

$$\|\text{uC}_{\varphi} - K\|_{H^\infty \to Z} \geq \limsup_{|a| \to 1} \|(\text{uC}_{\varphi} - K)f_a\|_Z.$$
Therefore, we obtain
\[
\|uC_\varphi - K\|_{H^\infty \to Z} \geq \limsup_{j \to \infty} \|uC_\varphi (k_j)\|_Z - \limsup_{j \to \infty} \|K(k_j)\|_Z = A,
\]
\[
\|uC_\varphi - K\|_{H^\infty \to Z} \geq \limsup_{j \to \infty} \|(uC_\varphi - K)g_a\|_Z
\]
\[
\geq \limsup_{j \to \infty} \|uC_\varphi g_a\|_Z - \limsup_{j \to \infty} \|Kg_a\|_Z = B,
\]
and
\[
\|uC_\varphi - K\|_{H^\infty \to Z} \geq \limsup_{j \to \infty} \|(uC_\varphi - K)h_a\|_Z
\]
\[
\geq \limsup_{j \to \infty} \|uC_\varphi h_a\|_Z - \limsup_{j \to \infty} \|Kh_a\|_Z = C.
\]
Therefore, we obtain
\[
\|uC_\varphi\|_{e, H^\infty \to Z} = \inf_K \|uC_\varphi - K\|_{H^\infty \to Z} \geq \max \{A, B, C\}.
\]

Next, we will prove that \(\|uC_\varphi\|_{e, H^\infty \to Z} \geq \max \{E, F, G\}\). Let \(\{z_j\}_{j \in \mathbb{N}}\) be a sequence in \(\mathbb{D}\) such that \(|\varphi(z_j)| \to 1\) as \(j \to \infty\). Define
\[
k_j(z) = \frac{1 - |\varphi(z_j)|^2}{(1 - \varphi(z_j)z)} - \frac{5}{3} \frac{(1 - |\varphi(z_j)|^2)^2}{(1 - \varphi(z_j)z)^2} + \frac{2}{3} \frac{(1 - |\varphi(z_j)|^2)^3}{(1 - \varphi(z_j)z)^3},
\]
\[
p_j(z) = \frac{1}{(1 - \varphi(z_j)z^2)} - \frac{1}{3} \frac{(1 - |\varphi(z_j)|^2)^2}{(1 - \varphi(z_j)z)^2} + \frac{1}{3} \frac{(1 - |\varphi(z_j)|^2)^3}{(1 - \varphi(z_j)z)^3},
\]
and
\[
q_j(z) = \frac{1 - |\varphi(z_j)|^2}{(1 - \varphi(z_j)z)} - 2 \frac{(1 - |\varphi(z_j)|^2)^2}{(1 - \varphi(z_j)z)^2} + \frac{1}{3} \frac{(1 - |\varphi(z_j)|^2)^3}{(1 - \varphi(z_j)z)^3}.
\]
It is easy to see that all \(k_j, p_j\) and \(q_j\) belong to \(H^\infty\) and converge to 0 uniformly on compact subsets of \(\mathbb{D}\). Moreover,
\[
k_j(\varphi(z_j)) = 0, \quad k_j''(\varphi(z_j)) = 0, \quad |k_j'(\varphi(z_j))| = \frac{1}{3} \frac{|\varphi(z_j)|}{(1 - |\varphi(z_j)|^2)},
\]
\[
p_j'(\varphi(z_j)) = 0, \quad p_j''(\varphi(z_j)) = 0, \quad |p_j(\varphi(z_j))| = \frac{1}{3},
\]
\[
q_j(\varphi(z_j)) = 0, \quad q_j'(\varphi(z_j)) = 0, \quad |q_j''(\varphi(z_j))| = \frac{2|\varphi(z_j)|^2}{(1 - |\varphi(z_j)|^2)^2}.
\]
Then for any compact operator \(K : H^\infty \to Z\), by Lemma 2.1 we obtain
\[
\|uC_\varphi - K\|_{H^\infty \to Z} \geq \limsup_{j \to \infty} \|uC_\varphi (k_j)\|_Z - \limsup_{j \to \infty} \|K(k_j)\|_Z
\]
\[
\geq \limsup_{j \to \infty} \frac{(1 - |z_j|^2)|2u'(z_j)\varphi'(z_j) + u(z_j)\varphi''(z_j)||\varphi(z_j)|}{1 - |\varphi(z_j)|^2},
\]
\[
\|uC_\varphi - K\|_{H^\infty \to Z} \geq \limsup_{j \to \infty} \|uC_\varphi (p_j)\|_Z - \limsup_{j \to \infty} \|K(p_j)\|_Z
\]
\[
\geq \limsup_{j \to \infty} (1 - |z_j|^2)|u''(z_j)|,
\]
and
\[
\|uC_\varphi - K\|_{H^\infty \to Z} \geq \limsup_{j \to \infty} \|uC_\varphi (q_j)\|_Z - \limsup_{j \to \infty} \|K(q_j)\|_Z
\]
For any \( f \), therefore, we only need to prove that

\[
\| uC_{\varphi} \|_{e, H^\infty \to Z} = \inf_K \| uC_{\varphi} - K \|_{H^\infty \to Z} \\
\geq \limsup_{j \to \infty} \frac{(1 - |z_j|^2)|u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|^2}{(1 - |\varphi(z_j)|^2)^2}
\]

From the definition of the essential norm, we obtain

\[
\| uC_{\varphi} \|_{e, H^\infty \to Z} = \inf_K \| uC_{\varphi} - K \|_{H^\infty \to Z} \\
\geq \limsup_{j \to \infty} \frac{(1 - |z_j|^2)|2u'(z_j)\varphi'(z_j) + u(z_j)\varphi''(z_j)||\varphi(z_j)|}{1 - |\varphi(z_j)|^2}
\]

and

\[
\| uC_{\varphi} \|_{e, H^\infty \to Z} = \inf_K \| uC_{\varphi} - K \|_{H^\infty \to Z} \\
\geq \limsup_{j \to \infty} \frac{(1 - |z_j|^2)|2u'(z_j)\varphi'(z_j) + u(z_j)\varphi''(z_j)|}{1 - |\varphi(z_j)|^2}
\]

Hence,

\[
\| uC_{\varphi} \|_{e, H^\infty \to Z} \geq \max \{ E, F, G \}.
\]

Finally, we prove that

\[
\| uC_{\varphi} \|_{e, H^\infty \to Z} \leq \max \{ A, B, C \} \quad \text{and} \quad \| uC_{\varphi} \|_{e, H^\infty \to Z} \leq \max \{ E, F, G \}.
\]

For \( r \in [0, 1] \), set \( K_r : H(\mathbb{D}) \to H(\mathbb{D}) \) by

\[
(K_rf)(z) = f_r(z) = f(rz), \quad f \in H(\mathbb{D}).
\]

It is obvious that \( f_r \to f \) uniformly on compact subsets of \( \mathbb{D} \) as \( r \to 1 \). Moreover, the operator \( K_r \) is compact on \( H^\infty \) and \( \| K_r \|_{H^\infty \to H^\infty} \leq 1 \). Let \( \{ r_j \} \subset (0, 1) \) be a sequence such that \( r_j \to 1 \) as \( j \to \infty \). Then for all positive integers \( j \), the operator \( uC_{\varphi}K_{r_j} : H^\infty \to Z \) is compact. By the definition of the essential norm, we get

\[
\| uC_{\varphi} \|_{e, H^\infty \to Z} \leq \limsup_{j \to \infty} \| uC_{\varphi} - uC_{\varphi}K_{r_j} \|_{H^\infty \to Z}.
\]

Therefore, we only need to prove that

\[
\limsup_{j \to \infty} \| uC_{\varphi} - uC_{\varphi}K_{r_j} \|_{H^\infty \to Z} \leq \max \{ A, B, C \}
\]

and

\[
\limsup_{j \to \infty} \| uC_{\varphi} - uC_{\varphi}K_{r_j} \|_{H^\infty \to Z} \leq \max \{ E, F, G \}.
\]

For any \( f \in H^\infty \) such that \( \| f \|_{\infty} \leq 1 \), consider

\[
\| (uC_{\varphi} - uC_{\varphi}K_{r_j})f \|_Z = |u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| + \| u(f - f_{r_j}) \circ \varphi \|_{\ast \ast} + |u'(0)(f - f_{r_j})(\varphi(0)) + u(0)(f - f_{r_j})'((\varphi(0))\varphi'(0)|.
\]

(2.2)
Here $\|g\|_{ss} = \sup_{z \in \mathbb{D}}(1 - |z|^2)|g''(z)|$. It is obvious that

$$\lim_{j \to \infty} |u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| = 0$$

(2.3)

and

$$\lim_{j \to \infty} |u'(0)(f - f_{r_j})(\varphi(0)) + u(0)(f - f_{r_j})'(\varphi(0))\varphi'(0)| = 0.$$  

(2.4)

Now, we consider

$$\limsup_{j \to \infty} \|u \cdot (f - f_{r_j}) \circ \varphi\|_{ss} \leq Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6,$$  

(2.5)

where

$$Q_1 := \limsup_{j \to \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2)|(f - f_{r_j})'(\varphi(z))|2u'(z)\varphi'(z) + u(z)\varphi''(z)|,$$

$$Q_2 := \limsup_{j \to \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2)|(f - f_{r_j})'(\varphi(z))|2u'(z)\varphi'(z) + u(z)\varphi''(z)|,$$

$$Q_3 := \limsup_{j \to \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2)|(f - f_{r_j})(\varphi(z))|\varphi''(z)|,$$

$$Q_4 := \limsup_{j \to \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2)|(f - f_{r_j})(\varphi(z))|\varphi''(z)|,$$

$$Q_5 := \limsup_{j \to \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2)|(f - f_{r_j})''(\varphi(z))|\varphi'(z)|^2|u(z)|,$$

$$Q_6 := \limsup_{j \to \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2)|(f - f_{r_j})''(\varphi(z))|\varphi'(z)|^2|u(z)|,$$

and $N \in \mathbb{N}$ is large enough such that $r_j \geq \frac{1}{2}$ for all $j \geq N$. Since $uC_\varphi : H^\infty \to \mathcal{Z}$ is bounded, from the proof of Theorem 1 in [2], we see that $u \in \mathcal{Z}$,

$$\tilde{J}_1 := \sup_{z \in \mathbb{D}}(1 - |z|^2)|2u'(z)\varphi'(z) + u(z)\varphi''(z)| < \infty$$

and

$$\tilde{J}_2 := \sup_{z \in \mathbb{D}}(1 - |z|^2)|\varphi'(z)|^2|u(z)| < \infty.$$  

Since $r_jf'_{r_j} \to f'$, as well as $r_j^2f''_{r_j} \to f''$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$, we have

$$Q_1 \leq \tilde{J}_1 \limsup_{j \to \infty} \sup_{|w| \leq r_N} |f'(w) - r_jf'(r_jw)| = 0,$$  

(2.6)

$$Q_5 \leq \tilde{J}_2 \limsup_{j \to \infty} \sup_{|w| \leq r_N} |f''(w) - r_j^2f''(r_jw)| = 0,$$  

(2.7)

and

$$Q_3 \leq \|u\|_{\mathcal{Z}} \limsup_{j \to \infty} \sup_{|w| \leq r_N} |f(w) - f(r_jw)| = 0.$$  

(2.8)

Next, we consider $Q_2$. We have $Q_2 \leq \limsup_{j \to \infty}(S_1 + S_2)$, where

$$S_1 := \sup_{|\varphi(z)| > r_N} (1 - |z|^2)|f'(\varphi(z))|2u'(z)\varphi'(z) + u(z)\varphi''(z)|$$

and

$$S_2 := \sup_{|\varphi(z)| > r_N} (1 - |z|^2)r_j|f'(r_j\varphi(z))|2u'(z)\varphi'(z) + u(z)\varphi''(z)|.$$
First we estimate $S_1$. Using the fact that $\|f\|_{\infty} \leq 1$, we have

$$
S_1 = \sup_{|z| > r_N} (1 - |z|^2)|f'(\varphi(z))||u'(z)\varphi'(z) + u(z)\varphi''(z)| \frac{3(1 - |\varphi(z)|^2)}{3(1 - |\varphi(z)|^2)}
$$

$$
\leq \frac{\|f\|_{\infty}}{r_N} \sup_{|z| > r_N} \frac{(1 - |z|^2)|2u'(z)\varphi'(z) + u(z)\varphi''(z)||\varphi(z)|}{3(1 - |\varphi(z)|^2)}
$$

$$
\leq \frac{\|f\|_{\infty}}{r_N} \sup_{|z| > r_N} \sup_{|a| > r_N} \frac{(1 - |z|^2)|2u'(z)\varphi'(z) + u(z)\varphi''(z)||\varphi(z)|}{3(1 - |\varphi(z)|^2)}
$$

$$
\leq \sup_{|a| > r_N} \|uC_{\varphi}(f_a)\|_{Z} + \frac{5}{3} \sup_{|a| > r_N} \|uC_{\varphi}(g_a)\|_{Z} + \frac{2}{3} \sup_{|a| > r_N} \|uC_{\varphi}(h_a)\|_{Z}.
$$

From (2.9), we see that

$$
\limsup_{j \to \infty} S_1 \leq \limsup_{|a| \to 1} \|uC_{\varphi}(f_a)\|_{Z} + \limsup_{|a| \to 1} \|uC_{\varphi}(g_a)\|_{Z} + \limsup_{|a| \to 1} \|uC_{\varphi}(h_a)\|_{Z}
$$

$$
= A + B + C.
$$

Similarly, we have $\limsup_{j \to \infty} S_2 \leq A + B + C$, i.e., we get that

$$
Q_2 \leq A + B + C \leq \max \{A, B, C\}.
$$

(2.10)

From (2.9), we see that

$$
\limsup_{j \to \infty} S_1 \leq \limsup_{|a| \to 1} \frac{(1 - |z|^2)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{1 - |\varphi(z)|^2} = E.
$$

Similarly, we have $\limsup_{j \to \infty} S_2 \leq E$. Therefore,

$$
Q_2 \leq E.
$$

(2.11)

Also for $Q_4$, we have $Q_4 \leq \limsup_{j \to \infty} (S_3 + S_4)$, where

$$
S_3 := \sup_{|z| > r_N} (1 - |z|^2)|f(\varphi(z))||u''(z)|, \quad S_4 := \sup_{|z| > r_N} (1 - |z|^2)|f(r_j(\varphi(z)))||u''(z)|.
$$

After a calculation, we have

$$
S_3 = \sup_{|z| > r_N} (1 - |z|^2)|f(\varphi(z))||u''(z)|
$$

$$
\leq \|f\|_{\infty} \sup_{|z| > r_N} \frac{1}{3}(1 - |z|^2)|u''(z)|
$$

$$
\leq \sup_{|z| > r_N} \sup_{|a| > r_N} \frac{1}{3}(1 - |z|^2)|u''(z)|
$$

$$
\leq \sup_{|a| > r_N} \|uC_{\varphi}(f_a)\|_{Z} + \sup_{|a| > r_N} \|uC_{\varphi}(g_a)\|_{Z} + \frac{1}{3} \sup_{|a| > r_N} \|uC_{\varphi}(h_a)\|_{Z}
$$

$$
\leq \sup_{|a| > r_N} \|uC_{\varphi}(f_a)\|_{Z} + \sup_{|a| > r_N} \|uC_{\varphi}(g_a)\|_{Z} + \sup_{|a| > r_N} \|uC_{\varphi}(h_a)\|_{Z}.
$$

(2.12)
Taking limit as \( N \to \infty \) we obtain
\[
\limsup_{j \to \infty} S_3 \lesssim \limsup_{|a| \to 1} \| uC_\varphi (f_a) \|_Z + \limsup_{|a| \to 1} \| uC_\varphi (g_a) \|_Z + \limsup_{|a| \to 1} \| uC_\varphi (h_a) \|_Z
= A + B + C.
\]
Similarly, we have \( \limsup_{j \to \infty} S_4 \lesssim A + B + C \), i.e., we get that
\[
Q_4 \lesssim A + B + C \lesssim \max \{ A, B, C \}. \tag{2.13}
\]
From (2.12), we see that
\[
\limsup_{j \to \infty} S_3 \lesssim \limsup_{|\varphi(z)| \to 1} (1 - |\varphi(z)|)u''(z) = F.
\]
Similarly, we have that \( \limsup_{j \to \infty} S_4 \lesssim F \). Therefore,
\[
Q_4 \lesssim F. \tag{2.14}
\]
Also, for \( Q_6 \), we have \( Q_6 \leq \limsup_{j \to \infty} (S_5 + S_6) \), where
\[
S_5 := \sup_{|\varphi(z)| > r_N} (1 - |z|^2)|f''(\varphi(z))|\varphi(z)^2|u(z)|, \quad S_6 := \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^2|f''(r_j \varphi(z))|\varphi(z)^2|u(z)|.
\]
After a calculation, we have
\[
S_5 \lesssim \| f \|_\infty \sup_{|\varphi(z)| > r_N} (1 - |z|^2)|\varphi(z)|^2|u(z)|\left(1 - |\varphi(z)|^2\right)^2
\lesssim \sup_{|\varphi(z)| > r_N} (1 - |z|^2)|\varphi(z)|^2|u(z)|\left(1 - |\varphi(z)|^2\right)^2
\lesssim \sup_{|a| > r_N} \| uC_\varphi (f_a - 2g_a + h_a) \|_Z
\lesssim \sup_{|a| > r_N} \left( \| uC_\varphi (f_a) \|_Z + \| uC_\varphi (g_a) \|_Z + \| uC_\varphi (h_a) \|_Z \right).
\tag{2.15}
\]
Taking limit as \( N \to \infty \) we obtain
\[
\limsup_{j \to \infty} S_5 \lesssim \limsup_{|a| \to 1} \| uC_\varphi (f_a) \|_Z + \limsup_{|a| \to 1} \| uC_\varphi (g_a) \|_Z + \limsup_{|a| \to 1} \| uC_\varphi (h_a) \|_Z
= A + B + C.
\]
Similarly, we get \( \limsup_{j \to \infty} S_6 \lesssim A + B + C \), i.e., we have
\[
Q_6 \lesssim A + B + C \lesssim \max \{ A, B, C \}. \tag{2.16}
\]
From (2.15), we obtain
\[
\limsup_{j \to \infty} S_5 \lesssim \limsup_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)|\varphi(z)|^2|u(z)|}{(1 - |\varphi(z)|^2)^2} = G.
\]
Similarly, we obtain \( \limsup_{j \to \infty} S_6 \lesssim G \). Therefore,
\[
Q_6 \lesssim G. \tag{2.17}
\]
Hence, by (2.2), (2.8), (2.10), (2.13) and (2.16) we get
\[
\limsup_{j \to \infty} \| uC_\varphi - uC_\varphi K_{r_j} \|_{H^\infty \to Z} = \limsup_{j \to \infty} \sup_{\| h \| \leq 1} \| (uC_\varphi - uC_\varphi K_{r_j})f \|_Z
= \limsup_{j \to \infty} \sup_{\| h \| \leq 1} \| u \cdot (f - f_{r_j}) \circ \varphi \|_{H^\infty} \lesssim \max \{ A, B, C \}. \tag{2.18}
\]
Similarly, by (2.2)-(2.8), (2.11), (2.14) and (2.17) we get
\[
\limsup_{j \to \infty} \| uC_\varphi - uC_\varphi K_j \|_{H^\infty \to Z} \lesssim \max \{ E, F, G \}.
\] (2.19)

Therefore, by (2.1), (2.18) and (2.19), we obtain
\[
\| uC_\varphi \|_{e, H^\infty \to Z} \lesssim \max \{ A, B, C \} \quad \text{and} \quad \| uC_\varphi \|_{e, H^\infty \to Z} \lesssim \max \{ E, F, G \}.
\]

This completes the proof of Theorem 2.2.

\textbf{Theorem 2.3.} Let \( u \in H(\mathbb{D}) \) and \( \varphi \) be an analytic self-map of \( \mathbb{D} \) such that \( uC_\varphi : H^\infty \to Z \) is bounded. Then
\[
\| uC_\varphi \|_{e, H^\infty \to Z} \approx \limsup_{n \to \infty} \| u\varphi^n \|_Z.
\]

\textbf{Proof.} First, we prove that
\[
\| uC_\varphi \|_{e, H^\infty \to Z} \geq \limsup_{n \to \infty} \| u\varphi^n \|_Z.
\]
Let \( n \) be any positive integer and \( f_n(z) = z^n \). Then \( \| f_n \|_\infty = 1 \) and \( f_n \) uniformly converges to zero on compact subsets of \( \mathbb{D} \). By Lemma 2.1, we have \( \lim_{n \to \infty} \| K f_n \|_Z = 0 \). Hence,
\[
\| uC_\varphi - K \| \geq \limsup_{n \to \infty} \| (uC_\varphi - K) f_n \|_Z \geq \limsup_{n \to \infty} \| uC_\varphi f_n \|_Z.
\]

Therefore, by the definition of essential norm we get
\[
\| uC_\varphi \|_{e, H^\infty \to Z} \geq \limsup_{n \to \infty} \| uC_\varphi f_n \|_Z = \limsup_{n \to \infty} \| u\varphi^n \|_Z.
\] (2.20)

Next, we prove that
\[
\| uC_\varphi \|_{e, H^\infty \to Z} \lesssim \limsup_{n \to \infty} \| u\varphi^n \|_Z.
\]
Since \( uC_\varphi : H^\infty \to Z \) is bounded, by Theorem 1 of [2] we see that
\[
P := \sup_{k \geq 0} \| u\varphi^k \|_Z < \infty.
\]
Consider the Maclaurin expansion of \( f_a \), where
\[
f_a(z) = (1 - |a|^2) \sum_{k=0}^\infty \bar{a}^k z^k.
\]
For any fix positive integer \( n \geq 2 \), it follows from the linearity of \( uC_\varphi \) and the triangle inequality that
\[
\| uC_\varphi f_a \|_Z \leq (1 - |a|^2) \sum_{k=0}^\infty |a|^k \| u\varphi^k \|_Z
\]
\[
= (1 - |a|^2) \sum_{k=0}^{n-1} |a|^k \| u\varphi^k \|_Z + (1 - |a|^2) \sum_{k=n}^\infty |a|^k \| u\varphi^k \|_Z
\]
\[
\leq P n (1 - |a|^2) + (1 - |a|^2) \sum_{k=n}^\infty |a|^k \| u\varphi^k \|_Z
\]
\[
\leq P n (1 - |a|^2) + 2 \sup_{k \geq n} \| u\varphi^k \|_Z.
\]
Letting $|a| \to 1$ in the above inequality leads to
\[
\limsup_{|a| \to 1} \| uC_\phi a \|_Z \leq 2 \sup_{k \geq n} \| u \varphi^k \|_Z
\]
for any positive integer $n \geq 2$. Thus,
\[
\limsup_{|a| \to 1} \| uC_\phi f_a \|_Z \lesssim \limsup_{k \to \infty} \| u \varphi^k \|_Z.
\]
Similarly, we can prove that
\[
\limsup_{|a| \to 1} \| uC_\phi g_a \|_Z \lesssim \limsup_{n \to \infty} \| u \varphi^n \|_Z, \quad \limsup_{|a| \to 1} \| uC_\phi h_a \|_Z \lesssim \limsup_{n \to \infty} \| u \varphi^n \|_Z.
\]
Hence,
\[
\max \{ A, B, C \} \lesssim \limsup_{n \to \infty} \| u \varphi^n \|_Z.
\]
Therefore, by Theorem 2.2 we obtain
\[
\| uC_\phi \|_{e,H^\infty \to Z} \lesssim \max \{ A, B, C \} \lesssim \limsup_{n \to \infty} \| u \varphi^n \|_Z.
\] (2.21)
By (2.20) and (2.21), we get the desired result. The proof is completed. \qed

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References


