On the extended multivalued Geraghty type contractions

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Abstract

In this paper we present some absolute retract results for modified Geraghty multivalued type contractions in \( b \)-metric space. Our results, generalize several existing results in the corresponding literature. We also present some examples to support the obtained results. ©2016 all rights reserved.

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1. Introduction and preliminaries

Let \( P(X) \) denote the collection of all nonempty subsets of a set \( X \neq \emptyset \), and \( F : X \to P(X) \) be multifunctions (multivalued mapping). Throughout the paper, set of all nonempty closed and bounded subsets of \( X \) will be represented by \( P_{b,cl}(X) \) under the assumption that \( X \) is equipped with a metric. Further, the set of all fixed point(s) of \( F \) will be denoted by \( \mathcal{F}_F \), that is,

\[
\mathcal{F}_F = \{ x \in X : x \in Fx \}.
\]

Let \((X, d)\) be a metric space and \( B(x_0, r) = \{ x \in X : d(x_0, x) < r \} \). For \( x \in X \) and \( A, B \subseteq X \), we set

\[
D : P(X) \times P(X) \to [0, \infty) \cup \{ +\infty \}, \text{ such that}
\]

\[
D(A, B) = \sup \{ D(a, B) : a \in A \} \text{ and } D(B, A) = \sup \{ D(b, A) : b \in B \}.
\]

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Let $H : P(X) \times P(X) \to [0, \infty) \cup \{+\infty\}$ be defined as

$$H(A, B) = \begin{cases} \max\{D(A, B), D(B, A)\}, & A \neq \emptyset \neq B, \\ 0, & A = \emptyset = B, \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that $H$ forms a metric and it is called the Hausdorff metric (for more details see e.g. [13, 14] and the references therein).

For non-empty sets $X, Y$, a mapping $\varphi : X \to Y$ is called a selection of $F : X \to P(Y)$, whenever $\varphi(x) \in Fx$ for all $x \in X$. A topological space $X$ is an absolute retract for metric spaces if for each metric space $Y$, $A \in P_d(Y)$ and continuous function $\psi : A \to X$, there exists a continuous function $\varphi : Y \to X$ such that $\varphi|_A = \psi$ (see [12]).

Let $\mathcal{M}$ be the collection of all metric spaces, $X \in \mathcal{M}$, $D \in P(\mathcal{M})$ and $F : X \to P_{bcl}(X)$ a lower semi-continuous multifunction. We say that $F$ has the selection property with respect to $D$ if for each $Y \in D$, continuous function $f : Y \to X$ and continuous functional $g : Y \to (0, \infty)$ such that

$$G(y) := F(f(y)) \cap B(f(y), g(y)) \neq \emptyset$$

for all $y \in Y$, $A \in P_d(Y)$, every continuous selection $\psi : A \to X$ of $G|_A$ admits a continuous extension $\varphi : Y \to X$, which is a selection of $G$. If $D = \mathcal{M}$, then we say that $F$ has the selection property and we denote this by $F \in Sp(X)$ (for more details see [13, 14]).

In this paper, we present some new results on absolute retract (see e.g. [4, 10, 12–14]) of the fixed points set of extended multivalued Geraghty type contractions. Our results combine, extend and generalize several existing results on the corresponding literature (see e.g. [1, 3, 8, 9, 11, 15, 16] and related references therein).

## 2. Fixed points set of extended multivalued Geraghty type contractions

In the all over this paper let $\Psi$ be the set of all increasing and continuous functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following property: $\psi(ct) \leq c\psi(t)$ for all $c > 1$ and $\psi(0) = 0$. We denote by $\Theta$ the family of all increasing functions $\theta : [0, \infty) \to (0, 1)$.

### Definition 2.1.
Let $F : X \to P_{bcl}(X)$ be a multivalued mapping and $\alpha : X \times X \to [0, \infty)$ be a given function. Then $F$ is said to be $\alpha$-admissible if

$$(T3) \quad \alpha(x, y) \geq 1 \text{ for all } y \in Fx \Rightarrow \alpha(y, z) \geq 1, \text{ for all } z \in Fy.$$

### Example 2.2.
Let $X = [1, 2]$ and $Fx = [x - \frac{1}{2}, 2]$. Define $\alpha(x, y) = 1$ if $x = y = 2$ and $\alpha(x, y) = 0$ otherwise. Clearly, $F$ is $\alpha$-admissible.

### Definition 2.3.
Let $(X, d)$ be a metric space and $F : X \to P_{bcl}(X)$ be a multivalued mapping. We say that $F$ is an extended multivalued Geraghty type contraction if there exist $\alpha : X \times X \to [0, \infty)$, $a \in [0, 1)$ and some $L \geq 0$ such that

$$\eta(a)D(x, F(x)) \leq d(x, y) \Rightarrow \alpha(x, y)\psi(H(Fx, Fy)) \leq \theta(\psi(M(x, y)))\psi(M(x, y)) + L\phi(N(x, y))$$

for all $x, y \in X$, where,

$$M(x, y) = \max\{d(x, y), D(x, Fx), D(y, Fy), \frac{D(x, Fy) + D(y, Fx)}{2}\}$$

and

$$N(x, y) = \min\{D(x, Fx), D(y, Fx)\}$$
and \( \eta(a) = \frac{1}{1+a}, \theta \in \Theta \) and \( \psi, \phi \in \Psi \).

Furthermore, we say that \( F \) is \textit{generalized multivalued Geraghty type contraction} if

\[
\alpha(x,y)\psi(H(Fx, Fy)) \leq \theta(\psi(M(x,y)))\psi(M(x,y)) + L\phi(N(x,y)) \tag{2.1}
\]

for all \( x, y \in X \), where, \( L, M(x,y), N(x,y), \alpha(x,y), \theta, \psi, \phi \) are defined as above.

\textbf{Remark 2.4.} The functions belonging to \( \Theta \) are strictly smaller than 1. Then, the expression \( \theta(\psi(M(x,y))) \) in (2.1) satisfies

\[ \theta(\psi(M(x,y))) < 1 \]

for any \( x, y \in X \) with \( x \neq y \).

\textbf{Theorem 2.5.} Let \((X,d)\) be a complete metric space and \( F : X \to P_{b,d}(X) \) be a extended multivalued Geraghty type contraction such that

(i) \( F \) is \( \alpha \)-admissible;

(ii) there exists \( x_0 \in X \) and \( x_1 \in Fx_0 \) such that \( \alpha(x_0,x_1) \geq 1 \);

(iii) \( F \) is continuous.

Then \( F \) has a fixed point.

\textbf{Proof.} By condition (ii), there exists \( x_0 \in X \) and \( x_1 \in Fx_0 \) such that \( \alpha(x_0,x_1) \geq 1 \). If \( x_1 = x_0 \), as \( x_1 \in Fx_1 \), then \( x_1 \) is a fixed point of \( F \) and we have nothing to prove. First, we note that

\[
M(x_0,x_1) = \max\{d(x_0,x_1), D(x_0, Fx_0), D(x_1, Fx_1), \frac{D(x_0, Fx_1) + D(x_1, Fx_0)}{2}\}
\]

\[ = \max\{d(x_0,x_1), D(x_1, Fx_1)\} \]

Since \( \eta(a)D(x_0, Fx_0) \leq d(x_0, x_1) \), if \( M(x_0,x_1) = D(x_1, Fx_1) \), then

\[
\psi(D(x_1, Fx_1)) \leq \alpha(x_0,x_1)\psi(H(Fx_0, Fx_1)) \leq \theta(\psi(D(x_1, Fx_1)))\psi(D(x_1, Fx_1)) + L\phi(0)
\]

which is a contradiction. It follows that \( M(x_0,x_1) = d(x_0,x_1) \). Let \( q = \frac{1}{\sqrt{\theta(\psi(d(x_0,x_1)))}} > 1 \), then there exists \( x_2 \in Fx_1 \) such that

\[
\psi(d(x_1,x_2)) \leq q\alpha(x_0,x_1)\psi(H(Fx_0, Fx_1)). \tag{2.2}
\]

Using (2.1) with \( x = x_0 \) and \( y = x_1 \), by (2.2) we get

\[
\psi(d(x_1,x_2)) \leq \sqrt{\theta(\psi(d(x_0,x_1)))\psi(d(x_0,x_1))}. \tag{2.3}
\]

Now, by the properties of the function \( \psi \), we deduce

\[
\psi\left(\frac{d(x_1,x_2)}{\sqrt{\theta(\psi(d(x_0,x_1)))}}\right) \leq \frac{1}{\sqrt{\theta(\psi(d(x_0,x_1)))}} \psi(d(x_1,x_2)) < \psi(d(x_0,x_1))
\]

and so \( d(x_1,x_2) < \sqrt{\theta(\psi(d(x_0,x_1)))d(x_0,x_1)} < d(x_0,x_1) \). If \( x_2 \in Fx_2 \), then \( x_2 \) is a fixed point of \( F \). Assume that \( x_1 \neq x_2 \notin Fx_2 \). We have:

\[
M(x_1,x_2) = \max\{d(x_1,x_2), D(x_2, Fx_2)\}, N(x_1,x_2) = 0
\]

and \( \eta(a)D(x_1, Fx_1) \leq d(x_1,x_2) \). If \( M(x_1,x_2) = D(x_2, Fx_2) \), then

\[
0 < \psi(D(x_2, Fx_2)) \leq \alpha(x_1,x_2)\psi(H(Fx_1, Fx_2))
\]
which is a contradiction and hence $M(x_1, x_2) = d(x_1, x_2)$.

Put $q_1 = \frac{\sqrt{\theta(\psi(d(x_0, x_1)))}}{\psi(d(x_1, x_2))} > 1$ (by (2.3)). Then there exists $x_3 \in Fx_2$ such that

$$\psi(d(x_2, x_3)) < q_1 \alpha(x_1, x_2) \psi(HFx_1, Fx_2)).$$

Since $\eta(a)D(x_2, Fx_2) \leq d(x_2, x_3)$, by (2.1) with $x = x_2$ and $y = x_3$, we have

$$\psi(d(x_2, x_3)) < q_1 \alpha(x_1, x_2) \psi(HFx_1, Fx_2))$$

$$\leq q_1 \theta(\psi(M(x_1, x_2))) \psi(M(x_1, x_2)) + q_1 LN(x_1, x_2)$$

$$= q_1 \theta(\psi(d(x_1, x_2))) \psi(d(x_1, x_2))$$

$$\leq \sqrt{\theta(\psi(d(x_0, x_1))) \psi(d(x_0, x_1))}$$

$$\leq \sqrt{\theta(\psi(d(x_0, x_1)))} \psi(d(x_0, x_1)).$$

Since

$$\psi\left(\frac{d(x_2, x_3)}{\sqrt{\theta(\psi(d(x_0, x_1)))}}\right)^2 \leq \psi(d(x_2, x_3)) \leq \psi(d(x_0, x_1))$$

and $\psi$ is increasing, then

$$d(x_2, x_3) < \sqrt{\theta(\psi(d(x_0, x_1)))} d(x_0, x_1) < d(x_0, x_1).$$

By continuing this process, we obtain a sequence $\{x_n\}$ in $X$ such that $x_n \neq x_{n-1}$ and $d(x_n, x_{n+1}) < \sqrt{\theta(\psi(d(x_0, x_1)))} d(x_0, x_1)$ for all $n \in \mathbb{N}$.

Let $t = \sqrt{\theta(\psi(d(x_0, x_1)))}$, then $0 < t < 1$. By the triangle inequality for $n < m$, we have

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq (t^n \sum_{k=0}^{m-n-1} t^k) d(x_0, x_1)$$

$$\leq \frac{t^n}{1-t} d(x_0, x_1).$$

The previous inequality shows that $\{x_n\}$ is a Cauchy sequence in $(X, d)$. Since $(X, d)$ is a complete metric space, so there exists $x^* \in X$ such that $\lim_{n \to \infty} x_n = x^*$. The continuity of $F$ implies that

$$0 \leq D(x^*, Fx^*) = \lim_{n \to \infty} D(x_{n+1}, Fx^*) \leq \lim_{n \to \infty} H(Fx_n, Fx^*) = 0$$

and so $x^* \in Fx^*$.

Example 2.6. Let $X = [-1, \infty)$, $d(x, y) = |x - y|$ and for any $A, B \subset X$

$$D(A, B) = \sup \{D(a, B) : a \in A\},$$

$$H(A, B) = \max\{\sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A)\}.$$

Define a multivalued mapping $F : X \to P_{bd}(X)$ by $F(x) = [-1, \frac{x}{4}]$ for every $x \in X$. It is easy to see that $(X, d)$ is a complete metric space. We have

$$\eta(a)D(x, F(x)) \leq d(x, y), \quad \eta(a) = \frac{1}{1 + a}, \quad a \in [0, 1),$$

$$\leq \theta(\psi(D(x_2, Fx_2))) \psi(D(x_2, Fx_2))$$

$$< \psi(D(x_2, Fx_2)),$$
In this case, the pair \((X,d)\) is said to be a \(b\)-metric on a \(b\)-metric space. We consider next the following family of subsets given by

\[
\mathcal{P}(X) := \{Y | Y \subset X \text{ and } Y \neq \emptyset\}.
\]

In this case \(D\) is a generalized functional on a \(b\)-metric space \((X,d)\) defined by \(D : P(X) \times P(X) \to [0, \infty) \cup \{+\infty\},\)

\[
D(A,B) = \begin{cases} 
\inf\{d(a,b) | a \in A, b \in B\}, & A \neq \emptyset \neq B, \\
0, & A = \emptyset = B, \\
+\infty, & \text{otherwise}.
\end{cases}
\]

In particular, if \(x_0 \in X\) then \(D(x_0,B) := D(\{x_0\},B)\).

The following basic lemmas will be useful in the proof of main results.

**Lemma 3.2** ([7]). Let \((X,d)\) be a \(b\)-metric space. Then, we have

\[
D(x,A) \leq s[d(x,y) + D(y,A)] \quad \text{for all } x, y \in X \text{ and } A \subset X.
\]

**Lemma 3.3** ([7]). Let \((X,d)\) be a \(b\)-metric space and let \(\{x_k\}_{k=0}^n \subset X\). Then

\[
d(x_n, x_0) \leq sd(x_0, x_1) + \ldots + s^{n-1}d(x_{n-2}, x_{n-1}) + s^n d(x_{n-1}, x_n).
\]

We denote by \(\mathcal{F}\) the family of all functions \(\beta : [0, \infty) \to [0, \frac{1}{s^2})\) for some \(s > 1\).

**Definition 3.4.** Let \((X,d)\) be a complete \(b\)-metric space and \(F : X \to P_{b,cl}(X)\) be a multivalued mapping. We say that \(F\) is an extended multivalued Geraghty type contraction in \(b\)-metric space with \((s > 1)\), whenever there exist \(\alpha : X \times X \to [0, \infty), a \in [0, 1)\) and some \(L \geq 0\) such that for
\[ M(x, y) = \max\{d(x, y), D(x, Fx), D(y, Fy), \frac{D(x, Fy) + D(y, Fx)}{2s} \} \]

and
\[ N(x, y) = \min\{D(x, Fx), D(y, Fx)\}, \]
we have
\[ \eta(a)D(x, F(x)) \leq d(x, y) \implies \alpha(x, y)\psi(s^3H(Fx, Fy)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)) + L\phi(N(x, y)) \] (3.1)

for all \( x, y \in X \), where \( \eta(a) = \frac{1}{1+q}, \beta \in \mathcal{F} \) and \( \psi, \phi \in \Psi \).

**Theorem 3.5.** Let \((X, d)\) be a complete b-metric space with \((s > 1)\), and \( F : X \to P_{b,cl}(X) \) be a extended multivalued Geraghty type contraction such that

(i) \( F \) is \( \alpha \)-admissible;

(ii) there exists \( x_0 \in X \) and \( x_1 \in Fx_0 \) such that \( \alpha(x_0, x_1) \geq 1 \);

(iii) \( F \) is continuous.

Then \( F \) has a fixed point.

**Proof.** By condition (ii), there exists \( x_0 \in X \) and \( x_1 \in Fx_0 \) such that \( \alpha(x_0, x_1) \geq 1 \). If \( x_1 = x_0 \), as \( x_1 \in Fx_1 \), then \( x_1 \) is a fixed point of \( F \) and we have nothing to prove. First, we note that

\[ M(x_0, x_1) = \max\{d(x_0, x_1), D(x_0, Fx_0), D(x_1, Fx_1), \frac{D(x_0, Fx_1) + D(x_1, Fx_0)}{2s} \} = \max\{d(x_0, x_1), D(x_1, Fx_1)\}. \]

Since \( \eta(a)D(x_0, Fx_0) \leq d(x_0, x_1) \), if \( M(x_0, x_1) = D(x_1, Fx_1) \), then

\[ \psi(D(x_1, Fx_1)) \leq \alpha(x_0, x_1)\psi(s^3H(Fx_0, Fx_1)) \leq \beta(\psi(D(x_1, Fx_1)))\psi(D(x_1, Fx_1)) + L\phi(0) < \psi(D(x_1, Fx_1)), \]

which is a contradiction. It follows that \( M(x_0, x_1) = d(x_0, x_1) \). Let us take a real \( q \) such that \( 1 < q < s \). Then

\[ 0 < \psi(D(x_1, Fx_1)) \leq \alpha(x_0, x_1)\psi(H(Fx_0, Fx_1)) < q\alpha(x_0, x_1)\psi(s^3H(Fx_0, Fx_1)). \]

Hence, there exists \( x_2 \in Fx_1 \) such that

\[ \psi(d(x_1, x_2)) < q\alpha(x_0, x_1)\psi(s^3H(Fx_0, Fx_1)). \] (3.2)

Using (3.1) with \( x = x_0 \) and \( y = x_1 \), by (3.2) we get

\[ \psi(d(x_1, x_2)) < \frac{q}{s^2}\psi(d(x_0, x_1)). \] (3.3)

Now, by the properties of the function \( \psi \) and regarding the fact that \( \frac{q}{s^2} < 1 \), we deduce

\[ \psi\left(\frac{s^2}{q}d(x_1, x_2)\right) \leq \frac{s^2}{q}\psi(d(x_1, x_2)) < \psi(d(x_0, x_1)), \]

\[ d(x_1, x_2) \leq \frac{q}{s^2}d(x_0, x_1) < d(x_0, x_1). \]
If $x_2 \in Fx_2$, then $x_2$ is a fixed point of $F$. Assume that $x_1 \neq x_2 \notin Fx_2$. We have:

$$M(x_1, x_2) = \max\{d(x_1, x_2), D(x_2, Fx_2)\}, N(x_1, x_2) = 0$$

and $\eta(a)D(x_1, Fx_1) \leq d(x_1, x_2)$. If $M(x_1, x_2) = D(x_2, Fx_2)$, then

$$0 < \psi(D(x_2, Fx_2)) \leq \alpha(x_1, x_2)\psi(s^3 H(Fx_1, Fx_2))$$

$$\leq \theta(\psi(D(x_2, Fx_2)))\psi(D(x_2, Fx_2))$$

$$< \psi(D(x_2, Fx_2)),$$

which is a contradiction and hence $M(x_1, x_2) = d(x_1, x_2)$. Put

$$q_1 = \frac{q}{s^2}\psi(d(x_0, x_1)) \psi(d(x_1, x_2)).$$

By (3.3), we have $q_1 > 1$. Hence, there exists $x_3 \in Fx_2$ such that

$$\psi(d(x_2, x_3)) < q_1 \alpha(x_1, x_2)\psi(s^3 H(Fx_1, Fx_2)).$$

Since $\eta(a)D(x, Fx_2) \leq d(x_2, x_3)$, by (3.1) with $x = x_2$ and $y = x_3$, we have

$$\psi(d(x_2, x_3)) < q_1 \alpha(x_1, x_2)\psi(s^3 H(Fx_1, Fx_2))$$

$$\leq q_1 \beta(\psi(M(x_1, x_2)))\psi(M(x_1, x_2)) + q_1 L\psi(N(x_1, x_2))$$

$$\leq \frac{q_1}{s^2}\psi(d(x_1, x_2)).$$

So

$$\psi(d(x_2, x_3)) \leq \frac{q_1}{s^2}\psi(d(x_1, x_2)) \leq \left(\frac{q}{s^2}\right)^2\psi(d(x_0, x_1)).$$

By properties of $\psi$ we obtain

$$d(x_2, x_3) \leq \left(\frac{q}{s^2}\right)^2d(x_0, x_1).$$

By continuing this process, we obtain a sequence $\{x_n\}$ in $X$ such that $x_n \in Fx_{n-1}$, $x_n \neq x_{n-1}$ and $d(x_n, x_{n+1}) < \left(\frac{q}{s^2}\right)^n d(x_0, x_1)$ for all $n \in \mathbb{N}$. By the triangle inequality for $n < m$, we have

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} s^{-k-n+1} d(x_k, x_{k+1})$$

$$\leq \sum_{k=n}^{\infty} s^{-k-n+1} \left(\frac{q}{s^2}\right)^k d(x_0, x_1)$$

$$= \left[\frac{s\left(\frac{q}{s^2}\right)^n}{1 - s\left(\frac{q}{s^2}\right)}\right] d(x_0, x_1) \to 0 \text{ as } n \to \infty.$$
Example 3.6. Put \( X = \{1\} \cup \{m + \frac{1}{n+2} : m, n \in \mathbb{N}\} \) and define a metric \( d \) on \( X \) by

\[
d(x, y) = |x - y|.
\]

Define a mapping \( F \) on \( X \) by

\[
F(x) = \begin{cases} 
1 & \text{if } x = 1, \\
\frac{1}{n+2} & \text{if } x = m + \frac{1}{n}.
\end{cases}
\]

Then \( F \) satisfies in the assumptions of Theorem 3.5.

Proof. It is obvious that \((X, d)\) is a complete metric space and 1 is a unique fixed point of \( F \). If \( n < m \), we have

\[
\eta(a)D(m + \frac{1}{n+2}, F(m + \frac{1}{n+2})) < d(m + \frac{1}{n+2}, n + \frac{1}{n+2})
\]

\[
\eta(a)d(m + \frac{1}{n+2}, 7m + \frac{1}{n+2}) < d(m + \frac{1}{n+2}, n + \frac{1}{n+2})
\]

\[
\eta(a) | m + \frac{1}{n+2} - 7m - \frac{1}{n+2} | < | m + \frac{1}{n+2} - n - \frac{1}{n+2} |
\]

\[
\frac{1}{2} | -6m | \leq \eta(a) | -6m | < | m + \frac{1}{n+2} - n - \frac{1}{n+2} |.
\]

This is a contradiction. Therefore \( F \) satisfies in the assumptions of Theorem 3.5. \( \square \)

Example 3.7. Let \( X \) be the set of Lebesgue measurable functions on \([0, 1]\) such that \( \int_0^1 |x(t)|dt < 1 \). Define \( d : X \times X :\to [0, \infty) \) by

\[
d(x, y) = \int_0^1 |x(t) - y(t)|^2dt.
\]

Then, \( d \) is a \( b \)-metric on \( X \), with \( s = 2 \). The multivalued mapping \( T : X \to 2^X \) is defined by

\[
Tx(t) = \begin{cases} 
3x + 4, & \text{if } x(t) < -1, \\
[-x, 1], & \text{if } -1 \leq x(t) < 0, \\
\frac{1}{8} \ln(1 + x(t)), & \text{if } x(t) \geq 0.
\end{cases}
\]

Consider the mapping \( \alpha : X \times X :\to [0, \infty) \) by the following

\[
\alpha(x, y) = \begin{cases} 
2, & \text{if } y \leq x \leq -3, \\
1, & \text{if } x \geq y \geq 0, \\
0, & \text{otherwise}.
\end{cases}
\]

We take \( \beta : [0, \infty) :\to [0, \frac{1}{4}) \) and \( \psi : [0, \infty) :\to [0, \infty) \) as

\[
\psi(t) = t \quad \text{and} \quad \beta(t) = \frac{t^2 + 1}{4t^2 + 8}.
\]

Evidently, \( \psi \in \Psi \) and \( \beta \in F \). Moreover, \( T \) is \( \alpha \)-admissible, \( \alpha(1, T1) \geq 1 \) and \( T \) is continuous. Now, we prove that \( T \) is a generalized \( \alpha - \psi \)-Suzuki-Geraghty multivalued type contraction. For \( x(t) \geq 0 \), we have
\[
\alpha(x(t), y(t))\psi(s^3d(Tx(t), Ty(t))) \leq 2^3\left(\int_0^1 |Tx(t) - Ty(t)|^2 dt\right) \\
= 2^3\int_0^1 \frac{1}{8}\ln(1 + x(t)) - \frac{1}{8}\ln(1 + y(t)))^2 dt \\
= 2^{-3}\int_0^1 \ln(1 + x(t))^2 dt = 2^{-3}\int_0^1 \ln(1 + \frac{x(t) - y(t)}{1 + y(t)})^2 dt \\
\leq 2^{-3}\int_0^1 \ln(1 + |x(t) - y(t)|)^2 dt \leq 2^{-3}\int_0^1 |x(t) - y(t)|^2 dt \\
= 2^{-3}d(x, y) \leq \frac{d(x, y)^2 + 1}{4d(x, y)^2 + 8} d(x, y) = \beta(d(x, y)d(x, y)).
\]

For \(x(t) < 0\), by definition of \(Tx(t)\) and \(\alpha(x(t), y(t))\) the condition of (3.1) is satisfied. Thus, \(T\) is a generalized \(\alpha - \psi\)-Suzuki-Geraghty multivalued type contraction. By Theorem 3.5, \(T\) has a fixed point. Here \(0, -2\) are fixed points.

If in (3.2), \(F\) is a family of all functions \(\beta : [0, \infty) \to [0, \frac{1}{s}]\) for some \(s \geq 1\), we can deduce the following theorem.

**Theorem 3.8.** Let \((X, d)\) be a complete b-metric space and absolute retract for b-metric spaces, \(F : X \to P_b(X)\) an extended multivalued Geraghty type contraction, \(F\) is continuous, and \(F \in SP(X)\). If \(\alpha(x, y) \geq 1\) for all \(x \in X\) and \(y \in F(x)\), then \(F_F\) is an absolute retract for b-metric spaces.

**Proof.** Let \(Y\) be a b-metric space, \(A \in P_{cl}(Y)\) and \(\xi : A \to F_F\) a continuous function. Since \(X\) is an absolute retract for b-metric spaces, there exists a continuous function \(\varphi_0 : Y \to X\) such that \(\varphi_0|A = \xi\). Define the function \(g_0 : Y \to (0, \infty)\) by

\[
g_0(y) = \sup\{d(\varphi_0(y), z) | z \in F(\varphi_0(y))\} + 1
\]

for all \(y \in Y\). It is not difficult to see that \(g_0\) is continuous and

\[
F(\varphi_0(y)) \cap B(\varphi_0(y), g_0(y)) = F(\varphi_0(y))
\]

for all \(y \in A\) (see [14]). Also we observe that the function \(\xi : A \to F_F\) has the property \(\xi(y) \in F(\varphi_0(y))\) \((y \in A)\), so is a continuous selection of the multivalued mapping. Since \(F \in SP(X)\), there exists a continuous function \(\varphi_1 : Y \to X\) such that \(\varphi_1|A = \xi\) and \(\varphi_1(y) \in F(\varphi_0(y))\) for all \(y \in Y\). First, we note that

\[
M(\varphi_0(y), \varphi_1(y)) = \max\{d(\varphi_0(y), \varphi_1(y)), D(\varphi_0(y), F(\varphi_0(y))), D(\varphi_1(y), F(\varphi_1(y)))
\]

\[
= \frac{D(\varphi_0(y), F(\varphi_1(y))) + D(\varphi_1(y), F(\varphi_0(y)))}{2s}
\]

Since \(\eta(a)D(\varphi_0(y), F(\varphi_0(y))) \leq d(\varphi_0(y), \varphi_1(y))\), if \(M(\varphi_0(y), \varphi_1(y)) = D(\varphi_1(y), F(\varphi_1(y)))\), then

\[
\psi(D(\varphi_1(y), F(\varphi_1(y)))) \leq \alpha(\varphi_0(y), \varphi_1(y))\psi(s^3H(F(\varphi_0(y), F(\varphi_1(y))))
\]

\[
\leq \beta(\psi(D(\varphi_1(y), F(\varphi_1(y))))\psi(D(\varphi_1(y), F(\varphi_1(y)))) + L\phi(0)
\]

\[
< \psi(D(\varphi_1(y), F(\varphi_1(y))))
\]

which is contradiction. It follows that \(M(\varphi_0(y), \varphi_1(y)) = d(\varphi_0(y), \varphi_1(y))\). Let \(1 < q < s\) and \(r \in (1, \frac{q}{4})\), then

\[
\psi(D(\varphi_1(y), F(\varphi_1(y))) \leq \alpha(\varphi_0(y), \varphi_1(y))\psi(s^3H(F(\varphi_0(y), F(\varphi_1(y))))
\]

\[
\leq \beta(\psi(D(\varphi_0(y), \varphi_1(y))))\psi(d(\varphi_0(y), \varphi_1(y)))
\]

\[
< \frac{q}{s}\psi(d(\varphi_0(y), \varphi_1(y))).
\]
Now, by the property of $\psi \in \Psi$ and regarding the fact that $\frac{q}{s} < 1$ we have
\[ \psi\left( \frac{1}{q} D(\varphi_1(y), F\varphi_1(y)) \right) \leq \frac{1}{q} \psi(D(\varphi_1(y), F\varphi_1(y))) < \psi(d(\varphi_0(y), \varphi_1(y))). \]

Since $\psi$ is increasing, therefore
\[
D(\varphi_1(y), F\varphi_1(y)) \leq \frac{q}{s} d(\varphi_0(y), \varphi_1(y)) < \frac{q}{s} d(\varphi_0(y), \varphi_1(y)) + r^{-1}.
\]

Hence, $G_2(y) := F(\varphi_1(y)) \cap B(\varphi_1(y), \frac{q}{s} d(\varphi_0(y), \varphi_1(y)) + r^{-1}) \neq \emptyset$ for all $y \in Y$. Since we know that $F \in Sp(X)$, there exists a continuous function $\varphi_2 : Y \to X$ such that $\varphi_2|_A = \xi$ and $\varphi_2(y) \in G_2(y)$ for all $y \in Y$. Thus, $\varphi_2(y) \in F(\varphi_1(y))$ for all $y \in Y$ and
\[ d(\varphi_1(y), \varphi_2(y)) < \frac{q}{s} d(\varphi_0(y), \varphi_1(y)) + r^{-1}. \]

Similarly we have
\[ M(\varphi_1(y), \varphi_2(y)) = \max\{d(\varphi_1(y), \varphi_2(y)), D(\varphi_2(y), F\varphi_2(y))\}, N(\varphi_1(y), \varphi_2(y)) = 0. \]

If $M(\varphi_1(y), \varphi_2(y)) = D(\varphi_2(y), F\varphi_2(y))$, then
\[
0 < \psi(D(\varphi_2(y), F\varphi_2(y))) \leq \alpha(\varphi_1(y), \varphi_2(y)) \psi\left( s^3 H(F(\varphi_1(y), F(\varphi_2(y)))) \right) \\
\leq \beta(\psi(D(\varphi_2(y), F\varphi_2(y)))) \psi(D(\varphi_2(y), F\varphi_2(y))) \\
< \frac{q}{s} \psi(D(\varphi_2(y), F\varphi_2(y))) \\
< \psi(D(\varphi_2(y), F\varphi_2(y))),
\]
which is contradiction. It follows that $M(\varphi_1(y), \varphi_2(y)) = d(\varphi_1(y), \varphi_2(y))$.

Now, by the property of $\psi$ we have
\[ \psi\left( \frac{1}{q} D(\varphi_2(y), F\varphi_2(y)) \right) \leq \frac{1}{q} \psi(D(\varphi_2(y), F\varphi_2(y))) < \psi(d(\varphi_1(y), \varphi_2(y))). \]

Since $\psi$ is increasing, therefore
\[ D(\varphi_2(y), F\varphi_2(y)) \leq \frac{q}{s} d(\varphi_1(y), \varphi_2(y)) < \frac{q}{s} d(\varphi_1(y), \varphi_2(y)) + r^{-1}. \]

By (3.4) we have
\[ D(\varphi_2(y), F\varphi_2(y)) < \left( \frac{q}{s} \right)^2 d(\varphi_0(y), \varphi_1(y)) + r^{-2}. \]

Hence, $G_3(y) := F(\varphi_2(y)) \cap B(\varphi_2(y), \left( \frac{q}{s} \right)^2 d(\varphi_0(y), \varphi_1(y)) + r^{-2}) \neq \emptyset$. Since $F \in Sp(X)$, there exists a continuous function $\varphi_3 : Y \to X$ such that $\varphi_3|_A = \xi$ and $\varphi_3(y) \in G_3(y)$ for all $y \in Y$. Also, we have $d(\varphi_2(y), \varphi_3(y)) < \left( \frac{q}{s} \right)^2 d(\varphi_0(y), \varphi_1(y)) + r^{-2}$ and $\varphi_3(y) \in F(\varphi_2(y))$ for all $y \in Y$. By continuing this process, we obtain $\{ \varphi_n : Y \to X \}_{n \geq 0}$ a sequence of continuous functions such that $\varphi_n|_A = \xi$ and $d(\varphi_{n-1}(y), \varphi_n(y)) < \left( \frac{q}{s} \right)^{n-1} d(\varphi_0(y), \varphi_1(y)) + r^{-n+1}$ and $\varphi_n(y) \in F(\varphi_{n-1}(y))$ for all $y \in Y$ and $n \geq 1$. Now, for each $\lambda > 0$ we put
\[ Y_\lambda := \{ y \in Y : d(\varphi_0(y), \varphi_1(y)) < \lambda \}. \]

Since $\varphi_1(y) \in F(\varphi_0(y))$ and
\[ F(\varphi_0(y)) \cap B(\varphi_0(y), g_0(y)) = F(\varphi_0(y)), \]
$\varphi_1(y) \in B(\varphi_0(y), g_0(y))$. Hence, $d(\varphi_0(y), \varphi_1(y)) < \lambda_0 := g_0(y)$. Thus, $y \in Y_{\lambda_0}$. Since $Y_{\lambda_0}$ is open for each $\lambda > 0$, the family of sets $\{ Y_\lambda | \lambda > 0 \}$ is an open covering of $Y$ and we have
\[ d(\varphi_{n-1}(y), \varphi_n(y)) \leq \left( \frac{q}{s} \right)^{n-1} d(\varphi_0(y), \varphi_1(y)) + r^{-(n-1)} \]
for all $n \geq 1$ and $y \in Y$. Since $\frac{a}{b} < 1$, $r > 1$, and $X$ is complete, the sequence $\{\phi_n\}_{n \geq 0}$ converges uniformly on $Y_\lambda$ for all $\lambda > 0$. Let $\phi : Y \to X$ be the pointwise limit of $\{\phi_n\}_{n \geq 0}$ and note that $\phi$ is continuous and $\phi|_A = \xi$ because $\phi_n|_A = \xi$ for all $n \geq 0$. Since $F$ is continuous, hence $\phi(y) \in F(\phi(y))$ for all $y \in Y$. Therefore, $\phi : Y \to B$ is a continuous extension of $\xi$, that is, $B = \{x \in X : x \in F(x)\}$ is an absolute retract for $b$-metric spaces. \hfill $\Box$

4. Corollaries

By letting $\alpha(x, y) = 1$ for all $x, y \in X$, we get the following consequences:

**Corollary 4.1.** Let $(X, d)$ be a complete $b$-metric space and absolute retract for $b$-metric spaces, $F : X \to P_{b,d}(X)$, also there exists $a \in [0, 1)$ and some $L \geq 0$ such that,

$$
\eta(a)D(x, F(x)) \leq d(x, y) \implies \psi(s^3d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y)) + L\phi(N(x, y))
$$

(4.1)

for all $x, y \in X$, where $\eta(a) = \frac{1}{1+4a}$, $\beta \in \mathcal{F}$, and $\psi, \phi \in \Psi$ and

$$
N(x, y) = \min\{d(x, Tx), d(y, Tx)\},
$$

$F$ is continuous and $F \in SP(X)$. If $\alpha(x, y) \geq 1$ for all $x \in X$ and $y \in F(x)$, then $\mathcal{F}_F$ is an absolute retract for $b$-metric spaces.

If in (4.1), we let $L = 0$ then we obtain the following sequence.

**Corollary 4.2.** Let $(X, d)$ be a complete $b$-metric space and absolute retract for $b$-metric spaces, $F : X \to P_{b,d}(X)$, also there exist $a \in [0, 1)$ such that,

$$
\eta(a)D(x, F(x)) \leq d(x, y) \implies \psi(s^3d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))
$$

for all $x, y \in X$, where $\eta(a) = \frac{1}{1+4a}$, $\beta \in \mathcal{F}$ and $\psi, \phi \in \Psi$, $F$ is continuous, and $F \in SP(X)$. If $\alpha(x, y) \geq 1$ for all $x \in X$ and $y \in F(x)$, then $\mathcal{F}_F$ is an absolute retract for $b$-metric spaces.

5. Consequences

As it is expected, the main results of the paper yield several existing results in the literature by choosing the auxiliary functions $\alpha, \eta, \psi, \phi$ in a proper way. To list more results it is sufficient to take $d(x, y)$ instead of $M(x, y)$, and/or take $L = 0$. Notice also that, one can replace the single valued mapping instead of multivalued mapping to cover more results in the literature. Furthermore, by relaxing $b$-metric with metric, we observe more results as a consequence of our main results.

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References


