Fixed point theorems for generalized multivalued nonlinear $\mathcal{F}$-contractions

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Abstract

In this paper, we introduce certain new concepts of $\alpha$-$\eta$-lower semi-continuous and $\alpha$-$\eta$-upper semi-continuous mappings. By using these concepts, we prove some fixed point results for generalized multivalued nonlinear $\mathcal{F}$-contractions in metric spaces and ordered metric spaces. As an application of our results we deduce Suzuki-Wardowski type fixed point results and fixed point results for orbitally lower semi-continuous mappings in complete metric spaces. Our results generalize and extend many recent fixed point theorems including the main results of Minak et al. [G. Minak, M. Olgun, I. Altun, Carpathian J. Math., 31 (2015), 241–248], Altun et al. [I. Altun, G. Minak, M. Olgun, Nonlinear Anal. Model. Control, 21 (2016), 201–210] and Olgun et al. [M. Olgun, G. Minak, I. Altun, J. Nonlinear Convex Anal., 17 (2016), 579–587]. ©2016 All rights reserved.

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1. Introduction and preliminaries

Let $(\mathcal{X}, d)$ be a metric space. $2^{\mathcal{X}}$ denotes the family of all nonempty subsets of $\mathcal{X}$, $C(\mathcal{X})$ denotes the family of all nonempty, closed subsets of $\mathcal{X}$, $CB(\mathcal{X})$ denotes the family of all nonempty, closed, and bounded subsets of $\mathcal{X}$ and $K(\mathcal{X})$ denotes the family of all nonempty compact subsets of $\mathcal{X}$. It is clear that, $K(\mathcal{X}) \subseteq CB(\mathcal{X}) \subseteq C(\mathcal{X}) \subseteq P(\mathcal{X})$. For $\mathcal{A}, \mathcal{B} \in C(\mathcal{X})$, let

$$H(\mathcal{A}, \mathcal{B}) = \max \left\{ \sup_{x \in \mathcal{A}} D(x, \mathcal{B}), \sup_{y \in \mathcal{B}} D(y, \mathcal{A}) \right\},$$

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where $D(x, B) = \inf \{d(x, y) : y \in B\}$. Then $H$ is called generalized Pompeiu-Hausdorff distance on $C(X)$. It is well-known that $H$ is a metric on $CB(X)$, which is called Pompeiu-Hausdorff metric induced by $d$. For more details see [3, 11].

An interesting generalization of the Banach contraction principle to multivalued mappings is known as Nadler’s fixed point theorem [25]. After this, many authors extended Nadler’s fixed point theorem in many directions (see [10, 12, 21, 29] and references therein). In 2012, Samet et al. [28] defined $\alpha$-admissible mappings. This notion is generalized by many authors (see [20, 21]). Salimi et al. [27] generalized this idea by introducing the function $\eta$ and established fixed point theorems. Next, Asl et al. [8] extended these concepts to multivalued mappings by introducing the notion of $\alpha^*$-admissible mappings as follows:

**Definition 1.1** [8]. Let $T : X \to 2^X$ be a multivalued map on a metric space $(X, d)$, $\alpha : X \times X \to \mathbb{R}^+$ be a function, then $T$ is an $\alpha^*$-admissible mapping, if

$$\alpha(y, z) \geq 1 \implies \alpha_*(Ty, Tz) \geq 1, \quad y, z \in X,$$

where

$$\alpha_*(A, B) = \inf_{y \in A, z \in B} \alpha(y, z).$$

Hussain et al. [19] modified the notion of $\alpha^*$-admissible as follows:

**Definition 1.2** [19]. Let $T : X \to 2^X$ be a multivalued map on a metric space $(X, d)$, $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions where $\eta$ is bounded, then $T$ is an $\alpha^*$-admissible mapping with respect to $\eta$, if

$$\alpha(y, z) \geq \eta(y, z) \implies \alpha_*(Ty, Tz) \geq \eta_*(Ty, Tz), \quad y, z \in X,$$

where

$$\alpha_*(A, B) = \inf_{y \in A, z \in B} \alpha(y, z), \quad \eta_*(A, B) = \sup_{y \in A, z \in B} \eta(y, z).$$


**Definition 1.3** [4]. Let $T : X \to 2^X$ be a multivalued map on a metric space $(X, d)$, $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions. We say that $T$ is generalized $\alpha^*$-admissible mapping with respect to $\eta$, if

$$\alpha(y, z) \geq \eta(y, z) \implies \alpha(u, v) \geq \eta(u, v), \quad \text{for all } u \in Ty, v \in Tz.$$

In 2014, Hussain et al. [10] introduced the notion of $\alpha^*$-continuous mappings as follows:

**Definition 1.4** [10]. Let $(X, d)$ be a metric space, $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$ be functions. Then $T$ is an $\alpha^*$-continuous mapping on $X$, if for given $z \in X$ and sequence $\{z_n\}$ with

$$z_n \to z \quad \text{as} \quad n \to \infty, \quad \alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1}), \quad \text{for all } n \in \mathbb{N} \implies Tz_n \to Tz.$$

After that Hussain et al. [15] generalized Definition 1.4 to multivalued maps.

**Definition 1.5** [15]. Let $T : X \to 2^X$ be a multivalued map on a metric space $(X, d)$, $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions. We say that $T$ is $\alpha^*$-continuous multivalued mapping, if for given $z \in X$ and sequence $\{z_n\}$ with $z_n \to z$ as $n \to \infty$, $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$, for all $n \in \mathbb{N}$ we have $Tz_n \to Tz$. That is, $\lim_{n \to \infty} d(z_n, z) = 0$ and $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$ implies $\lim_{n \to \infty} H(Tz_n, Tz) = 0$.

Recently, Wardowski [31] defined $F$-contraction and proved a fixed point result as a generalization of the Banach contraction principle for this contraction. This idea has been extended in many directions (see [11, 14, 17] and references therein). Hussain et al. [18] broadened this idea to $\alpha^*$-$GF$-contraction with respect to a general family of functions $G$. Following Wardowski and Hussain, we denote by $F$, the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

1. $F$ is non-decreasing.
2. $F$ is continuous.
3. If $\lim_{u \to \infty} u^{-1} F(u) = \infty$, then $\lim_{u \to \infty} F(u) = \infty$.
4. $F$ is upper semicontinuous from above.
5. $F(t) \geq 0$ for all $t \in \mathbb{R}^+$.
6. $F$ is convex.
7. $F$ is lower semicontinuous.
8. $F$ is superadditive.
9. $F$ is subadditive.
10. $F$ is a supermodular function.
11. $F$ is a submodular function.
12. $F$ is a strongly supermodular function.
13. $F$ is a strongly submodular function.
14. $F$ is a modular function.
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71. $F$ is a modular function.
72. $F$ is a strongly supermodular function.
73. $F$ is a strongly submodular function.
74. $F$ is a modular function.
(F₁) \( F \) is strictly increasing;

(F₂) for all sequence \( \{ a_n \} \subseteq \mathbb{R}^+ \), \( \lim_{n \to \infty} a_n = 0 \), if and only if \( \lim_{n \to \infty} F(a_n) = -\infty \);

(F₃) there exists \( 0 < k < 1 \) such that \( \lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0 \),

(θ), if \( F \) also satisfies the following:

(F₄) \( F(\inf A) = \inf_{A} F(A) \) for all \( A \subseteq (0, \infty) \) with \( \inf A > 0 \),

(Θ), the set of all functions \( G : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying:

(θ) for all \( t_1, t_2, t_3, t_4 \in \mathbb{R}^+ \) with \( t_1 t_2 t_3 t_4 = 0 \) there exists \( \tau > 0 \) such that \( G(t_1, t_2, t_3, t_4) = \tau \).

On unifying the concepts of Wardowski’s and Nadlers, Altun et al. [5] gave the concept of multivalued \( F \)-contractions and established some fixed point results. On the other side, Minak et al. [23], extended the results of Wardowski as follows:

**Theorem 1.6** ([23]). Let \((X, d)\) be a complete metric space, \( T : \mathcal{X} \to K(\mathcal{X}) \) and \( F \in \mathcal{F}_\sigma \). If there exists \( \tau > 0 \) such that for any \( z \in \mathcal{X} \) with \( d(z, Tz) > 0 \), there exists \( y \in F_{\sigma}^z \) satisfying

\[
\tau + F(D(y, Tz)) \leq F(d(z, y)),
\]

where

\[
F_{\sigma}^z = \{ y \in Tz : F(d(z, y)) \leq F(D(z, Tz)) + \sigma \},
\]

then \( T \) has a fixed point in \( \mathcal{X} \) provided \( \sigma < \tau \) and \( z \to d(z, Tz) \) is lower semi-continuous.

**Theorem 1.7** ([23]). Let \((X, d)\) be a complete metric space, \( T : \mathcal{X} \to C(\mathcal{X}) \) and \( F \in \mathcal{F}_\sigma^* \). If there exists \( \tau > 0 \) such that for any \( z \in \mathcal{X} \) with \( d(z, Tz) > 0 \), there exists \( y \in F_{\sigma}^z \) satisfying

\[
\tau + F(D(y, Tz)) \leq F(d(z, y)),
\]

then \( T \) has a fixed point in \( \mathcal{X} \) provided \( \sigma < \tau \) and \( z \to d(z, Tz) \) is lower semi-continuous.

Minak et al. [23] also showed that \( F_{\sigma}^* \neq \emptyset \) in both cases when \( F \in \mathcal{F} \) and \( F \in \mathcal{F}_\sigma^* \). The aim of the present paper is to introduce the concept of \( \alpha-\eta \)-semicontinuous multivalued mappings and to prove fixed point theorem for multivalued nonlinear \( F \)-contractions that generalize the results of Altun et al. [6], Minak et al. [23], Olgun et al. [26] and Hussain et al. [18]. The following lemmas will be used in the sequel.

**Lemma 1.8** ([3]). Let \( T : \mathcal{X} \to \mathcal{Y} \) be a multivalued function, then the following statements are equivalent.

1. \( T \) is lower semi-continuous.
2. \( V \subset \mathcal{Y} \Rightarrow T^{-1}[\text{int}(V)] \) is open in \( \mathcal{X} \),

where \( \text{int}(V) \) denotes the interior of \( V \).

**Lemma 1.9** ([3]). Let \( T : \mathcal{X} \to \mathcal{Y} \) be a multivalued function, then the following statements are equivalent.

1. \( T \) is upper semi-continuous.
2. \( V \subset \mathcal{Y} \Rightarrow T^{-1}[\overline{V}] \) is closed in \( \mathcal{X} \),

where \( \overline{V} \) denotes the closure of \( V \).

2. Fixed point results for modified \( \alpha-\eta-GF \)-contraction

We begin this section with the following definitions.

**Definition 2.1.** Let \( T : \mathcal{X} \to 2^\mathcal{X} \) be a multivalued map on a metric space \((\mathcal{X}, d)\), \( \alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+ \) be
two functions. We say that $\mathcal{T}$ is $\alpha$-$\eta$ lower semi-continuous multivalued mapping on $\mathcal{X}$, if for given $z \in \mathcal{X}$ and sequence $\{z_n\}$ with
\[
\lim_{n \to \infty} d(z_n, z) = 0, \quad \alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1}), \quad \text{for all } n \in \mathbb{N},
\]
implies
\[
\lim_{n \to \infty} \inf D(z_n, \mathcal{T}z_n) \geq D(z, \mathcal{T}z).
\]

**Definition 2.2.** Let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a multivalued map on a metric space $(\mathcal{X}, d)$, $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. We say that $\mathcal{T}$ is $\alpha$-$\eta$ upper semi-continuous multivalued mapping on $\mathcal{X}$, if for given $z \in \mathcal{X}$ and sequence $\{z_n\}$ with
\[
\lim_{n \to \infty} d(z_n, z) = 0, \quad \alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1}), \quad \text{for all } n \in \mathbb{N},
\]
implies
\[
\lim_{n \to \infty} \sup D(z_n, \mathcal{T}z_n) \leq D(z, \mathcal{T}z).
\]

**Lemma 2.3.** Let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a multivalued map on a metric space $(\mathcal{X}, d)$, $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Then $\mathcal{T}$ is $\alpha$-$\eta$ continuous, if and only if it is $\alpha$-$\eta$ upper semi-continuous and $\alpha$-$\eta$ lower semi-continuous.

**Proof.** Suppose that $\mathcal{T}$ is $\alpha$-$\eta$ upper semi-continuous and $\alpha$-$\eta$ lower semi-continuous. Then there exists a sequence $\{z_n\}$ in $\mathcal{X}$ and $z \in \mathcal{X}$ with
\[
\lim_{n \to \infty} d(z_n, z) = 0, \quad \alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1}), \quad \text{for all } n \in \mathbb{N},
\]
implies
\[
\lim_{n \to \infty} \inf D(z_n, \mathcal{T}z_n) \geq D(z, \mathcal{T}z), \quad (2.1)
\]
and
\[
\lim_{n \to \infty} \sup D(z_n, \mathcal{T}z_n) \leq D(z, \mathcal{T}z). \quad (2.2)
\]

From (2.1) and (2.2), we get that $D(z_n, \mathcal{T}z_n) \to D(z, \mathcal{T}z)$ as $n \to \infty$. This is possible only when $\mathcal{T}z_n \to \mathcal{T}z$. Consequently, $\mathcal{T}$ is $\alpha$-$\eta$ continuous.

Conversely, suppose that $\mathcal{T}$ is $\alpha$-$\eta$ continuous. Then there exists a sequence $\{z_n\}$ in $\mathcal{X}$ and $z \in \mathcal{X}$ with $z_n \to z$ as $n \to \infty$ and $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$ for all $n \in \mathbb{N}$ implies $\mathcal{T}z_n \to \mathcal{T}z$ as $n \to \infty$. This implies that $D(z_n, \mathcal{T}z_n) \to D(z, \mathcal{T}z)$ as $n \to \infty$ or $\lim_{n \to \infty} D(z_n, \mathcal{T}z_n) = D(z, \mathcal{T}z)$. From here it follows that $\lim_{n \to \infty} \inf D(z_n, \mathcal{T}z_n) \geq D(z, \mathcal{T}z)$ and $\lim_{n \to \infty} \sup D(z_n, \mathcal{T}z_n) \leq D(z, \mathcal{T}z)$. Hence $\mathcal{T}$ is $\alpha$-$\eta$ upper semi-continuous and $\alpha$-$\eta$ lower semi-continuous.

**Remark 2.4.** As semi-continuity is a weaker property than continuity, an $\alpha$-$\eta$ upper semi-continuous and $\alpha$-$\eta$ lower semi-continuous mapping need not to be $\alpha$-$\eta$ continuous mapping, as shown in the examples below.

**Example 2.5.** Let $\mathcal{X} = \mathbb{R}$ with usual metric $d$. Then $(\mathcal{X}, d)$ is a metric space. Define $\mathcal{T}_1 : \mathcal{X} \to 2^{\mathcal{X}}$, $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ by
\[
\mathcal{T}_1 z = \begin{cases} 
\{0\} & \text{if } z \neq 0, \\
[-1, 1] & \text{if } z = 0,
\end{cases}
\]
\[
\alpha(y, z) = \begin{cases} 
1 & \text{if } z, y \neq 0, \\
0 & \text{if } z = y = 0,
\end{cases}
\]
and $\eta(z, y) = \frac{1}{2}$, for all $z, y \in \mathcal{X}$.

Firstly, we show that $\mathcal{T}_1$ is not lower semi-continuous multivalued map. For this, let $V = [-1, 1] \subset 2^{\mathcal{X}}$,
then $T_1^{-1}(\text{int}(V)) = T_1^{-1}((-1,1)) = \emptyset$ which is not open in $\mathbb{R}$, so by Lemma 1.8 $T_1$ is not lower semi-continuous. But $T_1$ is $\alpha\cdot \eta$ lower semi-continuous multivalued map. Indeed, $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$ for sequence $z_n$ of non-zero real numbers. Here arises two cases:

Case I. $z_n \to z = 0$.
If $z_n \to 0$, then $T_1 z_n = \{0\}$ and $T_1 z = [-1,1]$ such that $D(z_n, T_1 z_n) = D(0, \{0\}) = z_n$ and $D(z, T_1 z) = D(0, [-1,1]) = 0$. This implies that

$$\lim_{n \to \infty} \inf D(z_n, T z_n) = \lim_{n \to \infty} \inf z_n = z = 0 = D(z, T z).$$

Case II. $z_n \to z \neq 0$.
If $z_n \to z$, then $T_1 z_n = \{0\}$ and $T_1 z = \{0\}$ such that $D(z_n, T_1 z_n) = D(0, \{0\}) = z_n$ and $D(z, T_1 z) = z$. This implies that

$$\lim_{n \to \infty} \inf D(z_n, T_1 z_n) = \lim_{n \to \infty} \inf z_n = z = D(z, T_1 z).$$

On the other hand, in Case I we have

$$\lim_{n \to \infty} H(T_1 z_n, T_1 z) = 1.$$

Hence $T_1$ is not $\alpha\cdot \eta$-continuous multivalued map.

**Example 2.6.** Consider $\mathcal{X}$ the same as in Example 2.5. Define $T_2 : \mathcal{X} \to 2^\mathcal{X}$, $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ by

$$T_2 z = \begin{cases} \{-1,1\} & \text{if } z \neq 0, \\ \{0\} & \text{if } z = 0, \end{cases}$$

$$\alpha(z, y) = \begin{cases} 0 & \text{if } z, y \neq 0, \\ 2 & \text{if } z = y = 0, \end{cases}$$

and $\eta(z, y) = \frac{1}{4}$, for all $z, y \in \mathcal{X}$.

Firstly, we show that $T_2$ is not upper semi-continuous multivalued map. For this, let $V = [-1,1] \subset 2^\mathcal{X}$, then $T_2^{-1}(V) = T_2^{-1}([-1,1]) = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$, which is not closed in $\mathbb{R}$, so by Lemma 1.9 $T_2$ is not upper semi-continuous. But $T_2$ is $\alpha\cdot \eta$ upper semi-continuous multivalued map. Indeed, $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$ for sequence $z_n = 0$ for all $n \in \mathbb{N}$. Then $z_n$ approaches to $z = 0$ only. Therefore, If $z_n \to 0$, then $T_2 z_n = \{0\}$ and $T_2 z = \{0\}$. This implies that

$$\lim_{n \to \infty} \sup D(z_n, T_2 z_n) = 0 = D(z, T_2 z).$$

On the other hand,

$$\lim_{n \to \infty} H(T_2 z_n, T_2 z) = 1.$$

Hence $T_2$ is not $\alpha\cdot \eta$-continuous multivalued map.

**Remark 2.7.** Let $T : \mathcal{X} \to 2^\mathcal{X}$ be a multivalued map on a metric space $(\mathcal{X}, d)$. Let $f : \mathcal{X} \to \mathbb{R}$, defined by $f(z) = D(z, T z)$, for all $z \in \mathcal{X}$, be a lower semi-continuous mapping. Take $\alpha(z, y) = \eta(z, y)$, for all $z, y \in \mathcal{X}$, then for $z \in \mathcal{X}$ and a sequence $\{z_n\}$ with

$$\lim_{n \to \infty} d(z_n, z) = 0, \quad \alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1}) \quad \text{for all } n \in \mathbb{N},$$

we have

$$\lim_{n \to \infty} \inf f(z_n) \geq f(z),$$

and so

$$\lim_{n \to \infty} \inf D(z_n, T z_n) \geq D(z, T z).$$

This shows that $T$ is $\alpha\cdot \eta$ lower semi-continuous mapping. But if $T$ is $\alpha\cdot \eta$ lower semi-continuous mapping, then $f$ needs to be lower semi-continuous as shown in Example 2.12. Similarly, if $f : \mathcal{X} \to \mathbb{R}$ is upper semi-continuous mapping then, $T$ is $\alpha\cdot \eta$ upper semi-continuous mapping but not conversely.
Theorem 2.8. Let $(X,d)$ be a complete metric space and $\alpha, \eta : X \times X \rightarrow \mathbb{R}_+$ be two functions. Let $T : X \rightarrow K(X)$, $F \in \mathcal{F}$ and $G \in \mathcal{G}$ fulfilling the following assertions:

1. if for any $z \in X$ with $D(z, Tz) > 0$, there exists $y \in F^\sigma_z$ with $\alpha(z, y) \geq \eta(z, y)$ satisfying

$$G(D(z, Tz), D(y, Ty), D(z, Ty), D(y, Tz)) + F(D(y, Ty)) \leq F(d(z, y));$$

2. $T$ is generalized $\alpha_\ast$-admissible mapping with respect to $\eta$;
3. $T$ is $\alpha$-$\eta$ lower semi-continuous mapping;
4. there exists $z_0 \in X$ and $y_0 \in Tz_0$ such that $\alpha(z_0, y_0) \geq \eta(z_0, y_0)$.

Then $T$ has a fixed point in $X$ provided $\sigma < \tau$.

Proof. Let $z_0 \in X$, since $Tz \in K(X)$ for every $z \in X$, the set $F^\sigma_z$ is non-empty for any $\sigma > 0$, then there exists $z_1 \in F^\sigma_{z_0}$ and by hypothesis $\alpha(z_0, z_1) \geq \eta(z_0, z_1)$. Assume that $z_1 \notin Tz_1$, otherwise $z_1$ is the fixed point of $T$. Then, since $Tz_1$ is closed, $D(z_1, Tz_1) > 0$, so from condition (1), we have

$$G(D(z_0, Tz_0), D(z_1, Tz_1), D(z_0, 0), D(z_1, 0)) + F(D(z_1, Tz_1)) \leq F(d(z_0, z_1)). \quad (2.3)$$

Now for $z_1 \in X$ there exists $z_2 \in F^\sigma_{z_1}$ with $z_2 \notin Tz_2$, otherwise $z_2$ is the fixed point of $T$, since $Tz_2$ is closed, so $D(z_2, Tz_2) > 0$. Since $T$ is generalized $\alpha_\ast$-admissible mapping with respect to $\eta$, then $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$. Again by using condition (1), we get

$$G(D(z_1, Tz_1), D(z_2, Tz_2), D(z_1, Tz_2), D(z_2, Tz_1)) + F(D(z_2, Tz_2)) \leq F(d(z_1, z_2)).$$

On continuing recursively, we get a sequence $\{z_n\}_{n \in \mathbb{N}}$ in $X$ such that $z_{n+1} \in F^\sigma_{z_n}$, $z_{n+1} \notin Tz_{n+1}$, $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$ and

$$G(D(z_n, Tz_n), D(z_{n+1}, Tz_{n+1}), D(z_n, Tz_{n+1}), D(z_{n+1}, Tz_n)) + F(D(z_{n+1}, Tz_{n+1})) \leq F(d(z_n, z_{n+1})). \quad (2.4)$$

As $z_{n+1} \in Tz_n$, this implies that

$$G(D(z_n, Tz_n), D(z_{n+1}, Tz_{n+1}), D(z_n, Tz_{n+1}), 0) + F(D(z_{n+1}, Tz_{n+1})) \leq F(d(z_n, z_{n+1})).$$

From (2.4), we get that

$$G(D(z_n, Tz_n), D(z_{n+1}, Tz_{n+1}), D(z_n, Tz_{n+1}), 0) = \tau. \quad (2.4)$$

From equation (2.4), we get that

$$F(D(z_{n+1}, Tz_{n+1})) \leq F(d(z_n, z_{n+1})) - \tau. \quad (2.5)$$

Since $z_{n+1} \in F^\sigma_{z_n}$, we have

$$F(d(z_n, z_{n+1})) \leq F(D(z_n, Tz_n)) + \sigma. \quad (2.6)$$

Combining equations (2.5) and (2.6) gives

$$F(D(z_{n+1}, Tz_{n+1})) \leq F(D(z_n, Tz_n)) + \sigma - \tau. \quad (2.7)$$

Since $Tz_n$ and $Tz_{n+1}$ is compact, there exists $z_{n+1} \in Tz_n$ and $z_{n+2} \in Tz_{n+1}$ such that $d(z_n, z_{n+1}) = D(z_n, Tz_n)$ and $d(z_{n+1}, z_{n+2}) = D(z_{n+1}, Tz_{n+1})$, so equation (2.7) implies

$$F(d(z_{n+1}, z_{n+2})) \leq F(d(z_n, z_{n+1})) + \sigma - \tau. \quad (2.8)$$
By using equation (2.8), we get
\[ F(d(z_{n+1}, z_{n+2})) \leq F(d(z_n, z_{n+1})) + \sigma - \tau \]
\[ \leq F(d(z_{n-1}, z_n)) + 2\sigma - 2\tau \]
\[ \vdots \]
\[ \leq F(d(z_0, z_1)) + n\sigma - n\tau \]
\[ = F(d(z_0, z_1)) - n(\tau - \sigma). \]
\[ (2.9) \]

By letting limit as \( n \to \infty \) in equation (2.9), we get
\[ \lim_{n \to \infty} F(d(z_{n+1}, z_{n+2})) = -\infty, \] so by (F2), we obtain
\[ \lim_{n \to \infty} d(z_{n+1}, z_{n+2}) = 0. \]
\[ (2.10) \]

Now from (F3), there exists \( 0 < k < 1 \) such that
\[ \lim_{n \to \infty} \left[ d(z_{n+1}, z_{n+2}) \right]^k F(d(z_{n+1}, z_{n+2})) = 0. \]
\[ (2.11) \]

By equation (2.9), we get
\[ \lim_{n \to \infty} \left[ d(z_{n+1}, z_{n+2}) \right]^k [F(d(z_{n+1}, z_{n+2})) - d(z_0, z_1)] \leq -n(\tau - \sigma)d(z_{n+1}, z_{n+2})^k \leq 0. \]
\[ (2.12) \]

By taking limit as \( n \to \infty \) in equation (2.12) and applying equations (2.10) and (2.11), we have
\[ \lim_{n \to \infty} n[d(z_{n+1}, z_{n+2})]^k = 0. \]

This implies that there exists \( n_1 \in \mathbb{N} \) such that \( n[d(z_{n+1}, z_{n+2})]^k \leq 1 \), or \( d(z_{n+1}, z_{n+2}) \leq \frac{1}{n^{1/k}}, \) for all \( n > n_1 \). Next, for \( m > n > n_1 \) we have
\[ d(z_n, z_m) \leq \sum_{i=n}^{m-1} d(z_i, z_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}}, \]

since \( 0 < k < 1 \), \( \sum_{i=n}^{m-1} \frac{1}{i^{1/k}} \) converges. Therefore, \( d(z_n, z_m) \to 0 \) as \( m, n \to \infty \). Thus, \( \{z_n\} \) is a Cauchy sequence. Since \( X \) is complete, there exists \( z^* \in X \) such that \( z_n \to z^* \) as \( n \to \infty \). From equations (2.17) and (2.10), we have
\[ \lim_{n \to \infty} D(z_n, Tz_n) = 0. \]

Since \( T \) is \( \alpha-\eta \) lower semi-continuous mapping, then
\[ 0 \leq D(z, Tz) \leq \lim_{n \to \infty} \inf D(z_n, Tz_n) = 0. \]

Thus, \( T \) has a fixed point. \( \Box \)

**Theorem 2.9.** Let \( (X, d) \) be a complete metric space and \( \alpha, \eta : X \times X \to \mathbb{R}_+ \) be two functions. Let \( T : X \to C(X), F \in \mathcal{F}_\alpha \) and \( G \in \mathcal{G} \) satisfy all assertions of Theorem 2.8. Then \( T \) has a fixed point in \( X \).

**Proof.** Let \( z_0 \in X \), since \( Tz_0 \in C(X) \) for every \( z \in X \) and \( F \in \mathcal{F}_\alpha \), the set \( F_{\alpha} \) is non-empty for any \( \sigma > 0 \), then there exists \( z_1 \in F_{\alpha}^{z_0} \) and by hypothesis \( \alpha(z_0, z_1) \geq \eta(z_0, z_1) \). Assume that \( z_1 \notin Tz_1 \), otherwise \( z_1 \) is the fixed point of \( T \). Then, since \( Tz_1 \) is closed, \( D(z_1, Tz_1) > 0 \), so from condition (1) of Theorem 2.8 we have
\[ G(D(z_0, Tz_0), D(z_1, Tz_1), D(z_0, Tz_0)) + F(D(z_1, Tz_1)) \leq F(d(z_0, z_1)). \]

Now for \( z_1 \in X \) there exists \( z_2 \in F_{\alpha}^{z_1} \) with \( z_2 \notin Tz_2 \), otherwise \( z_2 \) is the fixed point of \( T \), since
$Tz_2$ is closed, so $D(z_2, Tz_2) > 0$. Since $T$ is generalized $\alpha$-admissible mapping with respect to $\eta$, then $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$. Again by using condition (1) of Theorem 2.8, we get
\[G(D(z_1, Tz_1), D(z_2, Tz_2), D(z_1, Tz_2), D(z_2, Tz_1)) + F(D(z_2, Tz_2)) \leq F(d(z_1, z_2)).\]

On continuing recursively, we get a sequence $\{z_n\}_{n \in \mathbb{N}}$ in $\mathcal{X}$ such that $z_{n+1} \in F^*_\alpha$, $z_{n+1} \notin Tz_{n+1}$, and $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$ and
\[G(D(z_n, Tz_n), D(z_{n+1}, Tz_{n+1}), D(z_n, Tz_{n+1}), D(z_{n+1}, Tz_n)) + F(D(z_{n+1}, Tz_{n+1})) \leq F(d(z_n, z_{n+1})).\]

The rest of the proof can be completed as the proof of Theorem 2.8. \hfill \square

**Corollary 2.10.** Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$ be two functions. Let $T : \mathcal{X} \rightarrow K(\mathcal{X})$ and $F \in \mathfrak{F}$ fulfill the conditions (2)-(4) of Theorem 2.8 and if for any $z \in \mathcal{X}$ with $D(z, TZ) > 0$, there exists $y \in F^*_\alpha$ with $\alpha(z, y) \geq \eta(z, y)$ satisfying
\[\tau + F(D(y, Ty)) \leq F(d(z, y)),\]
then $T$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$.

**Proof.** Define $G_L : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $G_L(t_1, t_2, t_3, t_4) = L \min\{t_1, t_2, t_3, t_4\} + \tau$, where $L \in \mathbb{R}^+$ and $\tau > 0$. Then $G_L \in \mathfrak{G}$ (see Example 2.1 of [13]). Therefore, the result follows by taking $G = G_L$ in Theorem 2.8. \hfill \square

**Corollary 2.11.** Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$ be two functions. Let $T : \mathcal{X} \rightarrow C(\mathcal{X})$ and $F \in \mathfrak{F}^*$ satisfy all conditions of Corollary 2.10. Then $T$ has a fixed point in $\mathcal{X}$.

**Proof.** By defining same $G_L$ as in Corollary 2.10 and using Theorem 2.9, we get the required result. \hfill \square

**Example 2.12.** Let $\mathcal{X} = \{\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{0\}$ with usual metric $d$. Then $(\mathcal{X}, d)$ is a metric space. Define $T : \mathcal{X} \rightarrow K(\mathcal{X})$, $\alpha, \eta : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$, $G : \mathbb{R}^4 \rightarrow \mathbb{R}^+$ and $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ by
\[
Tz = \begin{cases}
\{\frac{1}{2^n}\} & \text{if } z = \frac{1}{2^n}, \\
\{0\} & \text{if } z = 0,
\end{cases}
\]
\[
\alpha(z, y) = \begin{cases}
2 & \text{if } z = \frac{1}{2^n}, \\
\frac{1}{2} & \text{if } z = 0,
\end{cases}
\]
\[
\eta(z, y) = 1, \text{ for all } z, y \in \mathcal{X}, G(t_1, t_2, t_3, t_4) = \tau, \text{ where } \tau > 0 \text{ and } F(r) = \ln(r). \text{ Then}
\]
\[
D(z, TZ) = \begin{cases}
\frac{1}{2^n} & \text{if } z = \frac{1}{2^n}, \\
0 & \text{if } z = 0.
\end{cases}
\]

Let $D(z, TZ) > 0$, then $z = \frac{1}{2^n}$, so $Tz = \{\frac{1}{2^n}\}$. Thus for $y = \frac{1}{2^n} \in Tz$, we have
\[F(d(z, y)) - F(D(z, TZ)) = F\left(\frac{1}{2^n}\right) - F\left(\frac{1}{2^n}\right) = 0.
\]

Therefore, $y \in F^*_\sigma$ for $\sigma > 0$ with $\alpha(z, y) \geq \eta(z, y)$ and
\[
F(D(y, Ty)) - F(d(z, y)) = F\left(\frac{1}{2^n+1}\right) - F\left(\frac{1}{2^n}\right)
= \ln\left(\frac{1}{2^n+1}\right) - \ln\left(\frac{1}{2^n}\right)
= \ln\left(\frac{2^n}{2^n+1}\right) = \ln\left(\frac{1}{2}\right)
= - \ln 2.
\]

Hence $\tau + F(D(y, Ty)) \leq F(d(z, y))$ is satisfied for $0 < \sigma < \tau \leq \ln 2$. 

Since $\alpha(z, y) \geq \eta(z, y)$ when $z, y \in \left\{ \frac{1}{2^{n-r}} : n \in \mathbb{N} \right\}$, this implies that $\alpha(u, v) = 2 > 1 = \eta(u, v)$ for all $u \in T z$ and $v \in T y$. Hence $T$ is generalized $\alpha_\eta$-admissible mapping with respect to $\eta$.

Next, let $\lim_{n \to \infty} d(z_n, z) = 0$ and $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$, for all $n \in \mathbb{N}$, then $z_n \in \left\{ \frac{1}{2^{n-r}} : n \in \mathbb{N} \right\}$. This implies that $T z_n = \left\{ \frac{1}{2^n} \right\}$ and $D(z_n, T z_n) = \frac{1}{2^n}$, for all $n \in \mathbb{N}$. Here arises two cases:

Case I. $z_n \to z = 0$.

Then $T z = \{0\}$ and $D(z, T z) = 0$. Thus

$$\lim_{n \to \infty} \inf_{n \to \infty} D(z_n, T z_n) = \lim_{n \to \infty} \inf_{n \to \infty} \left( \frac{1}{2^n} \right) \geq 0 = D(z, T z).$$

Case II. $z_n \to z = \frac{1}{2^n}$.

Then $T z = \left\{ \frac{1}{2^n} \right\}$ and $D(z, T z) = \frac{1}{2^n}$. Thus

$$\lim_{n \to \infty} \inf_{n \to \infty} D(z_n, T z_n) = \lim_{n \to \infty} \inf_{n \to \infty} \left( \frac{1}{2^n} \right) = \frac{1}{2^n} = D(z, T z).$$

Hence $T$ is $\alpha_\eta$ lower semi-continuous mapping. Thus, all conditions of Corollary 2.10 (and Theorem 2.8) hold and 0 is a fixed point of $T$.

On the other hand, define $f : \mathcal{X} \to \mathbb{R}$, by $f(z) = D(z, T z)$, for all $z \in \mathcal{X}$. Then

$$\lim_{z \to 1} \inf_{z \to 1} f(z) = 0 \neq \frac{1}{2} = f(1).$$

Hence $f$ is not lower semi-continuous mapping at $z = 1$. That is, Theorems 1.6 and 1.7 can not be applied for this example.

Example 2.13. Consider the sequence $\{S_n\}_{n \in \mathbb{N}}$ as follows:

$$S_1 = 1,$$

$$S_2 = 1 + 2,$$

$$: :$$

$$S_n = 1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2},$$

$$: :$$

Let $\mathcal{X} = \{S_n : n \in \mathbb{N}\}$ with usual metric $d$. Then $(\mathcal{X}, d)$ is a metric space. Define $T : \mathcal{X} \to K(\mathcal{X})$, $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$, $\mathcal{G} : \mathbb{R}^4 \to \mathbb{R}^+$ and $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ by

$$T z = \begin{cases} \{S_{n-1}, S_{n+1}\} & \text{if } z = S_n, \ n > 2, \\
\{z\} & \text{otherwise,} \end{cases}$$

$$\alpha(z, y) = \begin{cases} 3 & \text{if } z \in \{S_n : n \geq 2\}, \\
1 & \text{otherwises,} \end{cases}$$

$$\eta(z, y) = 2, \text{ for all } z, y \in \mathcal{X}, \ \mathcal{G}(t_1, t_2, t_3, t_4) = L \min\{t_1, t_2, t_3, t_4\} + \tau, \text{ where } \tau = \frac{1}{e^\tau}, \ n \in \mathbb{N}, \ L \in \mathbb{R}^+ \text{ and } \mathcal{F}(r) = \ln(r).$$

Then

$$D(z, T z) = \begin{cases} |n| & \text{if } z = S_n, \ n > 2, \\
0 & \text{otherwise.} \end{cases}$$
Let $D(z, Tz) > 0$, then $z = S_n, n > 2$, so, $Tz = \{S_{n-1}, S_{n+1}\}$. Thus for $y = S_{n-1} \in Tz$, we have

$$F(d(z, y)) - F(D(z, Tz)) = F(|n|) - F(|n|) = 0.$$ 

Therefore, $y \in F^*_\sigma$ for $\sigma = \frac{1}{e^{n+1}}, n \in \mathbb{N}$ with $\alpha(z, y) \geq \eta(z, y)$ and

$$F(D(y, Ty)) - F(D(z, y)) = F(|n - 1| - F(|n|)
\leq \ln \left(\frac{|n - 1|}{|n|}\right)
< -\frac{1}{e^n}.$$ 

This implies that $\tau + F(D(y, Ty)) \leq F(d(z, y))$. Since $D(z, Ty) = 0$, we have,

$$G(D(z, Tz), D(y, Ty), D(z, Ty), D(y, Tz)) + F(D(y, Ty)) = \tau + F(D(y, Ty)) \leq F(d(z, y)).$$

Hence condition (1) of Theorem 2.8 is satisfied for $0 < \sigma = \frac{1}{e^{n+1}} < \tau = \frac{1}{e^n}$.

Since $\alpha(z, y) \geq \eta(z, y)$ when $z, y \in \{S_n : n \geq 2\}$, this implies that $\alpha(u, v) = 3 > 2 = \eta(u, v)$ for all $u \in Tz$ and $v \in Ty$. Hence $T$ is a generalized $\alpha, \eta$-admissible mapping with respect to $\eta$.

Next, let $\lim_{n \to \infty} d(z_n, z) = 0$ and $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$, for all $n \in \mathbb{N}$, then $z_n \in \{S_n : n \in \mathbb{N}, n \geq 2\}$. Here arises two cases:

Case I. $z_n \in \{S_n : n > 2\}$.
Then $Tz_n = \{S_{n-1}, S_{n+1}\}$ and $D(z_n, Tz_n) = |n|$, for all $n \in \mathbb{N}$.

Subcase I. $z_n \to z = S_n, n > 2$.
Then $Tz = \{S_{n-1}, S_{n+1}\}$ and $D(z, Tz) = |n|$. Thus

$$\lim_{n \to \infty} \inf_{n \to \infty} D(z_n, Tz_n) = \lim_{n \to \infty} \inf_{n \to \infty} (|n|) = |n| = D(z, Tz).$$

Subcase II. $z_n \to z = S_1$.
Then $Tz = \{S_1\}$ and $D(z, Tz) = 0$. Thus

$$\lim_{n \to \infty} \inf_{n \to \infty} D(z_n, Tz_n) = \lim_{n \to \infty} \inf_{n \to \infty} (|n|) \geq 0 = D(z, Tz).$$

Subcase III. $z_n \to z = S_2$.
Then $Tz = \{S_2\}$ and $D(z, Tz) = 0$. Thus

$$\lim_{n \to \infty} \inf_{n \to \infty} D(z_n, Tz_n) = \lim_{n \to \infty} \inf_{n \to \infty} (|n|) \geq 0 = D(z, Tz).$$

Case II. $z_n \in \{S_2\}$.
Then $z_n$ approaches to $S_2$ only. Therefore, $Tz_n = \{z_n\}$ and $Tz = \{z\}$. This implies that

$$\lim_{n \to \infty} \inf_{n \to \infty} D(z_n, Tz_n) = 0 = D(z, Tz).$$

Hence $T$ is $\alpha, \eta$ lower semi-continuous mapping. Thus, all the conditions of Theorem 2.8 hold and $\{S_1, S_2\}$ is set of fixed points of $T$. 

As an application of Theorems 2.8 and 2.9, we get the following results.

**Theorem 2.14.** Let \((X, d)\) be a complete metric space and \(\alpha, \eta : X \times X \to \mathbb{R}_+\) be two functions. Let \(T : X \to K(X), F \in \mathfrak{F}_*\) and \(G \in \mathfrak{G}\) fulfill the conditions (2) and (4) of Theorem 2.8. If for any \(y, z \in X\) with \(\alpha(z, y) \geq \eta(z, y)\) and \(H(Tz, Ty) > 0\) we have
\[
G(D(z, Tz), D(y, Ty), D(z, Ty), D(y, Tz)) + F(H(Tz, Ty)) \leq F(d(z, y)),
\]
then \(T\) has a fixed point in \(X\) provided \(T\) is \(\alpha\)-\(\eta\) continuous mapping.

**Proof.** By Lemma 2.3 we have \(T\) is \(\alpha\)-\(\eta\)-lower semi-continuous mapping. Also, for \(z \in X\) and \(y \in F_\sigma^\ast\) with \(D(z, Tz) > 0\) we have
\[
G(D(z, Tz), D(y, Ty), D(z, Ty), D(y, Tz)) + F(D(y, Ty)) \leq \frac{F(d(z, y))}{2},
\]
\[
G(D(z, Tz), D(y, Ty), D(z, Ty), D(y, Tz)) + F(H(Tz, Ty)) \leq F(d(z, y)).
\]
Thus, all the conditions of Theorem 2.8 are satisfied, so, \(T\) has a fixed point. \(\square\)

By similar arguments of Theorem 2.14, we state the following and omit its proof.

**Theorem 2.15.** Let \((X, d)\) be a complete metric space and \(\alpha, \eta : X \times X \to \mathbb{R}_+\) be two functions. Let \(T : X \to C(X), F \in \mathfrak{F}_*\) and \(G \in \mathfrak{G}\) satisfy all assertions of Theorem 2.14. Then \(T\) has a fixed point in \(X\).

On considering \(G = G_L\), as in Corollary 2.10, Theorems 2.14 and 2.15 reduce to the following corollaries.

**Corollary 2.16.** Let \((X, d)\) be a complete metric space and \(\alpha, \eta : X \times X \to \mathbb{R}_+\) be two functions. Let \(T : X \to K(X), F \in \mathfrak{F}\) fulfill the conditions (2) and (4) of Theorem 2.8. If for any \(y, z \in X\) with \(\alpha(z, y) \geq \eta(z, y)\) and \(H(Tz, Ty) > 0\) we have
\[
\tau + F(H(Tz, Ty)) \leq F(d(z, y)),
\]
then \(T\) has a fixed point in \(X\) provided \(T\) is \(\alpha\)-\(\eta\) continuous mapping.

**Corollary 2.17.** Let \((X, d)\) be a complete metric space and \(\alpha, \eta : X \times X \to \mathbb{R}_+\) be two functions. Let \(T : X \to C(X), F \in \mathfrak{F}\) satisfy all assertions of Corollary 2.16. Then \(T\) has a fixed point in \(X\).

**Theorem 2.18.** Let \((X, d)\) be a complete metric space, \(T : X \to K(X), F \in \mathfrak{F}\) and \(G \in \mathfrak{G}\). If for \(z \in X\) with \(D(z, Tz) > 0\), there exists \(y \in F_\sigma^\ast\) satisfying
\[
G(D(z, Tz), D(y, Ty), D(z, Ty), D(y, Tz)) + F(D(y, Ty)) \leq F(d(z, y)),
\]
then \(T\) has a fixed point in \(X\) provided \(\alpha < \tau\) and \(z \to D(z, Tz)\) is lower semi-continuous.

**Proof.** Define \(\alpha(z, y) = d(z, y) = \eta(z, y)\) for all \(z, y \in X\). Then \(\alpha(u, v) = d(z, y) = \eta(u, v)\), for all \(u \in Tz\) and \(v \in Ty\), that is, \(T\) is generalized \(\alpha\)-admissible mapping with respect to \(\eta\). Since \(z \to D(z, Tz)\) is lower semi-continuous, therefore by Remark 2.7 \(T\) is \(\alpha\)-\(\eta\)-lower semi-continuous. Thus, all the conditions of Theorem 2.8 holds. Hence \(T\) has a fixed point in \(X\). \(\square\)

**Theorem 2.19.** Let \((X, d)\) be a complete metric space, \(T : X \to C(X), F \in \mathfrak{F}\) and \(G \in \mathfrak{G}\). If for \(z \in X\) with \(D(z, Tz) > 0\), there exists \(y \in F_\sigma^\ast\) satisfying
\[
G(D(z, Tz), D(y, Ty), D(z, Ty), D(y, Tz)) + F(D(y, Ty)) \leq F(d(z, y)),
\]
then \(T\) has a fixed point in \(X\) provided \(\alpha < \tau\) and \(z \to D(z, Tz)\) is lower semi-continuous.

**Proof.** By defining \(\alpha(z, y)\) and \(\eta(z, y)\) the same as in proof of Theorem 2.18 and by using Theorem 2.8 we get the required result. \(\square\)

**Remark 2.20.** By taking \(G = G_L\), as in Corollary 2.11 in Theorems 2.18 and 2.19 we get Theorems 1.6 and 1.7.
3. Fixed point results for $\alpha$-$\eta$-$F$-contraction of Hardy-Rogers type

In this section we establish certain fixed point results for $\alpha$-$\eta$-$F$-contraction of Hardy-Rogers type.

**Theorem 3.1.** Let $(X, d)$ be a complete metric space and $\alpha, \eta : X \times X \to \mathbb{R}_+$ be two functions. Let $T : X \to K(X)$ and $F \in \mathcal{F}$ fulfill the following assertions:

1. $T$ is generalized $\alpha_*$-admissible mapping with respect to $\eta$;
2. $T$ is $\alpha$-$\eta$ lower semi-continuous mapping;
3. there exist $z_0 \in X$ and $y_0 \in Tz_0$ such that $\alpha(z_0, y_0) \geq \eta(z_0, y_0)$;
4. there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that

$$\lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,$$

and for any $z \in X$ with $D(z, Tz) > 0$, there exists $y \in F^2_\sigma$ with $\alpha(z, y) \geq \eta(z, y)$ satisfying

$$\tau(d(z, y)) + F(D(y, Ty)) \leq F(a_1d(z, y) + a_2D(z, Tz) + a_3D(y, Ty)) + a_4D(z, Ty) + a_5D(y, Tz),$$

where $a_1, a_2, a_3, a_4, a_5 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_5 \neq 1$.

Then $T$ has a fixed point in $X$.

**Proof.** Let $z_0 \in X$, since $Tz_0 \in K(X)$ for every $z \in X$, the set $F^2_\sigma$ is non-empty for any $\sigma > 0$, then there exists $z_1 \in F^2_\sigma$ and by hypothesis $\alpha(z_0, z_1) \geq \eta(z_0, z_1)$. Assume that $z_1 \notin Tz_1$, otherwise $z_1$ is the fixed point of $T$. Then, since $Tz_1$ is closed, $D(z_1, Tz_1) > 0$, so, from (4), we have

$$\tau(d(z_0, z_1)) + F(D(z_1, Tz_1)) \leq F(a_1d(z_0, z_1) + a_2D(z_0, Tz_0) + a_3D(z_1, Tz_1) + a_4D(z_0, Tz_1) + a_5D(z_1, Tz_0)).$$

Now for $z_1 \in X$ there exists $z_2 \in F^2_{\sigma^2}$ with $z_2 \notin Tz_2$, otherwise $z_2$ is the fixed point of $T$, since $Tz_2$ is closed, so, $D(z_2, Tz_2) > 0$. Since $T$ is generalized $\alpha_*$-admissible mapping with respect to $\eta$, then $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$. Again by using (4), we get

$$\tau(d(z_1, z_2)) + F(D(z_2, Tz_2)) \leq F(a_1d(z_1, z_2) + a_2D(z_1, Tz_1) + a_3D(z_2, Tz_2) + a_4D(z_1, Tz_2) + a_5D(z_2, Tz_1)).$$

On continuing recursively, we get a sequence $\{z_n\}_{n \in \mathbb{N}}$ in $X$ such that $z_{n+1} \in F^2_{\sigma^n}$, $z_{n+1} \notin Tz_{n+1}$, $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$ and

$$\tau(d(z_n, z_{n+1})) + F(D(z_{n+1}, Tz_{n+1})) \leq F(a_1d(z_n, z_{n+1}) + a_2D(z_n, Tz_n) + a_3D(z_{n+1}, Tz_{n+1}) + a_4D(z_n, Tz_{n+1}) + a_5D(z_{n+1}, Tz_n)).$$

As $z_{n+1} \in Tz_n$, this implies that

$$\tau(d(z_n, z_{n+1})) + F(D(z_{n+1}, Tz_{n+1})) \leq F(a_1d(z_n, z_{n+1}) + a_2D(z_n, Tz_n) + a_3D(z_{n+1}, Tz_{n+1}) + a_4D(z_n, Tz_{n+1}) + a_5D(z_{n+1}, Tz_n)).$$

As $z_{n+1} \in F^2_{\sigma^n}$, we have

$$F(d(z_n, z_{n+1})) \leq F(D(z_n, Tz_n)) + \sigma.$$  \hspace{1cm} (3.2)

As $Tz_n$ and $Tz_{n+1}$ is compact, there exist $z_{n+1} \in Tz_n$ and $z_{n+2} \in Tz_{n+1}$ such that $d(z_n, z_{n+1}) = D(z_n, Tz_n)$ and $d(z_{n+1}, z_{n+2}) = D(z_{n+1}, Tz_{n+1})$, so equations (3.1) and (3.2) imply

$$\tau(d(z_n, z_{n+1})) + F(d(z_{n+1}, z_{n+2})) \leq F(a_1d(z_n, z_{n+1}) + a_2d(z_n, z_{n+1}) + a_3d(z_{n+1}, z_{n+2}) + a_4d(z_n, z_{n+2})).$$
and
\[ \mathcal{F}(d(z_n, z_{n+1})) \leq \mathcal{F}(d(z_n, z_{n+1})) + \sigma. \] \tag{3.3}

Let \( d_n = d(z_n, z_{n+1}) \), for \( n \in \mathbb{N} \), then
\[ \tau(d_n) + \mathcal{F}(d_{n+1}) \leq \mathcal{F}((a_1 + a_2)d_n + a_3d_{n+1} + a_4d(z_n, z_{n+2}) \leq \mathcal{F}((a_1 + a_2 + a_4)d_n + (a_3 + a_4)d_{n+1}). \] \tag{3.4}

Assume that there exists \( n \in \mathbb{N} \) such that \( d_{n+1} \geq d_n \), then from (3.4), we get
\[ \tau(d_n) + \mathcal{F}(d_{n+1}) \leq \mathcal{F}(d_{n+1}). \]

This is a contradiction to the fact that \( \tau(d_n) > 0 \). Hence \( d_{n+1} < d_n \) for all \( n \in \mathbb{N} \). This shows that sequence \( \{d_n\} \) is decreasing. Therefore, there exists \( \delta \geq 0 \) such that \( \lim_{n \to \infty} d_n = \delta \). Now let \( \delta > 0 \). From (3.4), we get
\[ \tau(d_n) + \mathcal{F}(d_{n+1}) \leq \mathcal{F}(d_n). \] \tag{3.5}

Combining (3.3) and (3.5) gives
\[ \mathcal{F}(d_{n+1}) \leq \mathcal{F}(d_n) + \sigma - \tau(d_n) \leq \mathcal{F}(d_{n-1}) + 2\sigma - \tau(d_{n-1}) - \tau(d_{n-1}) - \cdots - \tau(d_0). \] \tag{3.6}

Let \( \tau(d_{p_n}) = \min\{\tau(d_0), \tau(d_1), \ldots, \tau(d_n)\} \) for all \( n \in \mathbb{N} \). From (3.6), we get
\[ \mathcal{F}(d_{n+1}) \leq \mathcal{F}(d_0) + n(\sigma - \tau(d_{p_n})). \] \tag{3.7}

From (3.6), we also get
\[ \mathcal{F}(D(z_{n+1}, T z_{n+1})) \leq \mathcal{F}(D(z_0, T z_0)) + n(\sigma - \tau(d_{p_n})). \]

Now consider the sequence \( \{\tau(d_{p_n})\} \). We distinguish two cases.

Case 1. For each \( n \in \mathbb{N} \), there is \( m > n \) such that \( \tau(d_{p_m}) > \tau(d_{p_n}) \). Then we obtain a subsequence \( \{d_{p_{n_k}}\} \) of \( \{d_{p_n}\} \) with \( \tau(d_{p_{n_k}}) > \tau(d_{p_{n_k+1}}) \) for all \( k \). Since \( d_{p_{n_k}} \to \delta^+ \), we deduce that
\[ \lim_{k \to \infty} \inf \tau(d_{p_{n_k}}) > \sigma. \]

Hence \( \mathcal{F}(d_{n_k}) \leq \mathcal{F}(d_0) + n(\sigma - \tau(d_{p_{n_k}})) \) for all \( k \). Consequently, \( \lim_{k \to \infty} \mathcal{F}(d_{n_k}) = -\infty \) and by (F2), we obtain \( \lim_{k \to \infty} d_{p_{n_k}} = 0 \), which contradicts that \( \lim_{n \to \infty} d_n > 0 \).

Case 2. There is \( n_0 \in \mathbb{N} \) such that \( \tau(d_{p_{m_0}}) > \tau(d_{p_m}) \) for all \( m > n_0 \). Then \( \mathcal{F}(d_{m}) \leq \mathcal{F}(d_0) + m(\sigma - \tau(d_{p_{n_0}})) \) for all \( m > n_0 \). Hence \( \lim_{m \to \infty} \mathcal{F}(d_m) = -\infty \), so \( \lim_{m \to \infty} d_m = 0 \), which contradicts that \( \lim_{n \to \infty} d_m > 0 \).

Thus, \( \lim_{n \to \infty} d_n = 0 \). From (F3), there exists \( 0 < r < 1 \) such that
\[ \lim_{n \to \infty} (d_n)^r \mathcal{F}(d_n) = 0. \]

By (3.7), we get for all \( n \in \mathbb{N} \)
\[ (d_n)^r \mathcal{F}(d_n) - (d_n)^r \mathcal{F}(d_0) \leq (d_n)^r n(\sigma - \tau(d - p_n)) \leq 0. \] \tag{3.8}

By letting \( n \to \infty \) in (3.8), we obtain
\[ \lim_{n \to \infty} n(d_n)^r = 0 \]

This implies that there exists \( n_1 \in \mathbb{N} \) such that \( n(d_n)^r \leq 1 \), or, \( d_n \leq \frac{1}{n^{1/r}} \), for all \( n > n_1 \). Rest of the proof can be completed as in Theorem 2.8.
Following the arguments in the proof of Theorem 3.1 and taking $F \in \mathcal{F}_*$, we obtain the following result.

**Theorem 3.2.** Let $(X, d)$ be a complete metric space and $\alpha, \eta : X \times X \rightarrow \mathbb{R}_+$ be two functions. Let $T : X \rightarrow C(X)$ and $F \in \mathcal{F}_*$ satisfy all conditions of Theorem 3.1. Then $T$ has a fixed point in $X$.

By taking $a_1 = 1$ and $a_2 = a_3 = 0$ in Theorems 3.1 and 3.2 respectively, we get the following.

**Corollary 3.3.** Let $(X, d)$ be a complete metric space and $\alpha, \eta : X \times X \rightarrow \mathbb{R}_+$ be two functions. Let $T : X \rightarrow K(X)$ and $F \in \mathcal{F}_*$ fulfill the following assertions:

1. $T$ is generalized $\alpha$-admissible mapping with respect to $\eta$;
2. $T$ is $\alpha, \eta$ lower semi-continuous mapping;
3. there exist $z_0 \in X$ and $y_0 \in Tz_0$ such that $\alpha(z_0, y_0) \geq \eta(z_0, y_0)$;
4. there exist $\sigma > 0$ and a function $\tau : (0, \infty) \rightarrow (\sigma, \infty)$ such that

$$\lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,$$

and for any $z \in X$ with $D(z, Tz) > 0$, there exists $y \in F^*_\sigma$ with $\alpha(z, y) \geq \eta(z, y)$ satisfying

$$\tau(d(z, y)) + F(D(y, Ty)) \leq F(d(z, y)).$$

Then $T$ has a fixed point in $X$.

**Corollary 3.4.** Let $(X, d)$ be a complete metric space and $\alpha, \eta : X \times X \rightarrow \mathbb{R}_+$ be two functions. Let $T : X \rightarrow C(X)$ and $F \in \mathcal{F}_*$ satisfy all conditions of Corollary 3.3. Then $T$ has a fixed point in $X$.

By taking $a_1 = a_2 = a_3 = 0$ and $a_4 = a_5 = 1/2$ in Theorems 3.1 and 3.2 respectively, we get the following results for $F$-contraction of Chatterjea type.

**Corollary 3.5.** Let $(X, d)$ be a complete metric space and $\alpha, \eta : X \times X \rightarrow \mathbb{R}_+$ be two functions. Let $T : X \rightarrow K(X)$ and $F \in \mathcal{F}_*$ fulfill the following assertions:

1. $T$ is generalized $\alpha$-admissible mapping with respect to $\eta$;
2. $T$ is $\alpha, \eta$ lower semi-continuous mapping;
3. there exist $z_0 \in X$ and $y_0 \in Tz_0$ such that $\alpha(z_0, y_0) \geq \eta(z_0, y_0)$;
4. there exist $\sigma > 0$ and a function $\tau : (0, \infty) \rightarrow (\sigma, \infty)$ such that

$$\lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,$$

and for any $z \in X$ with $D(z, Tz) > 0$, there exists $y \in F^*_\sigma$ with $\alpha(z, y) \geq \eta(z, y)$ satisfying

$$\tau(d(z, y)) + F(D(y, Ty)) \leq F\left(\frac{D(z, Ty) + D(y, Tz)}{2}\right).$$

Then $T$ has a fixed point in $X$.

**Corollary 3.6.** Let $(X, d)$ be a complete metric space and $\alpha, \eta : X \times X \rightarrow \mathbb{R}_+$ be two functions. Let $T : X \rightarrow C(X)$ and $F \in \mathcal{F}_*$ satisfy all conditions of Corollary 3.5. Then $T$ has a fixed point in $X$.

If we choose $a_4 = a_5 = 0$ in Theorems 3.1 and 3.2 respectively, we obtain the following results for $F$-contraction of Reich-type.

**Corollary 3.7.** Let $(X, d)$ be a complete metric space and $\alpha, \eta : X \times X \rightarrow \mathbb{R}_+$ be two functions. Let $T : X \rightarrow K(X)$ and $F \in \mathcal{F}_*$ fulfill the following assertions:

1. $T$ is generalized $\alpha$-admissible mapping with respect to $\eta$;
2. $T$ is $\alpha$-$\eta$ lower semi-continuous mapping;
3. there exist $z_0 \in X$ and $y_0 \in Tz_0$ such that $\alpha(z_0, y_0) \geq \eta(z_0, y_0)$;
4. there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that
   \[
   \lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,
   \]
   and for any $z \in X$ with $D(z, Tz) > 0$, there exists $y \in F_\sigma$ with $\alpha(z, y) \geq \eta(z, y)$ satisfying
   \[
   \tau(d(z, y)) + F(D(y, Ty)) \leq F(a_1 d(z, y) + a_2 D(z, Tz) + a_3 D(y, Ty)),
   \]
   where $a_1, a_2, a_3, a_4, a_5 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 = 1$ and $a_3 \neq 1$.

Then $T$ has a fixed point in $X$.

**Corollary 3.8.** Let $(X, d)$ be a complete metric space and $\alpha, \eta : X \times X \to \mathbb{R}_+$ be two functions. Let $T : X \to C(X)$ and $F \in \mathcal{F}$ satisfy all conditions of Corollary 3.7. Then $T$ has a fixed point in $X$.

As an application of Theorems 3.1 and 3.2, we obtain the following.

**Theorem 3.9.** Let $(X, d)$ be a complete metric space and $\alpha, \eta : X \times X \to \mathbb{R}_+$ be two functions. Let $T : X \to K(X)$ and $F \in \mathcal{F}$ fulfill the conditions (1) and (3) of Theorem 3.1 and the following assertions:
1. $T$ is $\alpha$-$\eta$ continuous mapping;
2. there exists a function $\tau : (0, \infty) \to (0, \infty)$ such that
   \[
   \lim_{t \to s^+} \inf \tau(t) > 0, \quad \text{for all } s \geq 0,
   \]
   and for any $y, z \in X$ with $\alpha(z, y) \geq \eta(z, y)$ and $H(Tz, Ty) > 0$ satisfying
   \[
   \tau(d(z, y)) + F(H(Tz, Ty)) \leq F(a_1 d(z, y) + a_2 D(z, Tz) + a_3 D(y, Ty)) + a_4 D(z, Ty) + a_5 D(y, Tz)),
   \]
   where $a_1, a_2, a_3, a_4, a_5 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

Then $T$ has a fixed point in $X$.

**Proof.** By Lemma 2.3 we have $T$ is $\alpha$-$\eta$-lower semi continuous mapping. Also, for $z \in X$ and $y \in F_\sigma$ with $D(z, Tz) > 0$, we have
\[
F(D(y, Ty)) \leq F(H(Tz, Ty)) \leq F(a_1 d(z, y) + a_2 D(z, Tz) + a_3 D(y, Ty)) + a_4 D(z, Ty) + a_5 D(y, Tz)) - \tau(d(z, y)).
\]

Thus, all conditions of Theorem 3.1 are satisfied. Hence $T$ has a fixed point. \hfill \Box

By similar arguments of Theorem 3.9 and using Theorem 3.2, we state the following theorem.

**Theorem 3.10.** Let $(X, d)$ be a complete metric space and $\alpha, \eta : X \times X \to \mathbb{R}_+$ be two functions. Let $T : X \to K(X)$ and $F \in \mathcal{F}$ satisfy all conditions of Theorem 3.9. Then $T$ has a fixed point in $X$.

**Theorem 3.11.** Let $(X, d)$ be a complete metric space, $T : X \to K(X)$ and $F \in \mathcal{F}$. If there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that
   \[
   \lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,
   \]
   and for any $z \in X$ with $D(z, Tz) > 0$, there exists $y \in F_\sigma$ satisfying
   \[
   \tau(d(z, y)) + F(D(y, Ty)) \leq F(a_1 d(z, y) + a_2 D(z, Tz) + a_3 D(y, Ty)) + a_4 D(z, Ty) + a_5 D(y, Tz)),
   \]
   where $a_1, a_2, a_3, a_4, a_5 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$, then $T$ has a fixed point in $X$ provided $z \to D(z, Tz)$ is lower semi-continuous.
Proof. Define $\alpha(z, y) = d(z, y) = \eta(z, y)$ for all $z, y \in \mathcal{X}$. Then by using Remark 2.7 and Theorem 3.1, we get the required result. \hfill \Box

**Theorem 3.12.** Let $(\mathcal{X}, d)$ be a complete metric space, $T : \mathcal{X} \to C(\mathcal{X})$ and $F \in \mathfrak{F}_s$ satisfy all assertions of Theorem 3.11. Then $T$ has a fixed point in $\mathcal{X}$.

Proof. Define $\alpha(z, y) = d(z, y) = \eta(z, y)$ for all $z, y \in \mathcal{X}$. Then by using Remark 2.7 and Theorem 3.2, we get the required result. \hfill \Box

By taking $a_1 = 1$ and $a_2 = a_3 = a_4 = a_5 = 0$ in Theorems 3.11 and 3.12, we get the following corollaries.

**Corollary 3.13** (Theorem 11 of [6]). Let $(\mathcal{X}, d)$ be a complete metric space, $T : \mathcal{X} \to K(\mathcal{X})$ and $F \in \mathfrak{F}_s$. If there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that

$$
\lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,
$$

and for any $z \in \mathcal{X}$ with $D(z, T z) > 0$, there exists $y \in F_x^z$ satisfying

$$
\tau(d(z, y)) + F(D(y, T y)) \leq F(d(z, y)),
$$

then $T$ has a fixed point in $\mathcal{X}$ provided $z \to D(z, T z)$ is lower semi-continuous.

**Corollary 3.14** (Theorem 10 of [6]). Let $(\mathcal{X}, d)$ be a complete metric space, $T : \mathcal{X} \to C(\mathcal{X})$ and $F \in \mathfrak{F}_s$ satisfy all assertions of Corollary 3.13. Then $T$ has a fixed point in $\mathcal{X}$.

By taking $a_1 = a_2 = a_3 = 0$ and $a_4 = a_5 = 1/2$ in Theorems 3.11 and 3.12, we get the following.

**Corollary 3.15.** Let $(\mathcal{X}, d)$ be a complete metric space, $T : \mathcal{X} \to K(\mathcal{X})$ and $F \in \mathfrak{F}_s$. If there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that

$$
\lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,
$$

and for any $z \in \mathcal{X}$ with $D(z, T z) > 0$, there exists $y \in F_x^z$ satisfying

$$
\tau(d(z, y)) + F(D(y, T y)) \leq F\left(\frac{D(z, T y) + D(y, T z)}{2}\right),
$$

then $T$ has a fixed point in $\mathcal{X}$ provided $z \to D(z, T z)$ is lower semi-continuous.

**Corollary 3.16.** Let $(\mathcal{X}, d)$ be a complete metric space, $T : \mathcal{X} \to C(\mathcal{X})$ and $F \in \mathfrak{F}_s$ satisfy all assertions of Corollary 3.15. Then $T$ has a fixed point in $\mathcal{X}$.

By choosing $a_4 = a_5 = 0$ in Theorems 3.11 and 3.12, we get the following.

**Corollary 3.17.** Let $(\mathcal{X}, d)$ be a complete metric space, $T : \mathcal{X} \to K(\mathcal{X})$ and $F \in \mathfrak{F}_s$. If there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that

$$
\lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,
$$

and for any $z \in \mathcal{X}$ with $D(z, T z) > 0$, there exists $y \in F_x^z$ satisfying

$$
\tau(d(z, y)) + F(D(y, T y)) \leq F(a_1 d(z, y) + a_2 D(z, T z) + a_3 D(y, T y)),
$$

where $a_1, a_2, a_3 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 = 1$ and $a_3 \neq 1$, then $T$ has a fixed point in $\mathcal{X}$ provided $z \to D(z, T z)$ is lower semi-continuous.
Corollary 3.18. Let \((X, d)\) be a complete metric space, \(T : X \rightarrow C(X)\) and \(F \in \mathcal{F}_s\) satisfy all assertions of Corollary 3.17 Then \(T\) has a fixed point in \(X\).

Remark 3.19. Corollary 3.13 is a generalization of Theorem 2.3 of [26]. In fact, if \(\tau\) is a constant, then \(T\) is a multivalued \(F\)-contraction and every multivalued \(F\)-contraction is multivalued nonexpansive and every multivalued nonexpansive map is upper semi-continuous, then \(T\) is upper semi-continuous. Therefore, the function \(z \rightarrow D(z, Tz)\) is lower semi-continuous. On the other hand for any \(z \in X\) with \(D(z, Tz) > 0\) and \(y \in F_y\), we have

\[
\tau(d(z, y)) + F(D(y, Ty)) \leq \tau(d(z, y)) + F(H(Tz, Ty)) \leq F(d(z, y)).
\]

Hence \(T\) satisfies all conditions of Corollary 3.13 Similarly, Corollary 3.14 generalizes Theorem 2.5 of [26].

Remark 3.20. If we take \(T\), a single self-mapping on \(X\), Theorems 3.11 and 3.12 reduce to Theorem 1 of [30].

4. Fixed point results in partially ordered metric space

Let \((X, d, \preceq)\) be a partially ordered metric space and \(T : X \rightarrow 2^X\) be a multivalued mapping. For \(A, B \in 2^X\), \(A \preceq B\) implies that \(a \preceq b\) for all \(a \in A\) and \(b \in B\). We say that \(T\) is monotone increasing, if \(Ty \preceq Tz\), for all \(y, z \in X\), for which \(y \preceq z\). There are many applications in differential and integral equations of monotone mappings in ordered metric spaces (see [2] [7] [10] [17] and references therein). In this section, from Sections 2 and 3, we derive the following new results in partially ordered metric spaces.

Theorem 4.1. Let \((X, d, \preceq)\) be a complete partially ordered metric space, \(T : X \rightarrow K(X)\), \(F \in \mathcal{F}\) and \(G \in \mathcal{G}\) fulfill the following assertions:

1. if for any \(z \in X\) with \(D(z, Tz) > 0\), there exists \(y \in F_y\) with \(z \preceq y\) satisfying
   \[G(D(z, Tz), D(y, Ty), D(z, Ty), D(y, Tz)) + F(D(y, Ty)) \leq F(d(z, y));\]
2. \(T\) is monotone increasing;
3. there exist \(z_0 \in X\) and \(y_0 \in Tz_0\) such that \(z_0 \preceq y_0\);
4. for given \(z \in X\) and sequence \(\{z_n\}\) with \(z_n \rightarrow z\) as \(n \rightarrow \infty\) and \(z_n \preceq z_{n+1}\) for all \(n \in \mathbb{N}\), we have
   \[\lim_{n \rightarrow \infty} \inf D(z_n, Tz_n) \geq D(z, Tz),\]
then \(T\) has a fixed point in \(X\) provided \(\sigma < \tau\).

Proof. Define \(\alpha, \eta : X \times X \rightarrow [0, \infty)\) by

\[
\alpha(z, y) = \begin{cases} 2 & z \preceq y, \\ 0 & \text{otherwise,} \end{cases} \quad \eta(z, y) = \begin{cases} 1 & z \preceq y, \\ 0 & \text{otherwise,} \end{cases}
\]

then for \(z, y \in X\) with \(z \preceq y\), \(\alpha(y, z) \geq \eta(y, z)\) implies \(u \preceq v\) for all \(u \in Tz\) and \(v \in Ty\). Hence \(\alpha(u, v) = 2 = \eta(u, v)\), for all \(u \in Tz\) and \(v \in Ty\) and \(\alpha(u, v) = \eta(u, v) = 0\) otherwise. This shows that \(T\) is generalized \(\alpha\)-admissible mapping with respect to \(\eta\). Thus, all the conditions of Theorem 2.8 are satisfied and \(T\) has a fixed point.

By similar arguments as in Theorem 4.1, we state the following.

Theorem 4.2. Let \((X, d, \preceq)\) be a complete partially ordered metric space, \(T : X \rightarrow C(X)\), \(F \in \mathcal{F}_s\) and \(G \in \mathcal{G}\) fulfill all conditions of Theorem 4.1 Then \(T\) has a fixed point in \(X\) provided \(\sigma < \tau\).

Theorem 4.3. Let \((X, d, \preceq)\) be a complete partially ordered metric space, \(T : X \rightarrow K(X)\), \(F \in \mathcal{F}\) and \(G \in \mathcal{G}\) fulfill the conditions (2) and (3) of Theorem 4.1 and the following assertions:
1. If for any \( z, y \in X \) with \( z \preceq y \) and \( H(Tz, Ty) > 0 \) satisfying
\[
G(D(z, Tz), D(y, Ty), D(z, Ty), D(y, Tz)) + F(H(Tz, Ty)) \leq F(d(z, y));
\]
2. for given \( z \in X \) and sequence \( \{z_n\} \) with \( z_n \to z \) as \( n \to \infty \) and \( z_n \preceq z_{n+1} \) for all \( n \in \mathbb{N} \), we have \( Tz_n \to Tz \),
then \( T \) has a fixed point in \( X \).

**Theorem 4.4.** Let \((X, d, \preceq)\) be a complete partially ordered metric space, \( T : X \to C(X) \), \( F \in \mathcal{F}_* \) and \( G \in \mathcal{G} \) fulfill all conditions of Theorem 4.3. Then \( T \) has a fixed point in \( X \).

By taking \( G = G_L \), as in Corollary 4.5, Theorems 4.1–4.4 reduce to the following.

**Corollary 4.5.** Let \((X, d, \preceq)\) be a complete partially ordered metric space, \( T : X \to K(X) \) and \( F \in \mathcal{F}_* \) satisfy conditions (2)-(4) of Theorem 4.1 and if for any \( z \in X \) with \( D(z, Tz) > 0 \), there exists \( y \in F^* \) with \( z \preceq y \) satisfying
\[
\tau + F(D(y, Ty)) \leq F(d(z, y)),
\]
then \( T \) has a fixed point in \( X \) provided \( \sigma < \tau \).

**Corollary 4.6.** Let \((X, d, \preceq)\) be a complete partially ordered metric space, \( T : X \to C(X) \) and \( F \in \mathcal{F}_* \) satisfy all conditions of Corollary 4.5. Then \( T \) has a fixed point in \( X \) provided \( \sigma < \tau \).

**Corollary 4.7.** Let \((X, d, \preceq)\) be a complete partially ordered metric space, \( T : X \to K(X) \) and \( F \in \mathcal{F}_* \) fulfill conditions (2)-(4) of Theorem 4.1 and if for any \( z \in X \) with \( z \preceq y \) and \( H(Tz, Ty) > 0 \) we have
\[
\tau + F(H(Tz, Ty)) \leq F(d(z, y)),
\]
then \( T \) has a fixed point in \( X \).

**Corollary 4.8.** Let \((X, d, \preceq)\) be a complete partially ordered metric space, \( T : X \to C(X) \) and \( F \in \mathcal{F}_* \) satisfy all conditions of Corollary 4.7. Then \( T \) has a fixed point in \( X \) provided \( \sigma < \tau \).

**Theorem 4.9.** Let \((X, d)\) be a complete metric space, \( T : X \to K(X) \) and \( F \in \mathcal{F}_* \) fulfill the following assertions:
1. \( T \) is monotone increasing;
2. there exist \( z_0 \in X \) and \( y_0 \in Tz_0 \) such that \( z_0 \preceq y_0 \);
3. for given \( z \in X \) and sequence \( \{z_n\} \) with \( z_n \to z \) as \( n \to \infty \) and \( z_n \preceq z_{n+1} \) for all \( n \in \mathbb{N} \) we have
\[
\lim_{n \to \infty} \inf D(z_n, Tz_n) \geq D(z, Tz);
\]
4. there exist \( \sigma > 0 \) and a function \( \tau : (0, \infty) \to (\sigma, \infty) \) such that
\[
\lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,
\]
and for any \( z \in X \) with \( D(z, Tz) > 0 \), there exists \( y \in F^* \) with \( z \preceq y \) satisfying
\[
\tau(d(z, y)) + F(D(y, Ty)) \leq F(a_1 d(z, y) + a_2 D(z, Tz) + a_3 D(y, Ty) + a_4 D(z, Ty) + a_5 D(y, Tz)),
\]
where \( a_1, a_2, a_3, a_4, a_5 \in [0, +\infty) \) such that \( a_1 + a_2 + a_3 + 2a_4 = 1 \) and \( a_3 \neq 1 \).
Then \( T \) has a fixed point in \( X \).
Theorem 4.10. Let \((X,d,\preceq)\) be a complete partially ordered metric space, \(T : X \to C(X)\) and \(F \in \mathcal{F}\) fulfill all conditions of Theorem 4.9. Then \(T\) has a fixed point in \(X\) provided \(\sigma < \tau\).

Theorem 4.11. Let \((X,d,\preceq)\) be a complete partially ordered metric space, \(T : X \to K(X)\) and \(F \in \mathcal{F}\) fulfill the conditions (1) and (2) of Theorem 4.9 and the following assertions:

1. for given \(z \in X\) and sequence \(\{z_n\}\) with \(z_n \to z\) as \(n \to \infty\) and \(z_n \preceq z_{n+1}\) for all \(n \in \mathbb{N}\), we have \(Tz_n \to Tz\);
2. there exist \(\sigma > 0\) and a function \(\tau : (0,\infty) \to (\sigma,\infty)\) such that
   \[
   \lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,
   \]
   and for any \(z,y \in X\) with \(z \preceq y\) and \(H(Tz,Ty) > 0\), satisfying
   \[
   \tau(d(z,y)) + F(H(Tz,Ty)) \leq F(a_1d(z,y) + a_2D(z,Tz) + a_3D(y,Ty) + a_4D(z,Ty) + a_5D(y,Tz)),
   \]
where \(a_1,a_2,a_3,a_4,a_5 \in [0,\infty)\) such that \(a_1 + a_2 + a_3 + 2a_4 = 1\) and \(a_3 \neq 1\).
Then \(T\) has a fixed point in \(X\).

Theorem 4.12. Let \((X,d,\preceq)\) be a complete partially ordered metric space, \(T : X \to C(X)\) and \(F \in \mathcal{F}\) fulfill all conditions of Theorem 4.11. Then \(T\) has a fixed point in \(X\) provided \(\sigma < \tau\).

5. Suzuki-Wardowski type fixed point results

In this section we establish certain fixed point results for Suzuki-Wardowski type multivalued \(F\)-contractions.

Theorem 5.1. Let \((X,d)\) be a complete metric space, \(T : X \to K(X)\) and \(F \in \mathcal{F}\). If for \(z,y \in X\) with \(\frac{1}{2}D(z,Tz) \leq d(z,y)\) and \(D(z,Tz) > 0\), we have
   \[
   \tau + F(D(y,Ty)) \leq F(d(z,y)),
   \]
then \(T\) has a fixed point in \(X\) provided \(z \to D(z,Tz)\) is lower semi-continuous.

Proof. Suppose that \(G = G_{\mathcal{F}}\) as in Corollary 2.10. Let \(z \in X\) with \(D(z,Tz) > 0\) and \(y \in F_z, \sigma < \tau\). Then \(y \in Tz\), therefore we have \(\frac{1}{2}D(z,Tz) \leq D(z,Tz) \leq d(z,y)\). So, by using (5.1), we get
   \[
   G(D(z,Tz),D(y,Ty),D(z,Ty),D(y,Tz)) + F(D(y,Ty)) = \tau + F(D(y,Ty)) \leq F(d(z,y)).
   \]
Thus, all conditions of Theorem 2.18 hold and \(T\) has a fixed point.

Theorem 5.2. Let \((X,d)\) be a complete metric space, \(T : X \to C(X)\) and \(F \in \mathcal{F}\). If for \(z,y \in X\) with \(\frac{1}{2}D(z,Tz) \leq d(z,y)\) and \(D(z,Tz) > 0\), we have
   \[
   \tau + F(D(y,Ty)) \leq F(d(z,y)),
   \]
then \(T\) has a fixed point in \(X\) provided \(z \to D(z,Tz)\) is lower semi-continuous.

Proof. By taking \(G = G_{\mathcal{F}}\) as in Corollary 2.10 and by using Theorem 2.19 we get the required result.

Theorem 5.3. Let \((X,d)\) be a complete metric space, \(T : X \to K(X)\) and \(F \in \mathcal{F}\). If for \(z,y \in X\) with \(\frac{1}{2}D(z,Tz) \leq d(z,y)\) and \(H(Tz,Ty) > 0\), we have
   \[
   \tau + F(H(Tz,Ty)) \leq F(d(z,y)),
   \]
then \(T\) has a fixed point in \(X\).
Proof. Since every multivalued $F$-contraction is multivalued nonexpansive and every multivalued nonexpansive map is upper semi-continuous, then $\mathcal{T}$ is upper semi-continuous. Therefore, the function $z \to D(z, Tz)$ is lower semi-continuous (see the Proposition 4.2.6 of [3]). Also, for $z, y \in \mathcal{X}$ with $\frac{1}{2}D(z, Tz) \leq d(z, y)$ and $D(z, Tz) > 0$ we have

$$\tau + F(D(y, Ty)) \leq \tau + F(H(Tz, Ty)) \leq F(d(z, y)).$$

Thus, all conditions of Theorem 3.11 hold and $\mathcal{T}$ has a fixed point. \qed

By similar arguments as in Theorem 5.3, we state the following theorem and omit its proof.

**Theorem 5.4.** Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $F \in \mathcal{F}$. If for $z, y \in \mathcal{X}$ with $\frac{1}{2}D(z, Tz) \leq d(z, y)$ and $H(Tz, Ty) > 0$, we have

$$\tau + F(H(Tz, Ty)) \leq F(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

By considering $\mathcal{T}$ a single-valued mapping in Theorem 5.3, we get the following.

**Corollary 5.5.** Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ and $F \in \mathcal{F}$. If for $z, y \in \mathcal{X}$ with $\frac{1}{2}d(z, Tz) \leq d(z, y)$ and $d(Tz, Ty) > 0$, we have

$$\tau + F(d(Tz, Ty)) \leq F(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

**Remark 5.6.** Corollary 5.5 is a generalization of the Corollary 3.1 of [18]. In fact, let Corollary 3.1 of [18] holds, then $\frac{1}{2}d(z, Tz) \leq d(z, Tz) \leq d(z, y)$. This implies that $\tau + F(d(Tz, Ty)) \leq F(d(z, y))$. Hence $\mathcal{T}$ satisfies all conditions of Corollary 5.5 and $\mathcal{T}$ has a fixed point.

**Theorem 5.7.** Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ be a continuous mapping and $F \in \mathcal{F}$. If there exists a function $\tau : (0, \infty) \to (0, \infty)$ such that

$$\lim_{t \to s^+} \inf \tau(t) > 0, \quad \text{for all } s \geq 0,$$

and for $z, y \in \mathcal{X}$ with $\frac{1}{2}D(z, Tz) \leq d(z, y)$ and $H(Tz, Ty) > 0$, we have

$$\tau(d(z, y)) + F(H(Tz, Ty)) \leq F(a_1d(z, y) + a_2D(z, Tz) + a_3D(y, Ty) + a_4D(z, Ty) + a_5D(y, Tz)),$$

(5.2)

where $a_1, a_2, a_3, a_4, a_5 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$, then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Proof. Let $\lim_{t \to s^+} \inf \tau(t) > \sigma > 0$, and for all $s \geq 0$ also suppose that $z \in \mathcal{X}$ with $D(z, Tz) > 0$ and $y \in F^*_\sigma, \sigma < \tau$. Then $\lim_{t \to s^+} \inf \tau(t) > 0$ and $y \in Tz$, therefore we have $\frac{1}{2}D(z, Tz) \leq D(z, Tz) \leq d(z, y)$. So, by using (5.2), we get

$$\tau(d(z, y)) + F(D(y, Ty)) \leq \tau(d(z, y)) + F(H(Tz, Ty)) \leq F(a_1d(z, y) + a_2D(z, Tz) + a_3D(y, Ty) + a_4D(z, Ty) + a_5D(y, Tz)).$$

Since $\mathcal{T}$ is continuous, then $\mathcal{T}$ is upper semi-continuous. Therefore, the function $z \to D(z, Tz)$ is lower semi-continuous (see the Proposition 4.2.6 of [3]). Thus, all conditions of Theorem 3.11 hold and $\mathcal{T}$ has a fixed point. \qed
Theorem 5.8. Let \((X,d)\) be a complete metric space, \(T : X \to C(X)\) be a continuous mapping and \(F \in \mathfrak{F}_s\). If there exists a function \(\tau : (0, \infty) \to (\sigma, \infty)\) such that
\[
\lim_{t \to s^+} \inf \tau(t) > 0, \quad \text{for all } s \geq 0,
\]
and for \(z, y \in X\) with \(\frac{1}{2}D(z, Tz) \leq d(z, y)\) and \(H(Tz, Ty) > 0\), we have
\[
\tau(d(z, y)) + F(H(Tz, Ty)) \leq F(a_1 d(z, y) + a_2 D(z, Tz) + a_3 D(y, Ty) + a_4 D(z, Ty) + a_5 D(y, Tz)),
\]
where \(a_1, a_2, a_3, a_4, a_5 \in [0, +\infty)\) such that \(a_1 + a_2 + 3a_3 + 2a_4 = 1\) and \(a_3 \neq 1\. Then \(T\) has a fixed point in \(X\).

Proof. By using the same arguments as in Theorem 5.7 and by using Theorem 3.12, we get the required result.

6. Applications to orbitally lower semi-continuous mappings

Let \(z_0 \in X\) be any point. Then an orbit \(O(z_0)\) of a mapping \(T : X \to 2^X\) at a point \(z_0\) is a set
\[
O(z_0) = \{ z_{n+1} : z_{n+1} \in Tz_n, \ n = 0, 1, 2, \ldots \}.
\]
Recall that a function \(g : X \to \mathbb{R}\) is called \(T\)-orbitally lower semi-continuous, if for any sequence \(\{z_n\}\) in \(X\) with \(z_{n+1} \in Tz_n\) for all \(n = 0, 1, 2, \ldots\), \(g(z) \leq \liminf_{n \to \infty} g(z_n)\), whenever \(\lim_{n \to \infty} z_n = z\) \(9\). Many authors extended Nadler’s fixed point theorem for lower semi-continuous mappings (see \([13, 22, 23]\) and references therein). In this section, as an application of our results proved in Sections 1 and 2, we deduce certain fixed point theorems.

Theorem 6.1. Let \((X, d)\) be a complete metric space, \(T : X \to K(X)\), \(F \in \mathfrak{F}\) and \(G \in \mathfrak{G}\). If for \(z \in O(w), w \in X\) with \(D(z, Tz) > 0\), there exists \(y \in \mathcal{F}^*_\sigma\) satisfying
\[
G(D(z, Tz), D(y, Ty), D(z, Ty), D(y, Tz)) + F(D(y, Ty)) \leq F(d(z, y)), \quad (6.1)
\]
then \(T\) has a fixed point in \(X\) provided \(\sigma < \tau\) and \(z \to D(z, Tz)\) is \(T\)-orbitally lower semi-continuous.

Proof. Define \(\alpha, \eta : X \times X \to \mathbb{R}_+\) by
\[
\alpha(z, y) = \begin{cases} 2 & \text{if } z, y \in O(w), \\ 0 & \text{otherwise}, \end{cases} \quad \text{and } \eta(z, y) = 1, \ \forall z, y \in X.
\]

Then \(\alpha(z, y) \geq \eta(z, y)\), when \(z, y \in O(w)\). Since \(z \to D(z, Tz)\) is \(T\)-orbitally lower semi-continuous, so for any sequence \(\{z_n\}\) in \(X\) with \(z_{n+1} \in Tz_n\) and \(\lim_{n \to \infty} d(z_n, z) = 0\), we have
\[
D(z, Tz) \leq \lim_{n \to \infty} \inf D(z_n, Tz_n).
\]
This implies that \(T\) is \(\alpha, \eta\)-orbitally lower semi-continuous mapping. Now let \(\alpha(z, y) \geq \eta(z, y)\), then \(z, y \in O(w)\). So, for all \(u \in Tz\) and \(v \in Ty\) we have \(u, v \in O(w)\). Therefore, \(\alpha(u, v) = 2 > 1 = \eta(u, v)\). This shows that \(T\) is generalized \(\alpha, \eta\)-admissible mapping with respect to \(\eta\). Also, from equation (6.1), for any \(z \in X\) with \(D(z, Tz) > 0\), there exists \(y \in \mathcal{F}^*_\sigma\) with \(\alpha(z, y) \geq \eta(z, y)\), we have
\[
G(D(z, Tz), D(y, Ty), D(z, Ty), D(y, Tz)) + F(D(y, Ty)) \leq F(d(z, y)).
\]
Thus, all the conditions of Theorem 2.8 are satisfied and so \(T\) has a fixed point. 

\(\square\)
By similar arguments as in Theorem 6.1, we state the following theorem and omit its proof.

**Theorem 6.2.** Let \((X, d)\) be a complete metric space, \(T : X \to C(X)\), \(F \in \mathfrak{F}\), and \(G \in \mathfrak{G}\). If for \(z \in O(w), w \in X\) with \(D(z, Tz) > 0\), there exists \(y \in \mathcal{F}^*_\sigma\) satisfying
\[
G(D(z, Tz), D(y, Ty), D(z, Ty), D(y, Tz)) + F(D(y, Ty)) \leq F(d(z, y)),
\]
then \(T\) has a fixed point in \(X\) provided \(\sigma < \tau\) and \(z \to D(z, Tz)\) is \(T\)-orbitally lower semi-continuous.

**Theorem 6.3.** Let \((X, d)\) be a complete metric space, \(T : X \to K(X)\), \(F \in \mathfrak{F}\) and \(G \in \mathfrak{G}\). If for \(z, y \in O(w)\) with \(H(Tz, Ty) > 0\) we have
\[
G(D(z, Tz), D(y, Ty), D(z, Ty), D(y, Tz)) + F(H(Tz, Ty)) \leq F(d(z, y)),
\]
then \(T\) has a fixed point in \(X\) provided \(T\) is orbitally continuous.

**Proof.** By defining \(\alpha(z, y), \eta(z, y)\) the same as in the proof of Theorem 6.1 and applying Theorem 2.14, we get the required result.

**Theorem 6.4.** Let \((X, d)\) be a complete metric space, \(T : X \to K(X)\) and \(F \in \mathfrak{F}\). If for \(z, y \in O(w)\) with \(H(Tz, Ty) > 0\) satisfying
\[
G(D(z, Tz), D(y, Ty), D(z, Ty), D(y, Tz)) + F(H(Tz, Ty)) \leq F(d(z, y)),
\]
then \(T\) has a fixed point in \(X\) provided \(T\) is orbitally continuous.

**Proof.** By defining \(\alpha(z, y), \eta(z, y)\) the same as in the proof of Theorem 6.1 and applying Theorem 2.15, we get the required result.

By taking \(G = G_L\), as in Corollary 2.11, Theorems 6.1, 6.2, 6.3 and 6.4 reduce to the following.

**Corollary 6.5.** Let \((X, d)\) be a complete metric space, \(T : X \to K(X)\) and \(F \in \mathfrak{F}\). If for \(z \in O(w), w \in X\) with \(D(z, Tz) > 0\), there exists \(y \in \mathcal{F}^*_\sigma\) satisfying
\[
\tau + F(D(y, Ty)) \leq F(d(z, y)),
\]
then \(T\) has a fixed point in \(X\) provided \(\sigma < \tau\) and \(z \to D(z, Tz)\) is \(T\)-orbitally lower semi-continuous.

**Corollary 6.6.** Let \((X, d)\) be a complete metric space, \(T : X \to C(X)\) and \(F \in \mathfrak{F}\). If for \(z \in O(w), w \in X\) with \(D(z, Tz) > 0\), there exists \(y \in \mathcal{F}^*_\sigma\) satisfying
\[
\tau + F(D(y, Ty)) \leq F(d(z, y)),
\]
then \(T\) has a fixed point in \(X\) provided \(\sigma < \tau\) and \(z \to D(z, Tz)\) is \(T\)-orbitally lower semi-continuous.

**Corollary 6.7.** Let \((X, d)\) be a complete metric space, \(T : X \to K(X)\) and \(F \in \mathfrak{F}\). If for \(z, y \in O(w)\) with \(H(Tz, Ty) > 0\) satisfying
\[
\tau + F(H(Tz, Ty)) \leq F(d(z, y)),
\]
then \(T\) has a fixed point in \(X\) provided \(T\) is orbitally continuous.

**Corollary 6.8.** Let \((X, d)\) be a complete metric space, \(T : X \to C(X)\) and \(F \in \mathfrak{F}\). If for \(z, y \in O(w)\) with \(H(Tz, Ty) > 0\) satisfying
\[
\tau + F(H(Tz, Ty)) \leq F(d(z, y)),
\]
then \(T\) has a fixed point in \(X\) provided \(T\) is orbitally continuous.
Remark 6.9. If we take $\mathcal{T}$, a single mapping from $X$ to $X$, Theorems 6.3 and 6.4 reduce to the Theorem 4.1 of [18] and Corollaries 6.7 and 6.8 reduce to Corollary 4.1 of [18].

Theorem 6.10. Let $(X, d)$ be a complete metric space, $\mathcal{T} : X \to K(X)$ and $F \in \mathfrak{F}$. If there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that
\[
\lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,
\]
and for any $z \in O(w), w \in X$ with $D(z, Tz) > 0$, there exists $y \in F(z)$ satisfying
\[
\tau(d(z, y)) \leq \mathcal{F}(d(y, Tz)) \leq \mathcal{F}(a_1 d(z, y) + a_2 D(z, Tz) + a_3 D(y, Tz) + a_4 D(z, Ty) + a_5 D(y, Tz)),
\]
where $a_1, a_2, a_3, a_4, a_5 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$, then $\mathcal{T}$ has a fixed point in $X$ provided $z \to D(z, Tz)$ is $T$-orbitally lower semi-continuous.

Proof. By defining $\alpha(z, y), \eta(z, y)$ the same as in the proof of Theorem 6.1 and applying Theorem 3.11 we get the required result.

Theorem 6.11. Let $(X, d)$ be a complete metric space, $\mathcal{T} : X \to C(X)$ and $F \in \mathfrak{F}$, satisfying all conditions of Theorem 6.10. Then $\mathcal{T}$ has a fixed point in $X$.

References


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