Quadratic $\rho$-functional inequalities in complex matrix normed spaces

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Abstract

In this paper, we solve the following quadratic $\rho$-functional inequalities

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \|\rho(2f(x + y) + 2f(x - y) - f(x) - f(y))\|,$$

where $\rho$ is a fixed complex number with $|\rho| < 1$, and

$$\|2f(x + y) + 2f(x - y) - f(x) - f(y)\| \leq \|\rho(2f(x + y) + 2f(x - y) - f(x) - f(y))\|,$$

where $\rho$ is a fixed complex number with $|\rho| < \frac{1}{2}$. By using the direct method, we prove the Hyers-Ulam stability of these inequalities in complex matrix normed spaces, and prove the Hyers-Ulam stability of quadratic $\rho$-functional equations associated with these inequalities in complex matrix normed spaces. ©2016 All rights reserved.

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1. Introduction and preliminaries

The first stability problem concerning with the group homomorphisms was raised by Ulam [13] and affirmatively solved by Hyers [5]. Hyers’ result was generalized by Aoki [11] for additive mappings and by

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The functional equation
\[ f(x + y) + f(x - y) = 2f(x) + 2f(y), \]
(1.1)
is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The Hyers-Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof [12] for mappings from a normed space to a Banach space. Cholewa [2] noticed that Skof’s theorem remains true if the domain is replaced by an Abelian group. In 1992, Czerwik [3] gave a generalization of the Skof–Cholewa’s result.

The following functional equation
\[ 2f\left(\frac{x + y}{2}\right) + 2f\left(\frac{x - y}{2}\right) = f(x) + f(y), \]
(1.2)
is called a Jensen-type quadratic equation (see [6]). In [6], Jang et al. proved the Hyers-Ulam stability of the equation (1.2) in fuzzy Banach spaces. In 2014, Wang et al. [14] investigated some stability results for Jensen-type quadratic functional equation (1.2) in intuitionistic fuzzy normed spaces.

In this paper, we consider the following two quadratic $\rho$-functional inequalities
\[ \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \|\rho(2f\left(\frac{x + y}{2}\right) + 2f\left(\frac{x - y}{2}\right) - f(x) - f(y))\|, \]
(1.3)
where $\rho$ is a fixed complex number with $|\rho| < 1$, and
\[ \|2f\left(\frac{x + y}{2}\right) + 2f\left(\frac{x - y}{2}\right) - f(x) - f(y)\| \leq \|\rho(f(x + y) + f(x - y) - 2f(x) - 2f(y))\|, \]
(1.4)
where $\rho$ is a fixed complex number with $|\rho| < \frac{1}{2}$, in complex matrix Banach spaces. More precisely, we solve the problem of the quadratic $\rho$-functional inequalities (1.3) and (1.4), and prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequalities (1.3) and (1.4) in complex matrix Banach spaces by using the direct method. Moreover, we prove the Hyers-Ulam stability of quadratic $\rho$-functional equations associated with the quadratic $\rho$-functional inequalities (1.3) and (1.4) in complex matrix Banach spaces.

Following [7] [8] [10], we will also use the following notations. The set of all $(m \times n)$-matrices in $X$ will be denoted by $M_{m,n}(X)$. When $m = n$, the matrix $M_{n,n}(X)$ will be written as $M_n(X)$. The symbol $e_{ij} \in M_{1,n}(C)$ will denote the row vector whose $j$-th component is 1 and the other components are 0. Similarly, $E_{ij} \in M_{n}(C)$ will denote the $n \times n$ matrix whose $(i,j)$-component is 1 and the other components are 0. The $n \times n$ matrix whose $(i,j)$-component is $x$ and the other components are 0 will be denoted by $E_{ij} \otimes x \in M_n(X)$. For $x \in M_n(X)$, $y \in M_k(X),$
\[ x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}. \]

Let $(X, \|\cdot\|)$ be a normed space. Note that $(X, \|\cdot\|_n)$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer $n$ and $\|AxB\|_k \leq \|A\|\|B\||\|x\|_n$ holds for $A \in M_{k,n}$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}$, and that $(X, \{\|\cdot\|_n\})$ is a matrix Banach space if and only if $X$ is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space.

A matrix normed space $(X, \|\cdot\|_n)$ is called an $L^\infty$-matrix normed space if $\|x \oplus y\|_{n+k} = \max\{\|x\|_n, \|y\|_k\}$ holds for all $x \in M_n(X)$ and all $y \in M_k(X)$.

Let $E, F$ be vector spaces. For a given mapping $h : E \to F$ and a given positive integer $n$, define $h_n : M_n(E) \to M_n(F)$ by
\[ h_n([x_{ij}]) = [h(x_{ij})] \]
for all $[x_{ij}] \in M_n(E)$. 

Lemma 1.1 ([8, 10]). Let \((X, \{\| \cdot \|_n\})\) be a matrix normed space. Then

1. \(\|E_{kl} \otimes x\|_n = \|x\|\) for \(x \in X\);
2. \(\|x_{kl}\| \leq \|\Delta_{ij}\|_n \leq \sum_{i,j=1}^n \|\Delta_{ij}\|\) for \([\Delta_{ij}] \in M_n(X)\);
3. \(\lim_{n \to \infty} x_n = x\) if and only if \(\lim_{n \to \infty} x_{ijn} = x_{ij}\) for \(x_n = [x_{ijn}], x = [x_{ij}] \in M_k(X)\).

Throughout this paper, let \((X, \{\| \cdot \|_n\})\) be a matrix normed space and \((Y, \{\| \cdot \|_n\})\) be a matrix Banach space.

2. Stability of the quadratic \(\rho\)-functional inequality \([1.3]\) in complex matrix normed spaces

In this section, we prove the Hyers-Ulam stability of the quadratic \(\rho\)-functional inequality \([1.3]\) in complex matrix normed spaces. We assume that \(\rho\) is a fixed complex number with \(|\rho| < 1\).

Lemma 2.1. Let \(V\) and \(W\) be complex normed spaces. A mapping \(f : V \to W\) satisfies

\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \|\rho(2f(x + y) + 2f(x - y) - f(x) - f(y))\|
\]

for all \(x, y \in V\) if and only if \(f : V \to W\) is quadratic.

Proof. The proof is similar to the proof of [9, Lemma 2.2].

Corollary 2.2. A mapping \(f : V \to W\) satisfies

\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| = \|\rho(2f(x + y) + 2f(x - y) - f(x) - f(y))\|
\]

for all \(x, y \in V\) if and only if \(f : V \to W\) is quadratic.

Theorem 2.3. Let \(r, \theta\) be positive real numbers with \(r < 2\), and let \(f : X \to Y\) be a mapping such that

\[
\|f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\|_n \\
\leq \|\rho(2f_n([x_{ij}] + [y_{ij}]) + 2f_n([x_{ij}] - [y_{ij}]) - f_n([x_{ij}]) - f_n([y_{ij}]))\|_n \\
+ \sum_{i,j=1}^n \theta(\|x_{ij}\|^r + \|y_{ij}\|^r)
\]

for all \(x = [x_{ij}], y = [y_{ij}] \in M_n(X)\). Then there exists a unique quadratic mapping \(Q : X \to Y\) such that

\[
\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^n \frac{2\theta}{4 - 2^r}\|x_{ij}\|^r
\]

for all \(x = [x_{ij}] \in M_n(X)\).

Proof. When \(n = 1\), \((2.1)\) is equivalent to

\[
\|f(a + b) + f(a - b) - 2f(a) - 2f(b)\| \leq \|\rho(2f(a + b) + 2f(a - b) - f(a) - f(b))\|
\]

\[
+ \theta(\|a\|^r + \|b\|^r)
\]
for all \(a, b \in X\). By letting \(a = b = 0\) in (2.3), we get \(\|2f(0)\| \leq |\rho|\|2f(0)\|\), implying that \(f(0) = 0\). Next, by letting \(b = a\) in (2.3), we obtain

\[
\|f(2a) - 4f(a)\| \leq 2\theta\|a\|^{r}
\]

(2.4)

for all \(a \in X\). It follows from (2.4) that

\[
\|f(a) - \frac{1}{4}f(2a)\| \leq \frac{1}{2}\theta\|a\|^{r}
\]

for all \(a \in X\). Hence

\[
\|\frac{1}{4^l}f(2^{l}a) - \frac{1}{4^m}f(2^{m}a)\| \leq \sum_{j=l}^{m-1} \|\frac{1}{4^j}f(2^{j}a) - \frac{1}{4^{j+1}}f(2^{j+1}a)\|
\]

\[
\leq \frac{1}{2} \sum_{j=l}^{m-1} \frac{2^{rj}}{4^j}\theta\|a\|^{r}
\]

(2.5)

for all nonnegative integers \(m\) and \(l\) with \(m > l\) and all \(a \in X\). It follows from (2.5) that the sequence \(\{\frac{f(2^{n}a)}{4^n}\}\) is a Cauchy sequence in \(Y\) for all \(a \in X\). Since \(Y\) is complete, the sequence \(\{\frac{f(2^{n}a)}{4^n}\}\) is convergent. So one can define the mapping \(Q : X \to Y\) by

\[
Q(a) = \lim_{n \to \infty} \frac{1}{4^n}f(2^{n}a)
\]

(2.6)

for all \(a \in X\). Moreover, by letting \(l = 0\) and passing the limit \(m \to \infty\) in (2.5), we get

\[
\|f(a) - Q(a)\| \leq \frac{2\theta}{4 - 2^r}\|a\|^{r}
\]

(2.7)

for all \(a \in X\).

Now, we show that the mapping \(Q\) is quadratic. It follows from (2.3) and (2.6) that

\[
\|Q(a + b) + Q(a - b) - 2Q(a) - 2Q(b)\| = \lim_{n \to \infty} \frac{1}{4^n}\|f(2^{n}(a + b)) + f(2^{n}(a - b)) - 2f(2^{n}a) - 2f(2^{n}b)\|
\]

\[
\leq \lim_{n \to \infty} \frac{1}{4^n}\|\rho(f(\frac{2^{n}(a + b)}{2}) + f(\frac{2^{n}(a - b)}{2}) - f(2^{n}a) - f(2^{n}b))\|
\]

\[
+ \lim_{n \to \infty} \frac{2^{rn}}{4^n}\theta(\|a\|^{r} + \|b\|^{r})
\]

\[
= \|\rho(2Q(\frac{a + b}{2}) + 2Q(\frac{a - b}{2}) - Q(a) - Q(b))\|
\]

for all \(a, b \in X\). Thus, by Lemma 2.1 the mapping \(Q : X \to Y\) is quadratic.

To prove the uniqueness of \(Q\), let \(Q' : X \to Y\) be another quadratic mapping satisfying (2.2). Let \(n = 1\). Then we get

\[
\|Q(a) - Q'(a)\| = \frac{1}{4}Q(2^{n}a) - \frac{1}{4}Q'(2^{n}a)
\]

\[
\leq \frac{1}{4}Q(2^{n}a) - \frac{1}{4}f(2^{n}a) + \frac{1}{4}Q'(2^{n}a) - \frac{1}{4}f(2^{n}a)
\]

\[
\leq 4\theta \frac{2^{rn}}{4 - 2^r}\|a\|^{r}
\]

for all \(a \in X\). By letting \(n \to \infty\) in the above inequality, we get \(Q(a) = Q'(a)\) for all \(a \in X\), which gives the conclusion.
By Lemma [1.1] and (2.7), we get
\[ \|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2\theta}{4 - 2^r} \|x_{ij}\|^r \]
for all \( x = [x_{ij}] \in M_n(X) \). Thus \( Q : X \to Y \) is a unique quadratic mapping satisfying (2.2), as desired. This completes the proof of the theorem. \( \square \)

**Theorem 2.4.** Let \( r, \theta \) be positive real numbers with \( r > 2 \), and let \( f : X \to Y \) be a mapping satisfying (2.1) for all \( x = [x_{ij}], y = [y_{ij}] \in M_n(X) \). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that
\[ \|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2\theta}{4 - 2^r} \|x_{ij}\|^r \]  \( (2.8) \)
for all \( x = [x_{ij}] \in M_n(X) \).

**Proof.** It follows from (2.4) that
\[ \|f(a) - 4f\left(\frac{a}{2}\right)\| \leq \frac{2}{2^r} \theta \|a\|^r \]
for all \( a \in X \). Hence
\[ \|4^i f\left(\frac{a}{2}\right) - 4^m f\left(\frac{a}{2^m}\right)\| \leq \sum_{j=l}^{m-1} \|4^j f\left(\frac{a}{2^j}\right) - 4^{j+1} f\left(\frac{a}{2^{j+1}}\right)\| \]
\[ \leq \frac{2}{2^r} \sum_{j=l}^{m-1} \frac{4^j}{2^{rj}} \theta \|a\|^r \] \( (2.9) \)
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( a \in X \). It follows from (2.9) that the sequence \( \{4^m f\left(\frac{a}{2^m}\right)\} \) is a Cauchy sequence in \( Y \) for all \( a \in X \). Since \( Y \) is complete, the sequence \( \{4^m f\left(\frac{a}{2^m}\right)\} \) is convergent. So one can define the mapping \( Q : X \to Y \) by
\[ Q(a) = \lim_{n \to \infty} 4^n f\left(\frac{a}{2^n}\right) \]
for all \( a \in X \). Moreover, by letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.9), we get
\[ \|f(a) - Q(a)\| \leq \frac{2\theta}{2^r - 4} \|a\|^r \]
for all \( a \in X \). The rest of the proof is similar to that of Theorem 2.3 and thus it is omitted. \( \square \)

By the triangle inequality, we obtain
\[ \|f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\|_n \]
\[ = \rho(2f_n\left(\frac{[x_{ij}] + [y_{ij}]}{2}\right) + 2f_n\left(\frac{[x_{ij}] - [y_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}]))\|_n \]
\[ \leq \|f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\|_n \]
\[ - \rho(2f_n\left(\frac{[x_{ij}] + [y_{ij}]}{2}\right) + 2f_n\left(\frac{[x_{ij}] - [y_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}]))\|_n. \]

As corollaries of Theorems 2.3 and 2.4, we obtain the Hyers-Ulam stability results for the quadratic \( \rho \)-functional equation associated with the quadratic \( \rho \)-functional inequality (1.3) in complex matrix Banach spaces.
Corollary 2.5. Let \( r, \theta \) be positive real numbers with \( r < 2 \), and let \( f : X \to Y \) be a mapping such that

\[
\| f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}]) \| \\
= \rho (2f_n([x_{ij}] + [y_{ij}]) + 2f_n([x_{ij}] - [y_{ij}]) - f_n([x_{ij}]) - f_n([y_{ij}])) \leq \sum_{i,j=1}^{n} \theta (\|x_{ij}\|^{r} + \|y_{ij}\|^{r})
\]

for all \( x = [x_{ij}], y = [y_{ij}] \in M_n(X) \). Then there exists a unique quadratic mapping \( Q : X \to Y \) satisfying (2.2) for all \( x = [x_{ij}] \in M_n(X) \).

Corollary 2.6. Let \( r, \theta \) be positive real numbers with \( r > 2 \), and let \( f : X \to Y \) be a mapping satisfying (2.10) for all \( x = [x_{ij}], y = [y_{ij}] \in M_n(X) \). Then there exists a unique quadratic mapping \( Q : X \to Y \) satisfying (2.8) for all \( x = [x_{ij}] \in M_n(X) \).

Remark 2.7. If \( \rho \) is a real number such that \(-1 < \rho < 1\) and \( Y \) is a real Banach space, then all the assertions in this section remain valid.

3. Stability of the quadratic \( \rho \)-functional inequality (1.4) in complex matrix normed spaces

In this section, we prove the Hyers-Ulam stability of the quadratic \( \rho \)-functional inequality (1.4) in complex matrix normed spaces. We assume that \( \rho \) is a fixed complex number with \( |\rho| < \frac{1}{2} \).

Lemma 3.1. Let \( V \) and \( W \) be complex normed spaces. A mapping \( f : V \to W \) satisfies

\[
\|2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) - f(x) - f(y)\| \leq \rho (\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|)
\]

for all \( x, y \in V \) if and only if \( f : V \to W \) is quadratic.

Proof. The proof is similar to the proof of [9, Lemma 3.1].

Corollary 3.2. A mapping \( f : V \to W \) satisfies

\[
\|2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) - f(x) - f(y)\| = \rho (\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|)
\]

for all \( x, y \in V \) if and only if \( f : V \to W \) is quadratic.

Theorem 3.3. Let \( r, \theta \) be positive real numbers with \( r < 2 \), and let \( f : X \to Y \) be a mapping such that

\[
\|2f_n(\frac{[x_{ij}] + [y_{ij}]}{2}) + 2f_n(\frac{[x_{ij}] - [y_{ij}]}{2}) - f_n([x_{ij}]) - f_n([y_{ij}])\|_n
\leq \rho (f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}]))_n
\]

\[+ \sum_{i,j=1}^{n} \theta (\|x_{ij}\|^{r} + \|y_{ij}\|^{r})
\]

for all \( x = [x_{ij}], y = [y_{ij}] \in M_n(X) \). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that

\[
\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2r \theta}{4 - 2r} \|x_{ij}\|^{r}
\]

for all \( x = [x_{ij}] \in M_n(X) \).
Proof. When $n = 1$, (3.1) is equivalent to
\[
\|2f(\frac{a+b}{2}) + 2f(\frac{a-b}{2}) - f(a) - f(b)\| \\
\leq \|\rho(a+b) + f(a-b) - 2f(a) - 2f(b)\| + \theta(\|a\|^r + \|b\|^r)
\] (3.3)
for all $a, b \in X$. By letting $a = b = 0$ in (3.3), we get $\|2f(0)\| \leq |\rho|\|2f(0)\|$, implying that $f(0) = 0$. Next, by letting $b = 0$ in (3.3), we obtain
\[
\|f(2a) - 4f(a)\| \leq 2^r \theta \|a\|^r
\] (3.4)
for all $a \in X$. It follows from (2.4) that
\[
\|f(a) - \frac{1}{4}f(2a)\| \leq \frac{2^r}{4} \theta \|a\|^r
\]
for all $a \in X$. Hence
\[
\left\| \frac{1}{4}f(2^ja) - \frac{1}{4^m}f(2^ma) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{4^j}f(2^ja) - \frac{1}{4^{j+1}}f(2^{j+1}a) \right\|
\]
(3.5)
for all nonnegative integers $m$ and $l$ with $m > l$ and all $a \in X$. It follows from (3.5) that the sequence $\{\frac{f(2^na)}{4^n}\}$ is a Cauchy sequence in $Y$ for all $a \in X$. Since $Y$ is complete, the sequence $\{\frac{f(2^na)}{4^n}\}$ is convergent. So one can define the mapping $Q : X \to Y$ by
\[
Q(a) = \lim_{n \to \infty} \frac{1}{4^n}f(2^na)
\] (3.6)
for all $a \in X$. Moreover, by letting $l = 0$ and passing the limit $m \to \infty$ in (3.5), we get
\[
\|f(a) - Q(a)\| \leq \frac{2^r \theta}{4 - 2^r} \|a\|^r
\]
for all $a \in X$.

Now, we show that the mapping $Q$ is quadratic. It follows from (3.3) and (3.6) that
\[
\|2Q(\frac{a+b}{2}) + 2Q(\frac{a-b}{2}) - Q(a) - Q(b)\| = \lim_{n \to \infty} \frac{1}{4^n}\|2f(\frac{2^n(a+b)}{2}) + 2f(\frac{2^n(a-b)}{2}) - f(2^na) - f(2^nb)\|
\]
\[
\leq \lim_{n \to \infty} \frac{1}{4^n}\|\rho(2^n(a+b)) + f(2^n(a-b)) - 2f(2^na) - 2f(2^nb)\|
\]
\[
+ \lim_{n \to \infty} \frac{2^n}{4^n}\theta(\|a\|^r + \|b\|^r)
\]
\[
= \|\rho(Q(a+b) + Q(a-b) - 2Q(a) - 2Q(b))\|
\]
for all $a, b \in X$. Thus, by Lemma 3.1, the mapping $Q : X \to Y$ is quadratic. The rest of the proof is similar to that of Theorem 2.3 and thus it is omitted. \qed

**Theorem 3.4.** Let $r, \theta$ be positive real numbers with $r > 2$, and let $f : X \to Y$ be a mapping satisfying (3.1) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that
\[
\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^n \frac{2^r \theta}{2r - 4} \|x_{ij}\|^r
\] (3.7)
for all $x = [x_{ij}] \in M_n(X)$.
Proof. It follows from (3.4) that

\[ \| f(a) - 4f(\frac{a}{2}) \| \leq \theta \| a \|^r \]

for all \( a \in X \). Hence

\[
\| 4^l f(\frac{a}{2^l}) - 4^m f(\frac{a}{2^m}) \| \leq \sum_{j=l}^{m-1} \| 4^j f(\frac{a}{2^j}) - 4^{j+1} f(\frac{a}{2^{j+1}}) \| \\
\leq \sum_{j=l}^{m-1} \frac{4^j}{2^j} \theta \| a \|^r
\]

(3.8)

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( a \in X \). It follows from (3.8) that the sequence \( \{ 4^n f(\frac{a}{2^n}) \} \) is a Cauchy sequence in \( Y \) for all \( a \in X \). Since \( Y \) is complete, the sequence \( \{ 4^n f(\frac{a}{2^n}) \} \) converges. So one can define the mapping \( Q : X \to Y \) by

\[ Q(a) = \lim_{n \to \infty} 4^n f(\frac{a}{2^n}) \]

for all \( a \in X \). Moreover, by letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.8), we get

\[ \| f(a) - Q(a) \| \leq \frac{2^n \theta}{2^r - 4} \| a \|^r \]

for all \( a \in X \). The rest of the proof is similar to that of Theorem 3.3 and thus it is omitted. \( \square \)

By the triangle inequality, we obtain

\[
\| 2f_n\left(\frac{[x_{ij}] + [y_{ij}]}{2}\right) + 2f_n\left(\frac{[x_{ij}] - [y_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}]) \|_n \\
- \| \rho(f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}]) \|_n \\
\leq \| 2f_n\left(\frac{[x_{ij}] + [y_{ij}]}{2}\right) + 2f_n\left(\frac{[x_{ij}] - [y_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}]) \|_n \\
- \rho(f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}]) \|_n.
\]

As corollaries of Theorems 3.3 and 3.4, we obtain the Hyers-Ulam stability results for the quadratic \( \rho \)-functional equation associated with the quadratic \( \rho \)-functional inequality (1.4) in complex matrix Banach spaces.

**Corollary 3.5.** Let \( r, \theta \) be positive real numbers with \( r < 2 \), and let \( f : X \to Y \) be a mapping such that

\[ \| 2f_n\left(\frac{[x_{ij}] + [y_{ij}]}{2}\right) + 2f_n\left(\frac{[x_{ij}] - [y_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}]) \\
- \rho(f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}]) \|_n \leq \sum_{i,j=1}^{n} \theta(\| x_{ij} \|^r + \| y_{ij} \|^r)
\]

(3.9)

for all \( x = [x_{ij}], y = [y_{ij}] \in M_n(X) \). Then there exists a unique quadratic mapping \( Q : X \to Y \) satisfying (3.2) for all \( x = [x_{ij}] \in M_n(X) \).

**Corollary 3.6.** Let \( r, \theta \) be positive real numbers with \( r > 2 \), and let \( f : X \to Y \) be a mapping satisfying (3.9) for all \( x = [x_{ij}], y = [y_{ij}] \in M_n(X) \). Then there exists a unique quadratic mapping \( Q : X \to Y \) satisfying (3.7) for all \( x = [x_{ij}] \in M_n(X) \).

**Remark 3.7.** If \( \rho \) is a real number such that \(-1 < \rho < 1 \) and \( Y \) is a real Banach space, then all the assertions in this section remain valid.
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