Common best proximity results for multivalued proximal contractions in metric space with applications

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Abstract

The study of the best proximity points is an interesting topic of optimization theory. We introduce the notion of $\alpha_\ast$-proximal contractions for multivalued mappings on a complete metric space and establish the existence of common best proximity point for these mappings in the context of multivalued and single-valued mappings. As an application, we derive some best proximity point and fixed point results for multivalued and single-valued mappings on partially ordered metric spaces. Our results generalize and extend many known results in the literature. Some examples are provided to illustrate the results obtained herein. ©2016 All rights reserved.

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1. Introduction and preliminaries

Fixed point theory concerns with some techniques to find a solution of the pattern $T \mathbf{x} = \mathbf{x}$, where $T$ is a self-mapping defined on a subset $A$ of a metric space $(X, d)$. A well-known principle that guarantees a unique fixed point solution is the Banach contraction principle \cite{9}. Over the years, this principle has been generalized
in many ways (see \[5, 7, 15, 28, 29\]). An interesting generalization of the Banach contraction principle is for multivalued mappings and is known as Nadler’s fixed point theorem \[24\]. In 1982, Sessa \[31\] defined the concept of weakly commuting mappings to obtain common fixed point for pair of such mappings. Jungek generalized this idea, first to compatible mappings \[18\] and then to weakly compatible mappings \[19\]. A mapping \( T : A \rightarrow B \) does not necessarily have a fixed point, where \( A \) and \( B \) are nonempty subsets of a metric space \( X \). One can proceed to find an element \( x \in A \) in the sense that the distance \( d(x, Tx) \) is minimum. Fan’s best approximation theorem \[13\] asserts that if \( K \) is a nonempty, compact, and convex subset of a normed space \( X \) and \( T : K \rightarrow X \) is a continuous mapping, then there exists an element \( x \) satisfying the condition \( d(x, Tx) = \inf \{d(y, Tx) : y \in K\} \). A best approximation theorem guarantees the existence of an approximate solution, while a best proximity point theorem provides an approximate solution which is optimal in the sense that there exists an element \( x \) such that \( d(x, Tx) = \inf \{d(x, y) : x \in A \text{ and } y \in B\} \); the element \( x \) is called a best proximity point of \( T \). Moreover, if the mapping under consideration is a self-mapping, then a best proximity point is reduced to a fixed point. The existence of best proximity points is an interesting aspect of optimization theory and it has attracted the attention of many authors (see \[11, 16, 18, 15, 16, 20, 22\] and references therein). Moreover, the best proximity point theorems for several classes of multivalued mappings have been proved in \[11, 14, 30\].

For non-empty subsets \( A \) and \( B \) of the metric space \( X \), the following notions will be used:

\[
\text{dist}(A, B) = \inf \{d(a, b) : a \in A, b \in B\}, \quad D(x, B) = \inf \{d(x, b) : b \in B\},
\]

\[
A_0 = \{a \in A : d(a, b) = \text{dist}(A, B) \text{ for some } b \in B\},
\]

\[
B_0 = \{b \in B : d(a, b) = \text{dist}(A, B) \text{ for some } a \in A\},
\]

\(2^X\) is the set of all nonempty subsets of \( X \), \( CL(X)\) is the set of all nonempty closed subsets of \( X \), \( K(X)\) is the set of all compact subsets of \( X \) for every \( A, B \in CL(X) \), \( H(A, B) = \max \{\sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A)\} \) if the maximum exists and \( H(A, B) = 0 \) otherwise, and let \( \Psi \) be the collection of all non-decreasing functions \( \psi : [0, +\infty) \rightarrow [0, +\infty) \) such that \( \sum_{n=1}^{\infty} \psi^n(t) < +\infty \) for each \( t > 0 \), where \( \psi^n \) is the \( n \)th iterate of \( \psi \).

We present now the necessary definitions and results which will be useful in the sequel.

**Definition 1.1.** (\[23\].) Let \( A \) and \( B \) be nonempty subsets of a metric space \( (X, d) \). A point \( x \) is called a common best proximity point of mappings \( T_i : A \rightarrow B, (i = 1, 2, ..., n) \) if

\[
D(x, T_i x) = \text{dist}(A, B).
\]

**Lemma 1.2.** (\[5\].) Let \( (X, d) \) be a metric space and \( B \in CL(X) \). Then for each \( x \in X \) with \( d(x, B) > 0 \) and \( q > 1 \), there exists an element \( b \in B \) such that

\[
d(x, b) < qd(x, B).
\]

**Definition 1.3.** (\[6\].) Let \( (A, B) \) be a pair of nonempty subsets of a metric space \( (X, d) \) with \( A_0 \neq \emptyset \). Then the pair \( (A, B) \) is said to have the weak \( P \)-property if and only if for any \( x_1, x_2 \in A \) and \( y_1, y_2 \in B \),

\[
\begin{align*}
\left\{ \begin{array}{l}
\quad d(x_1, y_1) = \text{dist}(A, B) \\
\quad d(x_2, y_2) = \text{dist}(A, B)
\end{array} \right\} \quad \Rightarrow \quad d(x_1, x_2) \leq d(y_1, y_2).
\end{align*}
\]

**Definition 1.4.** (\[6\].) Let \( A \) and \( B \) be two nonempty subsets of a metric space \( (X, d) \). A mapping \( T : A \rightarrow 2^B \setminus \emptyset \) is called \( \alpha \)-proximal admissible if there exists a mapping \( \alpha : A \times A \rightarrow [0, \infty) \) such that

\[
\begin{align*}
\left\{ \begin{array}{l}
\quad \alpha(x_1, x_2) \geq 1 \\
\quad d(u_1, y_1) = \text{dist}(A, B) \\
\quad d(u_2, y_2) = \text{dist}(A, B)
\end{array} \right\} \quad \Rightarrow \quad \alpha(u_1, u_2) \geq 1,
\end{align*}
\]

where \( x_1, x_2, u_1, u_2 \in A, y_1 \in Tx_1 \) and \( y_2 \in Ty_2 \).
Definition 1.5 (6). Let \( A \) and \( B \) be two nonempty subsets of a metric space \((X, d)\). A mapping \( T : A \to CL(B) \) is said to be an \( \alpha-\psi \)-proximal contraction, if there exist \( \psi \in \Psi \) and \( \alpha : A \times A \to [0, \infty) \) such that

\[
\alpha(x, y)H(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in A. \tag{1.1}
\]

In this paper, we generalize the above mentioned notions for a pair of multivalued and single-valued mappings and define \( \alpha_\ast \)-proximal admissible with respect to \( \eta : A \times A \to [0, \infty) \), \( \alpha \)-proximal admissible with respect to \( \eta : A \times A \to [0, \infty) \) and prove common best proximity point theorems as well as fixed point theorems for these mappings. Our results generalize and improve the results of Ali et al. [6], Jungck ([18], [19]), Samet et al. [29], and Hussain et al. [17].

2. Common best proximity points for multivalued mappings

We begin this section with a definition.

Definition 2.1. Let \( A \) and \( B \) be nonempty subsets of a metric space \((X, d)\) and \( \mathcal{T}_1, \mathcal{T}_2 : A \to 2^B \setminus \emptyset \) be multivalued mappings. The pair \((\mathcal{T}_1, \mathcal{T}_2)\) is \( \alpha_\ast \)-proximal admissible with respect to \( \eta \) if there exist \( \alpha, \eta : A \times A \to [0, \infty) \) such that for \( z_1, z_2, u_1, u_2 \in A \),

\[
\begin{align*}
\alpha(z_1, z_2) &\geq \eta(z_1, z_2) \\
d(u_1, y_1) &\leq \text{dist}(A, B) \\
d(u_2, y_2) &\leq \text{dist}(A, B)
\end{align*}
\]

for all \( y_1 \in \mathcal{T}_1z_1 \) and \( y_2 \in \mathcal{T}_2z_2 \), \( i, j \in \{1, 2\} \). When \( \alpha(z_1, z_2) = 1 \) for all \( z_1, z_2 \in A \), the pair \((\mathcal{T}_1, \mathcal{T}_2)\) is called \( \eta_\ast \)-proximal sub-admissible, and when \( \eta(z_1, z_2) = 1 \) for all \( z_1, z_2 \in A \), the pair \((\mathcal{T}_1, \mathcal{T}_2)\) is called \( \alpha_\ast \)-proximal admissible.

Example 2.2. Consider \( X = \mathbb{R}^2 \) with the usual metric. Suppose \( A = \{(1, x) : 0 \leq x \leq 1\} \) and \( B = \{(0, x) : 0 \leq x \leq 1\} \). Define \( \mathcal{T}_1, \mathcal{T}_2 : A \to 2^B \setminus \emptyset \) by

\[
\begin{align*}
\mathcal{T}_1(1, x) &= \begin{cases} 
\{(0, 1)\} & x = 1, \\
\{(0, \frac{x}{2}) : 0 \leq a \leq x\} & \text{otherwise}, 
\end{cases} \\
\mathcal{T}_2(1, x) &= \begin{cases} 
\{(0, \frac{x}{2}) : 0 \leq a \leq x\} & x \in [0, \frac{1}{2}], \\
\{(0, \alpha^2) : 0 \leq a \leq x\} & x \in [\frac{1}{2}, 1], 
\end{cases}
\end{align*}
\]

and \( \alpha, \eta : A \times A \to [0, \infty) \) by

\[
\begin{align*}
\alpha((1, x), (1, y)) &= \begin{cases} 
\frac{4}{5} & x, y \in \left[0, \frac{1}{2}\right], \\
1/2 & \text{otherwise}, 
\end{cases} \\
\eta((1, x), (1, y)) &= \frac{3}{4}
\end{align*}
\]

for all \((1, x), (1, y) \in A \times A \). If \( z_1 = (1, x_1) \) and \( z_2 = (1, x_2) \) in \( A \), then \( \alpha(z_1, z_2) \geq \eta(z_1, z_2) \) if \( x_1, x_2 \in \left[0, \frac{1}{2}\right] \). So, \( \mathcal{T}_1z_1 = \{(0, \frac{x}{2}) : 0 \leq a \leq x_1\} \) and \( \mathcal{T}_2z_2 = \{(0, \frac{x}{2}) : 0 \leq a \leq x_2\} \). This shows that \( d(u_1, y_1) = 1 = \text{dist}(A, B) \) and \( d(u_2, y_2) = 1 = \text{dist}(A, B) \) for all \( y_1 \in \mathcal{T}_1x_1 \) and \( y_2 \in \mathcal{T}_2x_2 \), \( i, j \in \{1, 2\} \) if and only if \( u_1, u_2 \in \{(1, \frac{x}{2}) : 0 \leq x \leq \frac{1}{2}\}\). Hence \( \alpha(u_1, u_2) = \frac{4}{5} > \frac{3}{4} = \eta(u_1, u_2) \). Thus the pair \((\mathcal{T}_1, \mathcal{T}_2)\) is \( \alpha_\ast \)-proximal admissible with respect to \( \eta \).

Theorem 2.3. Let \( A \) and \( B \) be two nonempty closed subsets of a complete metric space \((X, d)\) such that \( A_0 \) is non-empty and \( \mathcal{T}, \mathcal{S} : A \to CL(B) \) be continuous multivalued mappings satisfying the following assertions:

1. \( \alpha(z_1, z_2) \geq \eta(z_1, z_2) \Rightarrow H(\mathcal{T}z_1, \mathcal{S}z_2) \leq \psi(d(z_1, z_2)) \);
2. \( \mathcal{T}z, \mathcal{S}z \subseteq B_0 \) for each \( z \in A_0 \) and \( (A, B) \) satisfies the weak P-property;
3. \((T, S)\) is \(\alpha_s\)-proximal admissible with respect to \(\eta\); 
4. there exists \(z_0, z_1, z_2 \in A_0\), \(y_1 \in T z_0\) and \(y_2 \in S z_0\) such that 
\[
d(z_1, y_1) = \text{dist}(A, B), \quad \alpha(z_0, z_1) \geq \eta(z_0, z_1)
\]
and 
\[
d(z_2, y_2) = \text{dist}(A, B), \quad \alpha(z_0, z_2) \geq \eta(z_0, z_2).
\]

Then the mappings \(T\) and \(S\) have a common best proximity point.

**Proof.** By the hypothesis, there exists \(z_0, z_1 \in A_0\) and \(y_1 \in T z_0\) such that 
\[
d(z_1, y_1) = \text{dist}(A, B), \quad \alpha(z_0, z_1) \geq \eta(z_0, z_1).
\]
(2.1)

If \(y_1 \in T z_1 \cap S z_1\), then \(z_1\) is the common best proximity point of \(T\) and \(S\). If \(y_1 \notin S z_1\), then from condition \([1]\) we have 
\[
0 < d(y_1, S z_1) \leq H(T z_0, S z_1) \leq \psi(d(z_0, z_1)).
\]
For \(q > 1\), it follows from Lemma \([1,2]\) that there exists \(y_2 \in S z_1\) such that 
\[
0 < d(y_1, y_2) < q d(y_1, S z_1) \\
\leq q H(T z_0, S z_1) \\
\leq q \psi(d(z_0, z_1)).
\]
(2.2)

As \(y_2 \in S z_1 \subseteq B_0\), there exists \(z_2 \neq z_1 \in A_0\) such that 
\[
d(z_2, y_2) = \text{dist}(A, B),
\]
(2.3)
otherwise, \(z_1\) is the common best proximity point of \(T\) and \(S\). As \((A, B)\) satisfies the weak P-property, (2.1) and (2.3) imply that 
\[
0 < d(z_1, z_2) \leq d(y_1, y_2).
\]
(2.4)

From (2.2) and (2.4), we have 
\[
0 < d(z_1, z_2) \leq q \psi(d(z_0, z_1)).
\]
Since \(\psi\) is non-decreasing, from the above inequality, we have 
\[
\psi(d(z_1, z_2)) \leq \psi(q \psi(d(z_0, z_1))).
\]

Put \(q_1 = \frac{\psi(q \psi(d(z_0, z_1)))}{\psi(d(z_1, z_2))}\). As the pair \((T, S)\) is \(\alpha_s\)-proximal admissible with respect to \(\eta\), so, \(\alpha(z_1, z_2) \geq \eta(z_1, z_2)\). Thus, we have 
\[
d(z_2, y_2) = \text{dist}(A, B), \quad \alpha(z_1, z_2) \geq \eta(z_1, z_2).
\]
(2.5)

Now, if \(y_2 \in T z_2 \cap S z_2\), then \(z_2\) is the common best proximity point of \(T\) and \(S\). If \(y_2 \notin T z_2\), then from condition \([1]\) we have 
\[
0 < d(T z_2, y_2) \leq H(T z_2, S z_1) \leq \psi(d(z_1, z_2)).
\]
For $q_1 > 1$, it follows from Lemma 1.2 that there exists $y_3 \in T z_2$ such that

$$0 < d(y_2, y_3) < q_1 d(y_2, T z_2) \leq q_1 H(S z_1, T z_2) \leq q_1 \psi(d(z_1, z_2)) = \psi(q \psi(d(z_0, z_1))).$$

(2.6)

As $y_3 \in T z_2 \subseteq B_0$, so there exists $z_3 \neq z_2 \in A_0$ such that

$$d(z_3, y_3) = \text{dist}(A, B),$$

(2.7)

otherwise, $z_2$ is the common best proximity point of $T$ and $S$. As $(A, B)$ satisfies the weak P-property, (2.5) and (2.7) imply that

$$0 < d(z_2, z_3) \leq d(y_2, y_3).$$

(2.8)

From (2.6) and (2.8), we have

$$0 < d(z_2, z_3) \leq \psi(q \psi(d(z_0, z_1))).$$

Since $\psi$ is strictly increasing, from the above inequality, we have

$$\psi(d(z_2, z_3)) < \psi^2(q \psi(d(z_0, z_1))).$$

Put $q_2 = \frac{\psi^2(q \psi(d(z_0, z_1)))}{\psi(d(z_2, z_3))}$. As the pair $(T, S)$ is $\alpha_\ast$-proximal admissible with respect to $\eta$, so, $\alpha(z_2, z_3) \geq \eta(z_2, z_3)$. Thus, we have

$$d(z_3, y_3) = \text{dist}(A, B), \quad \alpha(z_2, z_3) \geq \eta(z_2, z_3).$$

Now proceeding in the manner described above, we get a sequence $\{z_n\}$ in $A_0$ and $\{y_n\}$ in $B_0$ such that for $n \in \mathbb{N}$

$$y_{2n+1} \in T z_{2n} \quad \text{and} \quad y_{2n} \in T z_{2n-1},$$

(2.9)

where

$$d(z_{n+1}, y_{n+1}) = \text{dist}(A, B), \quad \alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1}), \quad \forall n \in \mathbb{N}$$

(2.10)

and

$$d(y_{n+1}, y_{n+2}) < \psi^n(\psi(q \psi(d(z_0, z_1))), \quad \forall n \in \mathbb{N}.$$  

(2.11)

As $y_{n+2} \in T z_{n+1} \cup S z_{n+1}$ and $T z_{n+1} \cup S z_{n+1} \subseteq B_0$ for all $n \in \mathbb{N}$, so there exists $z_{n+2} \neq z_{n+1} \in A_0$ such that

$$d(z_{n+2}, y_{n+2}) = \text{dist}(A, B), \quad \forall n \in \mathbb{N}.$$ 

(2.12)

Since $(A, B)$ satisfies the weak P-property, from (2.10) and (2.12), we have

$$d(z_{n+1}, z_{n+2}) \leq d(y_{n+1}, y_{n+2}), \quad \forall n \in \mathbb{N}.$$ 

(2.13)

From (2.11) and (2.13), we get

$$d(z_{n+1}, z_{n+2}) < \psi^n(\psi(q \psi(d(z_0, z_1))), \quad \forall n \in \mathbb{N}.$$ 

Now for $n > m$, we have

$$d(z_n, z_m) \leq \sum_{i=n}^{m-1} d(z_i, z_{i+1}) < \sum_{i=n}^{m-1} \psi^{i-1}(\psi(q \psi(d(z_0, z_1))).$$

Hence $\{z_n\}$ is a Cauchy sequence in $A$. Similarly, $\{y_n\}$ is a Cauchy sequence in $B$. Since $A$ and $B$ are closed subsets of a complete metric space $(X, d)$, there exist $z^* \in A$ and $y^* \in B$ such that $z_n \to z^*$ and $y_n \to y^*$ as $n \to \infty$. By taking limit as $n \to \infty$ in equation (2.12), we get that

$$d(z^*, y^*) = \text{dist}(A, B).$$
Since $\mathcal{T}$ and $\mathcal{S}$ are continuous, therefore from (2.9), we get that $y^* \in \mathcal{T} z^* \cap \mathcal{S} z^*$. Hence
\[
\text{dist}(\mathcal{A}, \mathcal{B}) \leq D(z^*, \mathcal{T} z^*) \leq d(z^*, y^*) = \text{dist}(\mathcal{A}, \mathcal{B})
\]
and
\[
\text{dist}(\mathcal{A}, \mathcal{B}) \leq D(z^*, \mathcal{S} z^*) \leq d(z^*, y^*) = \text{dist}(\mathcal{A}, \mathcal{B}).
\]
This implies that $D(z^*, \mathcal{T} z^*) = D(z^*, \mathcal{S} z^*) = \text{dist}(\mathcal{A}, \mathcal{B})$, that is, $z^*$ is a common best proximity point of $\mathcal{T}$ and $\mathcal{S}$.

\begin{example}
Consider $X, \mathcal{A}, \mathcal{B}, T_1, T_2 : \mathcal{A} \to 2^B \setminus \emptyset$ and $\alpha, \eta : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ as in Example 2.2. Then $\mathcal{A}_0 = \mathcal{A}, \mathcal{B}_0 = \mathcal{B}$, $\text{dist}(\mathcal{A}, \mathcal{B}) = 1$ and $T_1 z, T_2 z \subseteq B_0$ for each $z \in \mathcal{A}_0$. As $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{B}_0 = \mathcal{B}$, so for $z_1 = (1, x_1), z_2 = (1, x_2) \in \mathcal{A}$, there exist $y_1 = (0, x_1), y_2 = (0, x_2) \in \mathcal{B}$ such that $d(z_1, y_1) = d(z_2, y_2) = \text{dist}(\mathcal{A}, \mathcal{B})$ and $d(z_1, z_2) = |x_1 - x_2| = d(y_1, y_2)$. Hence the pair $(\mathcal{A}, \mathcal{B})$ satisfies the weak P-property and the pair $(T_1, T_2)$ is $\alpha$-proximal admissible map with respect to $\eta$ (see Example 2.2). Let $\psi(t) = \frac{t}{2}$ for all $t \geq 0$.
Note that $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$ if $x_1, x_2 \in [0, \frac{1}{2}]$. Therefore,
\[
H(T_1 z_1, T_2 z_2) = \frac{|x_1 - x_2|}{2} = \frac{1}{2} |x_1 - x_2| = \psi(d(z_1, z_2)).
\]
Also, for $z_0 = (1, \frac{1}{2}) \in \mathcal{A}_0, y_1 = (0, \frac{1}{4}) \in T_1 x_0$ and $y_2 = (0, \frac{1}{8}) \in T_2 x_0$, we have $z_1 = (1, \frac{1}{4}), z_2 = (1, \frac{1}{8}) \in \mathcal{A}_0$ such that $d(z_1, y_1) = d(z_2, y_2) = 1 = \text{dist}(\mathcal{A}, \mathcal{B}), \alpha(z_0, z_1) = \frac{4}{3} \geq \frac{3}{2} = \eta(z_0, z_1) \text{ and } \alpha(z_0, z_2) = \frac{4}{3} \geq \frac{3}{2} = \eta(z_0, z_2)$. Thus all the conditions of Theorem 2.3 are satisfied and $(1, 1)$ is a common best proximity point of $T_1$ and $T_2$.

The case $\eta(z_1, z_2) = 1$, reduces Theorem 2.3 to the following:

\begin{corollary}
Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space $(X, d)$ such that $\mathcal{A}_0$ is non-empty and $T, S : \mathcal{A} \to CL(B)$ be continuous multivalued mappings satisfying the following assertions:
1. $\alpha(z_1, z_2) \geq 1 \Rightarrow H(T z_1, S z_2) = \psi(d(z_1, z_2));$
2. $T z, S z \subseteq B_0$ for each $z \in \mathcal{A}_0$ and $(\mathcal{A}, \mathcal{B})$ satisfies the weak P-property;
3. $(T, S)$ is $\alpha$-proximal admissible;
4. there exist $z_0, z_1, z_2 \in \mathcal{A}_0, y_1 \in T z_0 \text{ and } y_2 \in S z_0$ such that
   \[
d(z_1, y_1) = \text{dist}(\mathcal{A}, \mathcal{B}), \quad \alpha(z_0, z_1) \geq 1
   \]
and
\[
d(z_2, y_2) = \text{dist}(\mathcal{A}, \mathcal{B}), \quad \alpha(z_0, z_2) \geq 1.
\]

Then the mappings $T$ and $S$ have a common best proximity point.

If we take $\alpha(z_1, z_2) = 1$ in Theorem 2.3 then we have the following:

\begin{corollary}
Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space $(X, d)$ such that $\mathcal{A}_0$ is non-empty and $T, S : \mathcal{A} \to CL(B)$ be continuous multivalued mappings satisfying the following assertions:
1. $\eta(z_1, z_2) \leq 1 \Rightarrow H(T z_1, S z_2) \leq \psi(d(z_1, z_2));$
2. $T z, S z \subseteq B_0$ for each $z \in \mathcal{A}_0$ and $(\mathcal{A}, \mathcal{B})$ satisfies the weak P-property;
3. $(T, S)$ is $\eta$-proximal subadmissible;
4. there exist $z_0, z_1, z_2 \in \mathcal{A}_0, y_1 \in T z_0 \text{ and } y_2 \in S z_0$ such that
   \[
d(z_1, y_1) = \text{dist}(\mathcal{A}, \mathcal{B}), \quad \eta(z_0, z_1) < 1
   \]
and
\[
d(z_2, y_2) = \text{dist}(\mathcal{A}, \mathcal{B}), \quad \eta(z_0, z_2) < 1.
\]

Then the mappings $T$ and $S$ have a common best proximity point.
In case, $T_1 = T_2,$ Definition 2.1 and Theorem 2.3 is reduced to the following:

**Definition 2.7.** Let $A$ and $B$ be two nonempty subsets of a metric space $(X,d)$ and $T : A \to 2^B \setminus \emptyset$ be a multivalued mapping. We say that $T$ is $\alpha, \eta$-proximal admissible with respect to $\eta$ if there exist two functions $\alpha, \eta : A \times A \to [0, \infty)$ such that for all $z_1, z_2, u_1, u_2 \in A,$

$$
\begin{align*}
\alpha(z_1, z_2) \geq \eta(z_1, z_2) & \quad \Rightarrow \quad \alpha(u_1, u_2) \geq \eta(u_1, u_2) \\
d(u_1, y_1) = \text{dist}(A, B) & \quad \Rightarrow \quad d(u_2, y_2) = \text{dist}(A, B)
\end{align*}
$$

for all $y_1 \in T z_1$ and $y_2 \in T z_2.$ When $\alpha(z_1, z_2) = 1$ for all $z_1, z_2 \in A,$ $T$ is called a $\eta$-proximal sub-admissible.

**Theorem 2.8.** Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X,d)$ such that $A_0$ is nonempty and $T : A \to CL(B)$ be a continuous multivalued mapping satisfying the following assertions:

1. $\alpha(z_1, z_2) \geq \eta(z_1, z_2) \Rightarrow H(T z_1, T z_2) \leq \psi(d(z_1, z_2));$
2. $T z \subseteq B_0$ for each $z \in A_0$ and $(A, B)$ satisfies the weak P-property;
3. $T$ is $\alpha, \eta$-proximal admissible with respect to $\eta;$
4. there exist $z_0, z_1 \in A_0,$ $y_1 \in T z_0$ such that

$$
\begin{align*}
d(z_1, y_1) = \text{dist}(A, B), \quad \alpha(z_0, z_1) \geq \eta(z_0, z_1).
\end{align*}
$$

Then the mapping $T$ has a best proximity point.

If we take $\eta(z_1, z_2) = 1$ in Theorem 2.8, then we have the following:

**Corollary 2.9.** Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X,d)$ such that $A_0$ is nonempty and $T : A \to CL(B)$ be a continuous multivalued mapping satisfying the following assertions:

1. $\alpha(z_1, z_2) \geq 1 \Rightarrow H(T z_1, T z_2) \leq \psi(d(z_1, z_2));$
2. $T z \subseteq B_0$ for each $z \in A_0$ and $(A, B)$ satisfies the weak P-property;
3. $T$ is $\alpha$-proximal admissible;
4. there exist $z_0, z_1 \in A_0,$ $y_1 \in T z_0$ such that

$$
\begin{align*}
d(z_1, y_1) = \text{dist}(A, B), \quad \alpha(z_0, z_1) \geq 1.
\end{align*}
$$

Then the mapping $T$ has a best proximity point.

**Remark 2.10.** The special case of Theorem 2.8 for $\alpha(z_1, z_2) = 1$ can be obtained as in Corollary 2.6.

**Remark 2.11.** When $\eta(z_1, z_2) = 1$ for all $z_1, z_2 \in A,$ Definition 2.7 reduces to Definition 10 in [6]. As the condition 1 is more general than the inequality (1.1) (see Remark 3.5 in [5]), so Corollary 2.9 extends Theorem 13 in [6].

**Remark 2.12.** When $A = B,$ Theorem 2.8 is reduced to the Theorem 3.3 in [5].

**Remark 2.13.** Note that the uniqueness of the common best proximity points of multivalued mappings $T$ and $S$ is not given in Theorem 2.3. Thus, we can present the following problem: Let $(X,d)$ be a complete metric space and $T, S : A \to CL(B)$ be continuous multivalued mappings satisfying all the assertions of Theorem 2.3. Does $T$ and $S$ have a unique common best proximity point? By adding a condition and taking mappings $T, S : A \to K(B),$ we can give a partial answer of this problem as follows:

**Theorem 2.14.** Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X,d)$ such that $A_0$ is non-empty and $T, S : A \to K(B)$ be continuous multivalued mappings satisfying all the assertions of Theorem 2.3 and also satisfy

H. $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$ for all common best proximity points of $T$ and $S.$
Then the mappings $\mathcal{T}$ and $\mathcal{S}$ have a unique common best proximity point.

Proof. We will only prove the part of uniqueness. Let $z_1, z_2$ be two common best proximity points of $\mathcal{T}$ and $\mathcal{S}$ such that $z_1 \neq z_2$, then by hypothesis H we have $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$ and $D(z_1, \mathcal{T}z_1) = \text{dist}(\mathcal{A}, \mathcal{B}) = D(z_1, \mathcal{S}z_1) = D(z_2, \mathcal{T}z_2) = D(z_2, \mathcal{S}z_2)$. Since $\mathcal{T}z_1$ and $\mathcal{S}z_2$ are compact, so there exist an element $u_1 \in \mathcal{T}z_1$ and $u_2 \in \mathcal{S}z_2$ such that

$$d(z_1, u_1) = D(z_1, \mathcal{T}z_1)$$

and

$$d(z_2, u_2) = D(z_2, \mathcal{S}z_2).$$

Since the pair $(\mathcal{T}, \mathcal{S})$ satisfies the weak $P$-property, so we have

$$d(z_1, z_2) = d(u_1, u_2).$$

So by using condition 1 and Lemma 1.2 there exists $q > 1$ such that

$$d(z_1, z_2) = d(u_1, u_2) < qD(u_1, \mathcal{S}z_2)$$

$$< qH(\mathcal{T}z_1, \mathcal{S}z_2)$$

$$< q\psi(d(z_1, z_2))$$

$$< qd(z_1, z_2),$$

which is a contradiction. This implies that $d(z_1, z_2) = 0$, consequently, $\mathcal{T}$ and $\mathcal{S}$ have a unique common best proximity point.

By similar arguments as in Theorem 2.14, we state the following:

**Theorem 2.15.** Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space $(X, d)$ such that $\mathcal{A}_0$ is nonempty and $\mathcal{T} : \mathcal{A} \rightarrow K(\mathcal{B})$ be a continuous multivalued mapping satisfying all the assertions of Theorem 2.8 with condition H, then $\mathcal{T}$ has a unique common best proximity point.

### 3. Common best proximity points for single-valued mappings

We start with the following definition:

**Definition 3.1.** Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of a metric space $(X, d)$ and $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{A} \rightarrow \mathcal{B}$ be mappings. The pair $(\mathcal{T}_1, \mathcal{T}_2)$ is $\alpha$-proximal admissible with respect to $\eta$ if there exist two functions $\alpha, \eta : \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ such that for $z_1, z_2, u_1, u_2 \in \mathcal{A}$,

$$\alpha(z_1, z_2) \geq \eta(z_1, z_2)$$

$$d(u_1, \mathcal{T}_1z_1) = \text{dist}(\mathcal{A}, \mathcal{B})$$

$$d(u_2, \mathcal{T}_2z_2) = \text{dist}(\mathcal{A}, \mathcal{B})$$

$$\Rightarrow \alpha(u_1, u_2) \geq \eta(u_1, u_2).$$

When $\alpha(z_1, z_2) = 1$ for all $z_1, z_2 \in \mathcal{A}$, the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is called $\eta$-proximal subadmissible and when $\eta(z_1, z_2) = 1$ for all $z_1, z_2 \in \mathcal{A}$, the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is called $\alpha$-proximal admissible.

**Example 3.2.** Consider $\mathcal{X} = \mathbb{R}^2$ with the usual metric. Let $\mathcal{A} = \{(-6, 0), (0, -6), (0, 5)\}$ and $\mathcal{B} = \{(-1, 0), (0, -1), (0, 0), (-1, 1), (1, 1)\}$ be closed subsets of $(X, d)$. Then $d(\mathcal{A}, \mathcal{B}) = 5$, $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{B}_0 = \mathcal{B}$. Define $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{A} \rightarrow \mathcal{B}$ by

$$\mathcal{T}_1(-6, 0) = (-1, 0),$$

$$\mathcal{T}_1(0, -6) = (0, -1),$$

$$\mathcal{T}_1(0, 5) = (1, 1),$$

$$\mathcal{T}_2(-6, 0) = (0, 0),$$

$$\mathcal{T}_2(0, -6) = (-1, 1),$$

$$\mathcal{T}_2(0, 5) = (1, 1),$$

and $\alpha, \eta : \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ by

$$\alpha(z_1, z_2) = \begin{cases} 1 & \text{if } y_1, y_2 \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\eta(z_1, z_2) = \frac{1}{2},$$

for all $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in \mathcal{A}$.\[\Box\]
Note that $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$ if $z_1, z_2 \in \{(0, -6), (0, 5)\}$. For $z_1 = (0, -6)$, $d(u_1, T_1 z_1) = \text{dist}(A, B)$ if $u_1 \in \{(0, -6)\}$ and $d(u_2, T_2 z_1) = \text{dist}(A, B)$ if $u_2 \in \{(0, 5)\}$. This implies that $\alpha(u_1, u_2) = 1 > \frac{1}{2} = \eta(u_1, u_2)$. For $z_2 = (0, 5)$, $d(u_1, T_1 z_1) = \text{dist}(A, B) = d(u_2, T_2 z_1) = \frac{1}{2} = \eta(u_1, u_2)$. Thus the pair $(T_1, T_2)$ is $\alpha$-proximal admissible with respect to $\eta$.

By Theorem 2.3, we immediately obtain the following result.

**Theorem 3.3.** Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X, d)$ such that $A_0$ is nonempty and let $T, S : A \rightarrow B$ be continuous mappings satisfying the following assertions for all $z_1, z_2 \in A$:

1. $\alpha(z_1, z_2) \geq \eta(z_1, z_2) \Rightarrow d(T z_1, S z_2) \leq \psi(d(z_1, z_2))$;
2. $T(A_0), S(A_0) \subseteq B_0$ and $(A, B)$ satisfies the weak P-property;
3. $(T, S)$ is $\alpha$-proximal admissible with respect to $\eta$;
4. there exist $z_0, z_1, z_2 \in A_0$ such that

$$d(z_1, T z_0) = \text{dist}(A, B), \quad \alpha(z_0, z_1) \geq \eta(z_0, z_1)$$

and

$$d(z_2, S z_0) = \text{dist}(A, B), \quad \alpha(z_0, z_2) \geq \eta(z_0, z_2).$$

Then the mappings $T$ and $S$ have a common best proximity point.

The case $A = B = X$ reduces Definition 3.1 and Theorem 3.3 into the following:

**Definition 3.4.** Let $(X, d)$ be a metric space and $T_1, T_2 : X \rightarrow X$ be mappings. The pair $(T_1, T_2)$ is $\alpha$-admissible with respect to $\eta$ if there exist functions $\alpha, \eta : X \times X \rightarrow [0, \infty)$ such that for all $z_1, z_2 \in X$,

$$\alpha(z_1, z_2) \geq \eta(z_1, z_2) \Rightarrow \alpha(T_1 z_1, T_2 z_2) \geq \eta(T_1 z_1, T_2 z_2).$$

When $\alpha(z_1, z_2) = 1$ for all $z_1, z_2 \in X$, the pair $(T_1, T_2)$ is called $\eta$-subadmissible and when $\eta(z_1, z_2) = 1$ for all $z_1, z_2 \in X$, the pair $(T_1, T_2)$ is called $\alpha$-admissible.

**Remark 3.5.** Definition 3.4 generalizes the concepts of compatibility and weak compatibility by Jungck [18] and [19]. Every weakly compatible pair is $\alpha$-admissible with respect to $\eta$. Indeed, let $(T_1, T_2)$ be weakly compatible pair. Then $T_1 (T_2 z) = T_2 (T_1 z)$ for all $z$ belonging to $C(T_1, T_2)$ as the set of all coincidence points of mappings $T_1$ and $T_2$. Define

$$\alpha(z_1, z_2) = \begin{cases} 1 & \text{if } z_1, z_2 \in C(T_1, T_2), \\ 0 & \text{otherwise}, \end{cases}$$

and $\eta(z_1, z_2) = \frac{1}{2}$ for all $z_1, z_2 \in X$.

Then $\alpha(z_1, z_2) > \eta(z_1, z_2)$ if $z_1, z_2 \in C(T_1, T_2)$. Since $(T_1, T_2)$ is weakly compatible pair, so for all $z_1, z_2 \in C(T_1, T_2)$, we have $T_1 (T_2 z_1) = T_1 (T_2 z_2) = T_2 (T_1 z_1)$ and $T_1 (T_2 z_2) = T_2 (T_1 z_2) = T_2 (T_2 z_2)$. This implies that $T_1 z_1, T_2 z_2 \in C(T_1, T_2)$. Hence $\alpha(T_1 z_1, T_2 z_2) = 1 > \frac{1}{2} = \eta(T_1 z_1, T_2 z_2)$, that is, the pair $(T_1, T_2)$ is $\alpha$-admissible with respect to $\eta$. But the converse is not true which is clear from the following:

**Example 3.6.** Consider $X = \mathbb{R}$ with the usual metric. Define $T_1, T_2 : X \rightarrow X$ by

$$T_1(z) = z^3, \quad T_2(z) = \frac{z^2}{4}$$

and $\alpha, \eta : X \times X \rightarrow [0, \infty)$ by

$$\alpha(z_1, z_2) = \begin{cases} 2 & \text{if } z_1, z_2 \geq 0, \\ 0 & \text{if } z_1, z_2 < 0, \end{cases} \quad \eta(z_1, z_1) = \frac{1}{4}$$

for all $z_1, z_2 \in X$. Note that $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$ when $z_1, z_2 \geq 0$. This implies that $\alpha(T_1 z_1, T_2 z_2) = 2 > \frac{1}{2} = \eta(T_1 z_1, T_2 z_2)$. Hence the pair $(T_1, T_2)$ is $\alpha$-admissible with respect to $\eta$. On the other hand, the coincidence points of $T_1$ and $T_2$ are 0 and $\frac{1}{2}$ such that $T_1 (T_2 \left(\frac{1}{2}\right)) \neq T_2 (T_1 \left(\frac{1}{2}\right)) = \left(\frac{1}{4}\left(\frac{1}{4}\right)^2\right)^2$. Thus, the pair $(T_1, T_2)$ is not weakly compatible.
Theorem 3.7. Let \((X,d)\) be a complete metric space and \(T,S:X \to X\) be continuous mappings satisfying the following assertions for all \(z_1,z_2 \in X:\)

1. \(\alpha(z_1,z_2) \geq \eta(z_1,z_2) \Rightarrow d(Tz_1,Sz_2) \leq \psi(d(z_1,z_2));\)
2. \((T,S)\) is \(\alpha\)-admissible with respect to \(\eta;\)
3. there exist \(z_0,z_1 \in X\) such that \(\alpha(z_0,Tz_0) \geq \eta(z_0,Tz_0)\) and \(\alpha(z_1,Sz_1) \geq \eta(z_1,Sz_1).\)

Then the mappings \(T\) and \(S\) have a common fixed point.

Taking \(\eta(z_1,z_2) = 1\) in Theorem 3.7 we get the following:

Corollary 3.8. Let \((X,d)\) be a complete metric space and \(T,S:X \to X\) be continuous mappings satisfying the following assertions for all \(z_1,z_2 \in X:\)

1. \(\alpha(z_1,z_2) \geq 1 \Rightarrow d(Tz_1,Sz_2) \leq \psi(d(z_1,z_2));\)
2. \((T,S)\) is \(\alpha\)-admissible;
3. there exist \(z_0,z_1 \in X\) such that \(\alpha(z_0,Tz_0) \geq 1\) and \(\alpha(z_1,Sz_1) \geq 1.\)

Then the mappings \(T\) and \(S\) have a common fixed point.

Remark 3.9. When \(T_1 = T_2 = T\) in Definition 3.4 we get Definition 2.1 in [28] and in case \(T = S,\) (with the help of Remark 3.5 in [5]), Corollary 3.8 generalizes Theorem 2.1 in [29].

When \(T_1 = T_2 = T,\) Definition 3.1 and Theorem 3.3 are reduced to Definition 8 in [15] and the following result, respectively.

Theorem 3.10. Let \(A\) and \(B\) be two nonempty closed subsets of a complete metric space \((X,d)\) such that \(A_0\) is nonempty and \(T : A \to B\) be a continuous mapping satisfying the following assertions for all \(z_1,z_2 \in A:\)

1. \(\alpha(z_1,z_2) \geq \eta(z_1,z_2) \Rightarrow d(Tz_1,Tz_2) \leq \psi(d(z_1,z_2));\)
2. \(T(A_0) \subseteq B_0\) and \((A,B)\) satisfies the weak \(P\)-property;
3. \(T\) is \(\alpha\)-proximal admissible with respect to \(\eta;\)
4. there exist \(z_0,z_1 \in A_0\) such that

\[
d(z_1,Tz_0) = \text{dist}(A,B), \quad \alpha(z_0,z_1) \geq \eta(z_0,z_1).
\]

Then \(T\) has a best proximity point.

Remark 3.11. The special cases of Theorems 3.3 and 3.10 for \(\eta(z_1,z_2) = 1\) and \(\alpha(z_1,z_2) = 1\) can be obtained as in Corollaries 2.5 and 2.6.

4. Generalization

In this section we generalize the results of Sections 2 and 3 for a sequence of mappings.

Definition 4.1. Let \(A\) and \(B\) be two nonempty subsets of a metric space \((X,d)\) and \(\{T_i : A \to 2^B \setminus \emptyset\}_{i=1}^{\infty}\) be a sequence of multivalued mappings. The sequence \(\{T_i\}\) is \(\alpha_s\)-proximal admissible with respect to \(\eta\) if there exist functions \(\alpha,\eta : A \times A \to [0,\infty)\) such that for \(z_1,z_2,u_1,u_2 \in A,\)

\[
\begin{align*}
\alpha(z_1,z_2) &\geq \eta(z_1,z_2) \\
d(u_i,y_i) &= \text{dist}(A,B) \quad \text{for all } y_i \in T_i z_1 \text{ and } u_i \in T_i z_2, \text{ and for all } i,j \in \mathbb{N}. \quad \text{When } \alpha(z_1,z_2) = 1 \text{ for all } z_1,z_2 \in A, \text{ the sequence } \{T_i\} \text{ is called } \eta_s\text{-proximal sub-admissible and when } \eta(z_1,z_2) = 1 \text{ for all } z_1,z_2 \in A, \text{ the sequence } \{T_i\} \text{ is called } \alpha_s\text{-proximal admissible.}
\end{align*}
\]
Theorem 4.2. Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X, d)$ such that $A_0$ is nonempty and $\{T_i : A \rightarrow CL(B)\}_{i=1}^{\infty}$ be a sequence of continuous multivalued mappings satisfying the following assertions:

1. $\alpha(z_1, z_2) \geq \eta(z_1, z_2) \Rightarrow H(T_i z_1, T_j z_2) \leq \psi(d(z_1, z_2))$ for each $i, j \in \mathbb{N}$;
2. $T_i z \subseteq B_0$, for each $z \in A_0$, $i \in \mathbb{N}$ and $(A, B)$ satisfies the weak P-property;
3. $\{T_i\}$ is $\alpha$-proximal admissible with respect to $\eta$;
4. there exist $z_0, z_i \in A$, and $y_i \in T_i z_0$ for each $i \in \mathbb{N}$ such that

$$d(z_i, y_i) = dist(A, B), \quad \alpha(z_0, z_i) \geq \eta(z_0, z_i).$$

Then the mappings $T_i$ have a common best proximity point.

Proof. It is similar to the proof of Theorem 2.3 and is omitted. \qed

Taking $\eta(z_1, z_2) = 1$ in Theorem 4.2 we get the following:

Corollary 4.3. Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X, d)$ such that $A_0$ is nonempty and $\{T_i : A \rightarrow CL(B)\}_{i=1}^{\infty}$ be a sequence of continuous multivalued mappings satisfying the following assertions:

1. $\alpha(z_1, z_2) \geq 1 \Rightarrow H(T_i z_1, T_j z_2) \leq \psi(d(z_1, z_2))$ for each $i, j \in \mathbb{N}$;
2. $T_i z \subseteq B_0$, for each $z \in A_0$, $i \in \mathbb{N}$ and $(A, B)$ satisfies the weak P-property;
3. $\{T_i\}$ is $\alpha$-proximal admissible;
4. there exists $z_0, z_i \in A$, and $y_i \in T_i z_0$ for each $i \in \mathbb{N}$ such that

$$d(z_i, y_i) = dist(A, B), \quad \alpha(z_0, z_i) \geq 1.$$

Then the mappings $T_i$ have a common best proximity point.

Taking $\alpha(z_1, z_2) = 1$ in Theorem 4.2 we get the following:

Corollary 4.4. Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X, d)$ such that $A_0$ is nonempty and $\{T_i : A \rightarrow CL(B)\}_{i=1}^{\infty}$ be a sequence of continuous multivalued mappings satisfying the following assertions:

1. $\eta(z_1, z_2) \leq 1 \Rightarrow H(T_i z_1, T_j z_2) \leq \psi(d(z_1, z_2))$ for each $i, j \in \mathbb{N}$;
2. $T_i z \subseteq B_0$, for each $z \in A_0$, $i \in \mathbb{N}$ and $(A, B)$ satisfies the weak P-property;
3. $\{T_i\}$ is $\eta$-proximal subadmissible;
4. there exist $z_0, z_i \in A$, and $y_i \in T_i z_0$ for each $i \in \mathbb{N}$ such that

$$d(z_i, y_i) = dist(A, B), \quad \eta(z_0, z_i) \leq 1.$$

Then the mappings $T_i$ have a common best proximity point.

Remark 4.5. The choice $A = B = X$ reduces Definition 4.1 and Theorem 4.2 into the Definition 3.1 and Theorem 3.2 in [5], respectively, and generalizes Theorem 4.1 in [17]. When $A = B = X$, Corollaries 4.3 and 4.4 generalize Corollaries 4.1 and 4.2 in [17], respectively.

Theorem 4.6. Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X, d)$ such that $A_0$ is nonempty and $\{T_i : A \rightarrow K(B)\}_{i=1}^{\infty}$ be a sequence of continuous multivalued mappings satisfying all assertions of Theorem 4.2 with condition $H$. Then the mappings $T_i$ have a unique common best proximity.
**Definition 4.7.** Let \( A \) and \( B \) be two nonempty subsets of a metric space \((X, d)\) and \( \{ T_i : A \to B \}_{i=1}^{\infty} \) be a sequence of mappings. The sequence \( \{ T_i \} \) is \( \alpha, \eta \)-proximal admissible with respect to \( \eta \) if there exists two functions \( \alpha, \eta : A \times A \to [0, \infty) \) such that for \( z_1, z_2, u_1, u_2 \in A \),

\[
\begin{align*}
\alpha(z_1, z_2) &\geq \eta(z_1, z_2) \\
 d(u_1, T_i z_1) &= \text{dist}(A, B) \\
 d(u_2, T_j z_2) &= \text{dist}(A, B)
\end{align*}
\]

for each \( i, j \in \mathbb{N} \). When \( \alpha(z_1, z_2) = 1 \) for all \( z_1, z_2 \in A \), the sequence \( \{ T_i \} \) is called \( \eta \)-proximal subadmissible and when \( \eta(z_1, z_2) = 1 \) for all \( z_1, z_2 \in A \), the sequence \( \{ T_i \} \) is called \( \alpha \)-proximal admissible.

From Definition 4.1 and Theorem 4.2, we obtain the following result for a sequence of single-valued mappings.

**Theorem 4.8.** Let \( A \) and \( B \) be two nonempty closed subsets of a complete metric space \((X, d)\) such that \( A_0 \) is nonempty and \( \{ T_i : A \to B \}_{i=1}^{\infty} \) be a sequence of continuous mappings satisfying the following assertions:

1. \( \alpha(z_1, z_2) \geq \eta(z_1, z_2) \Rightarrow d(T_i z_1, T_j z_2) \leq \psi(d(z_1, z_2)) \) for each \( i, j \in \mathbb{N} \);
2. \( T_i z \subseteq B_0 \) for each \( z \in A_0 \), \( i \in \mathbb{N} \) and \((A, B)\) satisfies the weak P-property;
3. \( \{ T_i \} \) is \( \alpha, \eta \)-proximal admissible with respect to \( \eta \);
4. there exist \( z_0, z_i \in A_0 \) such that for each \( i \in \mathbb{N} \)

\[
d(z_i, T_i z_0) = \text{dist}(A, B), \quad \alpha(z_0, z_i) \geq \eta(z_0, z_i).
\]

Then the mappings \( T_i \) have a common best proximity point.

5. **Common best proximity point results in partially ordered metric space**

Let \((X, d, \preceq)\) be a partially ordered metric space and \( A \) and \( B \) be two nonempty subsets of \( X \). The existence of best proximity point in the setting of a partially ordered metric space has been established in [2, 3, 10, 11, 25, 27]. In this section, we derive new results in partially ordered metric spaces as an application of our results in Sections 2 and 3. Recall that a mapping \( T : A \to B \) is said to be proximally increasing if it satisfies the condition

\[
\begin{align*}
z_1 \preceq z_2 &\quad \Rightarrow \quad d(u_1, T z_1) = \text{dist}(A, B) \\
 d(u_2, T z_2) &= \text{dist}(A, B)
\end{align*}
\]

where \( z_1, z_2, u_1, u_2 \in A \) (see [10]). Very recently, Pragadeeswarar et al. [27] defined the notion of proximal relation between two subsets of \( X \) as follows:

**Definition 5.1** ([27]). Let \( A \) and \( B \) be two nonempty subsets of a partially ordered metric space \((X, d, \preceq)\) such that \( A_0 \neq \emptyset \). Let \( B_1 \) and \( B_2 \) be two nonempty subsets of \( B_0 \). The proximal relation between \( B_1 \) and \( B_2 \) is denoted and defined by \( B_1 \preceq (1) B_2 \), if for every \( b_1 \in B_1 \) with \( d(a_1, b_1) = d(A, B) \), there exists \( b_2 \in B_2 \) with \( d(a_2, b_2) = d(A, B) \) such that \( a_1 \preceq a_2 \).

Now we present our main results of this section.

**Theorem 5.2.** Let \( A \) and \( B \) be two nonempty closed subsets of a partially ordered complete metric space \((X, d, \preceq)\) such that \( A_0 \) is nonempty and \( T, S : A \to CL(B) \) be continuous mappings satisfying the following assertions for all \( z_1, z_2 \in A \) with \( z_1 \preceq z_2 \):

1. \( H(T z_1, S z_2) \leq \psi(d(z_1, z_2)); \)
2. \( T z \subseteq B_0 \) for each \( z \in A_0 \) and \((A, B)\) satisfies the weak P-property;
3. \( z_1, z_2 \in A_0, \ z_1 \preceq z_2 \) implies \( T z_1 \preceq (1) S z_2 \);
4. there exist \( z_0, z_1, z_2 \in A_0, y_1 \in Tz_0 \) and \( y_2 \in Sz_0 \) such that

\[
d(z_1, y_1) = \text{dist}(A, B), \quad z_0 \preceq z_1\]

and

\[
d(z_2, y_2) = \text{dist}(A, B), \quad z_0 \preceq z_2.
\]

Then \( T \) and \( S \) have a common best proximity point.

Proof. Define \( \alpha, \eta : X \times X \rightarrow [0, \infty) \) by

\[
\alpha(z_1, z_2) = \begin{cases} 1 & z_1 \preceq z_2, \\ 0 & \text{otherwise}, \end{cases} \quad \eta(z_1, z_2) = \begin{cases} 1 & z_1 \preceq z_2, \\ 0 & \text{otherwise}. \end{cases}
\]

Since \( Tz_1 \preceq_{(1)} Sz_2 \), therefore for \( z_1, z_2, u_1, u_2 \in X, y_1 \in Tz_1, y_2 \in Sz_2 \) with

\[
\begin{align*}
\alpha(z_1, z_2) & \geq \eta(z_1, z_2) \\
d(u_1, y_1) & = \text{dist}(A, B) \\
d(u_2, y_2) & = \text{dist}(A, B)
\end{align*}
\]

we have \( u_1 \preceq u_2 \). This implies that \( \alpha(u_1, u_2) = 1 > \frac{1}{2} = \eta(u_1, u_2) \) for \( z_1 \preceq z_2 \) and \( \alpha(u_1, u_2) = 0 = \eta(u_1, u_2) \) otherwise. Thus, all the conditions of Theorem 2.3 are satisfied and hence mappings \( T \) and \( S \) have a common best proximity point. \( \square \)

By considering \( T = S \), Theorem 5.2 is reduced to the following:

**Theorem 5.3.** Let \( A \) and \( B \) be two nonempty closed subsets of a partially ordered complete metric space \( (X, d, \preceq) \) such that \( A_0 \) is non-empty and \( T : A \rightarrow CL(B) \) be a continuous mapping satisfying the following assertions for all \( z_1, z_2 \in A \) with \( z_1 \preceq z_2 \):

1. \( H(Tz_1, Tz_2) \leq \psi(d(z_1, z_2)) \);
2. \( Tz \subseteq B_0 \) for each \( z \in A_0 \) and \((A, B)\) satisfies the weak P-property;
3. \( z_1, z_2 \in A_0, z_1 \preceq z_2 \) implies \( Tz_1 \preceq_{(1)} Tz_2 \);
4. there exist \( z_0, z_1 \in A_0, y_1 \in Tz_0 \) such that

\[
d(z_1, y_1) = \text{dist}(A, B), \quad z_0 \preceq z_1.
\]

Then the mapping \( T \) has a best proximity point.

Following the arguments in the proof of Theorem 5.2, we obtain the following result.

**Theorem 5.4.** Let \( A \) and \( B \) be two nonempty closed subsets of a partially ordered complete metric space \( (X, d, \preceq) \) such that \( A_0 \) is non-empty and \( \{T_i : A \rightarrow CL(B)\}^\infty_{i=1} \) be sequence of continuous mappings satisfying the following assertions for all \( z_1, z_2 \in A \) with \( z_1 \preceq z_2 \):

1. \( H(T_i z_1, T_j z_2) \leq \psi(d(z_1, z_2)) \) for each \( i, j \in \mathbb{N} \);
2. \( T_i z \subseteq B_0 \) for each \( z \in A_0, i \in \mathbb{N} \) and \((A, B)\) satisfies the weak P-property;
3. \( z_1, z_2 \in A_0, z_1 \preceq z_2 \) implies \( T_i z_1 \preceq_{(1)} T_j z_2 \) for each \( i, j \in \mathbb{N} \);
4. there exist \( z_0, z_i \in A_0 \) and \( y_i \in T_i z_0 \) for each \( i \in \mathbb{N} \) such that

\[
d(z_i, y_i) = \text{dist}(A, B), \quad z_0 \preceq z_i.
\]

Then the mappings \( T_i \) have a common best proximity point.

For single valued mappings, from Theorems 5.2, 5.4 we obtain the following results.
Theorem 5.5. Let $A$ and $B$ be two nonempty closed subsets of a partially ordered complete metric space $(X,d,\preceq)$ such that $A_0$ is nonempty and $T,S : A \to B$ be continuous mappings satisfying the following assertions for all $z_1,z_2 \in A$ with $z_1 \preceq z_2$:

1. $d(Tz_1, Sz_2) \leq \psi(d(z_1,z_2))$;
2. $Tz,Sz \subseteq B_0$ for each $z \in A_0$ and $(A,B)$ satisfies the weak P-property;
3. $z_1,z_2 \in A_0$, $z_1 \preceq z_2$ implies $Tz_1 \preceq Sz_2$;
4. there exist $z_0,z_1,z_2 \in A_0$ such that

$$d(z_1,Tz_0) = \text{dist}(A,B), \quad z_0 \preceq z_1$$

and

$$d(z_2,Tz_0) = \text{dist}(A,B), \quad z_0 \preceq z_2.$$ 

Then $T$ and $S$ have a common best proximity point.

Theorem 5.6. Let $A$ and $B$ be two nonempty closed subsets of a partially ordered complete metric space $(X,d,\preceq)$ such that $A_0$ is non-empty and $T : A \to B$ be a continuous mapping satisfying the following assertions for all $z_1,z_2 \in A$ with $z_1 \preceq z_2$:

1. $d(Tz_1, Tz_2) \leq \psi(d(z_1,z_2))$;
2. $Tz \subseteq B_0$ for each $z \in A_0$ and $(A,B)$ satisfies the weak P-property;
3. $z_1,z_2 \in A_0$, $z_1 \preceq z_2$ implies $Tz_1 \preceq Tz_2$;
4. there exist $z_0,z_1 \in A_0$ such that

$$d(z_1,Tz_0) = \text{dist}(A,B), \quad z_0 \preceq z_1.$$ 

Then $T$ has a best proximity point.

Theorem 5.7. Let $A$ and $B$ be two nonempty closed subsets of a partially ordered complete metric space $(X,d,\preceq)$ such that $A_0$ is nonempty and $\{T_i : A \rightarrow B\}^\infty_i$ be sequence of continuous mappings satisfying the following assertions for all $z_1,z_2 \in A$ with $z_1 \preceq z_2$:

1. $d(T_iz_1, T_jz_2) \leq \psi(d(z_1,z_2))$ for each $i,j \in \mathbb{N}$;
2. $T_iz \subseteq B_0$ for each $z \in A_0$, $i \in \mathbb{N}$ and $(A,B)$ satisfies the weak P-property;
3. $z_1,z_2 \in A_0$, $z_1 \preceq z_2$ implies $T_iz_1 \preceq T_jz_2$ for each $i,j \in \mathbb{N}$;
4. there exist $z_0,z_i \in A_0$ for each $i \in \mathbb{N}$ such that

$$d(z_i, T_iz_0) = \text{dist}(A,B), \quad z_0 \preceq z_i.$$ 

Then the mappings $T_i$ have a common best proximity point.

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References


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