Multivalent guiding functions in the bifurcation problem of differential inclusions

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Abstract

In this paper we use the multivalent guiding functions method to study the bifurcation problem for differential inclusions with convex-valued right-hand part satisfying the upper Carathéodory and the sublinear growth conditions. ©2016 all rights reserved.

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1. Introduction and preliminaries

The base of the method of guiding functions was laid by Krasnosel’skii and Perov (see [17–19]). The method of multivalent guiding functions became one of the most important directions of its development in the case of differential equations (see [28]).

It is well-known that the application of topological degree methods to the study of various problems of the theory of differential inclusions is very effective (see [2–6]).

A number of works was devoted to the extension of the guiding functions method to the case of differential inclusions and this approach demonstrated its effectiveness to the study of periodic problems. The classical method of guiding potential was used by Borisovich et al. [2] and Górniewicz [5]. The method of integral guiding functions and some of its versions were developed in [8, 9, 13, 16, 27] and the method of multivalent

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guiding functions was extended to differential inclusions in [7 10 12]. For some other applications of the guiding functions method see, for example, [11 14 15 23].

Notice that now the bifurcation phenomena in dynamical systems governed by the various classes of differential inclusions were studied by not only the classical method of guiding functions [20], but also by the method of integral guiding functions [21][24][26][27].

In the present paper, developing the abstract approach proposed in [20], the method of multivalent guiding functions method see, for example, [11, 14, 15, 25].

In what follows we will use some known notions and notations from the theory of multivalued maps (multimaps) (see [2–6]). We recall some of them as follows.

Let \((X, d_X), (Y, d_Y)\) and \((Z, d_Z)\) be metric spaces. By the symbols \(P(Y)\) and \(K(Y)\) we denote the collections of all nonempty and, respectively, nonempty and compact subsets of the space \(Y\). If \(Y\) is a normed space, \(K\) denotes the collection of all nonempty convex compact subsets of \(Y\).

**Definition 1.1.** A multimap \(F : X \to P(Y)\) is called upper semicontinuous (u.s.c.) at the point \(x \in X\) if for each open set \(V \subset Y\) such that \(F(x) \subset Y\) there exists \(\delta > 0\) such that \(d_X(x, x') < \delta\) implies \(F(x') \subset V\). A multimap \(F : X \to P(Y)\) is called u.s.c.. if it is u.s.c. at each point \(x \in X\).

**Definition 1.2.** A multimap \(F : X \to P(Y)\) is called closed if its graph
\[ \Gamma_F = \{(x, y) | (x, y) \in X \times Y, \ y \in F(x)\} \]
is a closed subset of the space \(X \times Y\).

**Definition 1.3.** A multimap \(F : X \to P(Y)\) is called compact if its range \(F(X)\) is relatively compact in \(Y\).

**Remark 1.4.** If multimap \(F : X \to P(Y)\) is closed and compact, then it is u.s.c..

A multimap will be called multifunction if it is defined on a subset of \(\mathbb{R}\). Let \(I\) be a closed subset of \(\mathbb{R}\) endowed with the Lebesgue measure.

**Definition 1.5.** A multifunction \(F : I \to K(Y)\) is called measurable if, for each open subset \(V \subset Y\), its pre-image
\[ F^{-1}(V) = \{t \in I : F(t) \subset V\} \]
is the measurable subset of \(I\).

**Remark 1.6.** Each measurable multifunction \(F : I \to K(Y)\) has a measurable selection, i.e., there exists such measurable function \(f : I \to Y\), that \(f(t) \in F(t)\) for almost every (a.e.) \(t \in I\).

Let \(\Delta\) be a compact subset of \(\mathbb{R}\).

**Definition 1.7.** A multimap \(F : I \times \mathbb{R}^n \times \Delta \to K\) is called the upper Carathéodory multimap if
(i) for each \(x \in \mathbb{R}^n\), \(\mu \in \Delta\), multifunction \(F(\cdot, x, \mu) : I \to K\) is measurable;
(ii) for \(\mu\)-a.e. \(t \in I\) multimap \(F(t, \cdot, \cdot) : \mathbb{R}^n \times \Delta \to K\) is u.s.c..

**Definition 1.8.** A multimap \(F : I \times \mathbb{R}^n \times \Delta \to K\) satisfies the sublinear growth, if there is a positive Lebesgue integrable function \(\alpha(\cdot)\) such that for all \(x \in \mathbb{R}^n\), \(\mu \in \Delta\), at a.e. \(t \in I\)
\[ \|F(t, x, \mu)\| := \max_{y \in F(t, x, \mu)} \|y\| \leq \alpha(t)(1 + \|x\|). \]

In the sequel, we use some aspects of the bifurcation theory in the following situation (see [5 20]).
Let $A \subset U \subset \mathbb{R}^n$, where $A$ is compact and $U$ is open in $\mathbb{R}^n$. We identify the $n$-sphere $S^n = \{x \in \mathbb{R}^n \mid ||x|| = 1\}$ with $\mathbb{R}^n \cup \{\infty\}$. By $\Pi_k, k \geq 0$, we denote the $k$-th stable homotopy group of spheres, i.e.,

$$
\Pi_k := \lim_{n \to 0} \pi_{n+k}(S^n).
$$

Let $\mathcal{A}$ be a sheaf of Abelian groups over $Y$, $f : X \to Y$ be a closed surjection, the symbol $\mathcal{A}_y$ denotes the fibre of a sheaf $\mathcal{A}$ over $y \in Y$. By $\mathcal{A}^*$ we denote the inverse image of a sheaf $\mathcal{A}$ under a map $f$. For an integer $k \geq 1$, define

$$
s^0(f; \mathcal{A}) := \{y \in Y \mid H^0(f^{-1}(y); \mathcal{A}^*) \neq \mathcal{A}_y\},
$$

$$
s^k(f; \mathcal{A}) := \{y \in Y \mid H^k(f^{-1}(y); \mathcal{A}^*) \neq 0\},
$$

where $H^*(\cdot; \mathcal{A})$ denotes the Čech cohomology groups with coefficients in the sheaf $\mathcal{A}$ and, for integers $N \geq 1$, let us define the Vietoris indices of $f$ by

$$
i^N(f; \mathcal{A}) := \inf \{n \geq 0 \mid \max_{0 \leq k \leq N-1} \{rd_Y(s^k(f; \mathcal{A})) + k\} + 1 < n\},
$$

where for $A \subset Y$, $rd_Y(A) := \sup \{\dim C \mid C$ is closed in $Y$, $C \subset A\}$ and $\dim$ denotes the topological dimension of a set (see [H]).

We set $i(f; \mathcal{A}) = \sup_{N \geq 0} i^N(f; \mathcal{A})$. If a sheaf $\mathcal{A}$ is constant and equals $\mathbb{Z}$, then $i(f; \mathbb{Z})$ is denoted by $i(f)$.

**Definition 1.9.** Let $\nu : Z \to X$. We say that $\nu$ belongs to the class $\mathcal{V}$ ($\nu$ is a $\mathcal{V}$-map) if

(i) $\nu$ is the perfect surjection, i.e., the surjection with compact fibres;

(ii) $i(\nu) < \infty$.

We say that $\nu : Z \to X$ is a $\tilde{\mathcal{V}}$-map if $\nu$ is a $\mathcal{V}$-map and

(iii) $\dim \nu := \sup_{x \in X} \dim \nu^{-1}(x) < \infty$.

Let $(X, X'), (Y, Y')$ be pairs of spaces and $m \geq 0$. By $D_m(X, X'; Y, Y')$ (resp. $\tilde{D}_m(X, X'; Y, Y')$) we denote the class of all cotriads

$$(X, X') \xleftarrow{\nu} (Z, Z') \xrightarrow{\chi} (Y, Y'),$$

where $\nu$ is a $\mathcal{V}_m$-map (resp. $\tilde{\mathcal{V}}_m$-map) and $\chi$ is a continuous map. Additionally, we put

$$D(X, X'; Y, Y') := \bigcup_{m \geq 0} D_m(X, X'; Y, Y'),$$

$$(\tilde{D} = \bigcup_{m \geq 0} \tilde{D}_m);$$

hence $\tilde{D} \subset D$.

**Definition 1.10.** We say that cotriads

$$(X, X') \xleftarrow{\nu_i} (Z_i, Z'_i) \xrightarrow{\chi_i} (Y, Y'), \quad i = 1, 2$$

from $D(X, X'; Y, Y')$ (resp. $\tilde{D}$) are equivalent (written $(\nu_1, \chi_1) \approx (\nu_2, \chi_2)$) if there exists a cotriad

$$(X, X') \xleftarrow{\nu} (Z, Z') \xrightarrow{\chi} (Y, Y')$$

(resp. with finite-dimensional map $\nu$) and $\mathcal{V}_0$-maps $f_i : (Z, Z') \to (Z_i, Z'_i)$ such that $\nu_i \circ f_i = \nu$ and $\chi_i \circ f_i = \chi$, $i = 1, 2$.

**Definition 1.11.** Elements of the quotient

$$M(X, X'; Y, Y') = D(X, X'; Y, Y')/\approx$$

or

$$\tilde{M}(X, X'; Y, Y') = \tilde{D}(X, X'; Y, Y')/\approx$$

are called morphisms (resp. finite-dimensional morphisms).
By $M_m(X, X'; Y, Y')$ (resp. $\tilde{M}_m(X, X'; Y, Y')$, $m \geq 0$, we denote the set of all morphisms from $M(X, X'; Y, Y')$ (resp. $\tilde{M}$) which are represented by cotriads $(\nu, \chi) \in D(X, X'; Y, Y')$ (resp. $\tilde{D}$).

By $C(m, n)$, $m \geq n$, we denote the class of all pairs $(f, U)$, where $U$ is an open bounded subset of $\mathbb{R}^m$ and $f : (\overline{U}, \partial U) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ is a continuous map.

We say that $(f_0, U), (f_1, U)$ from $C(m, n)$ are homotopic if there is a homotopy $h : (\overline{U} \times [0, 1], \partial U \times [0, 1]) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ such that $h(\cdot, i) = f_i$, $i = 0, 1$.

Let $B^m$ be a closed ball centered at zero of radius $r = 1$, $(f, U) \in C(m, n)$ where $n \leq m < 2n - 2$. Assume, without loss of generality, that $\overline{U} \subset B^m$ and that $\dim \partial U \leq m - 1$. Consider the following sequence of Abelian groups and homomorphisms

$$\pi^{n-1}(\partial U) \xrightarrow{\delta_1} \pi^n(\overline{U}, \partial U) \xrightarrow{j^2} \pi^n(B^m, B^m \setminus U) \xrightarrow{i^1} \pi^n(B^m, S^{m-1}) \xrightarrow{\delta_2} \pi^{n-1}(S^{m-1})$$

in which $\delta_1$ denotes the coboundary homomorphism of the pair $(\overline{U}, \partial U)$, $j : (\overline{U}, \partial U) \to (B^m, B^m \setminus U)$, $i : (B^m, S^{m-1}) \to (B^m, B^m \setminus U)$ are the inclusions and $\delta_2$ is the coboundary homomorphism of the pair $(B^m, S^{m-1})$. Clearly $j^2$ is the excision isomorphism and $\delta_2$ is an isomorphism in view of the contractibility of $B^m$ and the exactness of the cohomotopy sequence of the pair $(B^m, S^{m-1})$.

Let

$$\kappa = \delta_2^{-1} \circ i^1 \circ (j^2)^{-1} \circ \delta_1$$

and let $\eta := [f|\partial U] \in \pi^{n-1}(\partial U)$, where $[f|\partial U]$ denotes the homotopy class of $f|\partial U$ and $\pi^{n-1}(\partial U)$ denotes $(n - 1)$-th cohomotopy group of $\partial U$. Without loss of generality we have identified here $[\partial U; \mathbb{R}^n \setminus 0]$ with $\pi^{n-1}(\partial U)$.

**Definition 1.12.** The generalized degree of $f$ on $U$ is the element

$$\deg(f, U) := \kappa(\eta) \in \pi^{n-1}(S^{m-1}) \cong \Pi_{m-n}.$$

Let $U$ be an open subset of $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^k$. Consider the problem of the bifurcation of solutions to the inclusion

$$0 \in \Phi(z, \lambda),$$

where $\Phi : U \to \mathbb{R}^n$ is a multifield corresponding to a multimap $F : U \to \mathbb{R}^n$, i.e., $\Phi(z) = z - F(z)$.

Let us make the following assumptions:

1. $\Phi \in \tilde{M}_m(U; \mathbb{R}^n)$ is a morphism such that $0 \in \Phi(0, \lambda)$ for all $\lambda \in \Lambda := \{\lambda \in \mathbb{R}^k \mid (0, \lambda) \in U\}$.

   We define the set of nontrivial solutions to (1.1) as

   $$S := \{(z, \lambda) \in U \setminus \Lambda \times \{0\} \mid \lambda \neq 0, 0 \in \Phi(z, \lambda)\},$$

and suppose that

2. the set of bifurcation points

   $$B(\Phi) := \{(0, \lambda) \in \Lambda \times \{0\} \mid (0, \lambda) \in \overline{S}\}$$

   is compact.

In order to define the bifurcation index of $\Phi$ we need some auxiliary objects. Let us consider an arbitrary continuous function $\alpha : \Lambda \to [0, \infty)$ such that, for $(0, \lambda) \notin B(\Phi)$,

$$0 < \alpha(\lambda) < d((0, \lambda), \partial U \cup \overline{S})$$

and

$$\alpha(\lambda) = 0$$
for \((\lambda, 0) \in B(\Phi)\). For instance we may put
\[
\alpha(\lambda) = \min \left\{ 1, \frac{1}{2} d((0, \lambda), \partial U \cup \overline{S}) \right\}.
\]
Next we set
\[
X := \{(z, \lambda) \in \mathbb{R}^m \mid \lambda \in \Lambda, \|z\| = \alpha(\lambda)\},
\]
\[
X^+ := \{(z, \lambda) \in \mathbb{R}^m \mid \lambda \in \Lambda, \|z\| < \alpha(\lambda)\}.
\]
Observe that \(X^+ \cup X \subset U\) and put
\[
X^- := U \setminus X^+.
\]
it is easy to see that \(S \subset X^-\) and \(B(\Phi) \subset X\).

Let \(f : U \to \mathbb{R}\) be a continuous function such that
\[
f(z, \lambda) = \begin{cases} < 0 & \text{for } (z, \lambda) \in X^-; \\ = 0 & \text{for } (z, \lambda) \in X^-; \\ > 0 & \text{for } (z, \lambda) \in X^+. \end{cases}
\]
Now we consider a morphism \(\Psi\) from \(\tilde{M}_n(U; \mathbb{R}^{n+1})\) such that, for all \((z, \lambda) \in U\), \(\Psi(z, \lambda) = \Phi(z, \lambda) \times \{f(z, \lambda)\}\).

Since, by (ii), the set of zeros of \(\Psi\) is compact, there is an open bounded set \(U'\) such that \(\overline{U'} \subset U\) and \(0 \not\in \Psi(z, \lambda)\) for \((z, \lambda) \in U \setminus U'\). Therefore \((\Psi, U') \in \tilde{M}(m, n + 1)\).

**Definition 1.13.** The bifurcation index \(\text{Bi}(\Phi)\) of \(\Phi\) is defined by the following formula
\[
\text{Bi}(\Phi) := \text{deg}(\Psi, U') \in \Pi_{k-1}.
\]
We will need the following version of the global bifurcation result of Kryszewski (see [5, 20]).

In addition to above assumptions let us suppose that

1. there is an open set \(U_1 \supset U\) and a morphism \(\Phi_1 \in \tilde{M}_n(U_1; \mathbb{R}^n)\) such that \(\Phi_1|U = \Phi\) and \(0 \in \Phi_1(0, \lambda)\) for all \((0, \lambda) \in \{0\} \times U_1 \cap \mathbb{R}^k\). Let
   \[
   S_1 := \{(z, \lambda) \in U_1 \mid z \neq 0, \ 0 \in \Phi_1(z, \lambda)\}.
   \]

**Lemma 1.14.** Let \(K\) be a compact subset of \(U_1\) such that \(B(\Phi) \subset K\) and \(\{0\} \times K \cap (\mathbb{R}^k \setminus \Lambda) = \emptyset\) (e.q. \(K = B(\Phi)\)). If \(\text{Bi}(\Phi) \neq 0\), then there is a nonempty connected set \(C \subset S_1\setminus K\) such that \(\overline{C} \cap K \neq \emptyset\) and at least one of the followings occurs:

1. \(C\) is unbounded;
2. \(\overline{C} \cap \partial U_1 \neq \emptyset\);
3. there is a point \(\lambda_* \in \mathbb{R}^k \setminus \Lambda\) such that \((0, \lambda_*) \in U_1\) and \((0, \lambda_*) \in \overline{C}\). Thus \(\Phi_1\) has bifurcation points outside \(U\) connected to \(K\) in \(S_1\).

2. Main result

We shall study the periodic problem for a family of differential inclusions of the following form:
\[
z'(t) \in F(t, z(t), \mu), \quad z(0) = z(T),
\]
under assumptions that

(H1) \(F : \mathbb{R} \times \mathbb{R}^n \times \Lambda \to K\nu(\mathbb{R}^n)\) is a \(T\)-periodic multimap \((T > 0)\), satisfying the upper Carathéodory conditions and the sublinear growth condition;
(H$_2$) for each $\mu \in \Lambda$ problem (2.1) admits a solution $z : [0, T] \to \mathbb{R}^n$ with $z(0) = z(T) = 0$.

By a solution of problem (2.1) we mean a pair $(z, \mu)$, satisfying inclusion (2.1) a.e. on $[0, T]$, where $z \in C([0, T]; \mathbb{R}^n)$ is a $T$-periodic absolutely continuous function, $\mu \in \Lambda$. From (H$_2$) it follows that $(0, \mu)$ is a solution to problem (2.1) for each $\mu \in \Lambda$. These solutions are called trivial. Let us denote by $S$ the set of all nontrivial solutions of problem (2.1).

Let $\mathbb{R}^n = \mathbb{R}^{n-2} \times \mathbb{R}^2$ be a metric space. Denote by $q$ the operator of projection on $\mathbb{R}^2$ and $p = i - q$, where $i$ is the identity map. The elements of $\mathbb{R}^2$ and $\mathbb{R}^{n-2}$ are denoted by $\xi$ and $\zeta$, respectively. Let $\varphi, \rho$ be polar coordinates in $\mathbb{R}^2$.

We consider the multivalent Riemann surface

$$\Pi = \{ (\varphi, \rho) : \varphi \in (-\infty, \infty), \rho \in (0, \infty) \}.$$ 

On $\Pi \times \mathbb{R}$ we define a continuously differentiable in the first argument and continuous in the second argument function $W(\xi, \mu)$ such that

$$\frac{\partial W(\varphi, \rho, \mu)}{\partial \varphi} > 0, \quad (\varphi, \rho) \in \Pi, \mu \in \Lambda,$$ 

$$(\varphi + 2\pi, \rho, \mu) = W(\varphi, \rho, \mu) + 2\pi, \quad (\varphi, \rho) \in \Pi, \mu \in \Lambda.$$ 

On $\mathbb{R}^{n-2} \times \Lambda$ let $V(\zeta, \mu)$ be a continuously differentiable in the first argument and continuous in the second argument function such that $\frac{\partial V(0, \mu)}{\partial \zeta} = 0$ and the following coercivity condition

$$\lim_{||\zeta|| \to \infty} V(\zeta, \mu) = +\infty$$ 

holds true.

For each $\mu \in \Lambda$, choose $\rho_1 := \rho_1(\mu)$, $\rho_2 := \rho_2(\mu)$ such that $0 \leq \rho_1 < \rho_2$ and for $\vartheta_0 := \min V(\zeta, \mu)$, take $\vartheta := \vartheta(\mu)$ such that $\vartheta > \vartheta_0$. We define the following domain

$$\Omega_\mu(\vartheta, \rho_1, \rho_2) = \{ z \in \mathbb{R}^n : V(pz, \mu) < \vartheta, \rho_1 < ||qz|| < \rho_2 \}.$$ 

We assume that on $[0, T]$ continuous functions $\alpha(t, \mu)$, $\beta(t, \mu)$ are given such that, for each $\mu \in \Lambda$ and a.e. $t \in [0, T]$ the following holds:

$$\sup_{z \in \Omega_\mu(\vartheta, \rho_1, \rho_2)} \sup_{y \in F(t, z, \mu)} \left( \frac{\partial W(qz, \mu)}{\partial qz}, qy \right) < \alpha(t, \mu),$$ 

$$\inf_{z \in \Omega_\mu(\vartheta, \rho_1, \rho_2)} \inf_{y \in F(t, z, \mu)} \left( \frac{\partial W(qz, \mu)}{\partial qz}, qy \right) > \beta(t, \mu).$$ 

Let us give the following definition.

**Definition 2.1.** A pair of functions $\{V(\zeta, \mu), W(\xi, \mu)\}$ with properties (2.2)-(2.6) is called the multivalent guiding function (MGF) for inclusion (2.1) on $\Omega_\mu(\vartheta, \rho_1, \rho_2)$ if the following conditions hold true:

$$\sup_{t \in [0, T]} \sup_{y \in F(t, z, \mu)} \left| \frac{\langle qy, qz \rangle}{||qz||} \right| < \frac{\rho_2 - \rho_1}{2T}, \quad z \in \Omega_\mu(\vartheta, \rho_1, \rho_2),$$ 

$$(\vartheta, \rho_1, \rho_2),$$ 

$$\left( \frac{\partial V(pz, \mu)}{\partial pz}, py \right) < 0, \quad y \in F(t, z, \mu), V(pz, \mu) \geq \vartheta, ||qz|| \leq \rho_2,$$ 

$$2\pi(N_\mu - 1) < \int_0^T \alpha(\tau, \mu) d\tau, \quad \int_0^T \beta(\tau, \mu) d\tau < 2\pi N_\mu,$$ 

where $N_\mu$ is an integer and $\alpha(t, \mu), \beta(t, \mu)$ are functions from (2.5), (2.6), respectively.
For all $\mu \in \Lambda$ and $\rho_0(\mu) = \rho_0 = (\rho_1 + \rho_2)/2$ we define
\[
G_\mu(\vartheta, \rho_0) = \{z \in \mathbb{R}^n : V(pz, \mu) < \vartheta, \|qz\| < \rho_0\},
\]
\[
\partial G_\mu(\vartheta, \rho_0) = \partial G_\zeta(\vartheta) \times \overline{G_\xi(\rho_0)} \cup \overline{G_\zeta(\vartheta)} \times \partial G_\xi(\rho_0).
\]

Let us define the map $\nabla V : \mathbb{R}^{n-2} \times \Lambda \rightarrow \mathbb{R}^{n-2}$ by
\[
\nabla V(\zeta, \mu) = \frac{\partial V_\zeta(\zeta, \mu)}{\partial \zeta}.
\]

We assume that for fixed $r > \eta > 0$, $\mu_0 \in \Lambda$ the following condition is satisfied
\[
\nabla V(pz, \mu) \neq 0
\]
for all $z \in \overline{G_\mu(\vartheta, \rho_0)}$ and $\mu : r - \eta \leq |\mu - \mu_0| \leq r + \eta$.

Now we are in position to formulate the main result of this paper.

**Theorem 2.2.** Suppose that conditions (H$_1$) and (H$_2$) are satisfied. Let \{V(\zeta, \mu), W(\xi, \mu)\} be MGF for inclusion \ref{2.1} on $\Omega(\vartheta, \rho_1, \rho_2)$ for each $\mu : |\mu - \mu_0| \geq r$.

Then one of the following cases occurs:

(i) there exists a sequence \{\{(y_n, \mu_n)\}_{n=1}^\infty\}, $\mu_n \rightarrow \pi$, $|\pi - \mu_0| = r$, $y_n \in \mathbb{R}^n$, $y_n \neq y_m$ for $n \neq m$, and a sequence $(z_n)$ of solutions of problem \ref{2.1} for $\mu = \mu_n$ such that $z_n(0) = z_n(T) = y_n \rightarrow 0$ and $z_n \rightarrow z$, where $z$ is a solution of problem \ref{2.1} for $\mu = \pi$ such that $z(0) = z(T) = 0$;

(ii) there exists a connected set $C$ of points $(y, \mu)$ with $y \neq 0$ such that

- $(0, \pi) \in \overline{C}$ where $|\pi - \mu_0| < r$,
- $C$ is unbounded or $\overline{C} \cap \partial U \neq \emptyset$ or $(0, \tilde{\mu}) \in \overline{C}$ for some $\tilde{\mu} \in \Lambda : |\tilde{\mu} - \mu_0| > r$,
- each point $(y, \mu) \in \overline{C}$ corresponds to a solution $z : [0, T] \rightarrow \mathbb{R}^n$ of inclusion \ref{2.1} with $z(0) = z(T) = y$. In particular, there is a sequence $(z_n)_{n=1}^\infty$ of solutions to inclusion \ref{2.1} for $\mu = \mu_n$, $z_n(0) = z_n(T) = y_n$, where $\mu_n \rightarrow \pi$ in $\Lambda$ with $|\pi - \mu_0| < r$, converging to a solution $z$ to inclusion \ref{2.1} for $\mu = \pi$, $z(0) = z(T) = 0$.

**Proof.**

**Claim 1.** First of all we shall show that the trajectory $z(t)$, starting at $\partial G_\mu(\vartheta, \rho_0)$, satisfies the following estimate
\[
z(t) \in G_\mu(\vartheta, \rho_2)
\]
for $t \in (0, T]$. We consider a component $\zeta(t)$ of the trajectory $z(t)$. If $\zeta(0)$ is an interior point of $G_\zeta(\vartheta)$, then there exists $\varepsilon_1 > 0$, such that
\[
\zeta(t) \in G_\zeta(\vartheta), \quad t \in (0, \varepsilon_1).
\]

Let us take $\zeta(0) \in \partial G_\zeta(\vartheta)$, i.e., $V(\zeta(0), \mu) = \vartheta$. Since $\|\xi(0)\| \leq \rho_0 < \rho_2$, from \ref{2.8} it follows that
\[
\langle \nabla V(\zeta(0), \mu), py \rangle < 0 \quad \text{for all} \quad y \in F(0, \zeta(0), \xi(0), \mu).
\]

Then for small $t > 0$ we have $V(\zeta(t), \mu) < \vartheta$. So for some $\varepsilon_1 > 0$ estimate \ref{2.12} holds true.

Considering the component $\xi(t)$, obviously for some $\varepsilon_2 > 0$ we obtain
\[
\xi(t) \in G_\xi(\rho_2), \quad t \in (0, \varepsilon_2).
\]

From \ref{2.12} and \ref{2.13} it follows that $z(t) \in G_\mu(\vartheta, \rho_2)$, $0 < t < \min\{\varepsilon_1, \varepsilon_2\}$. It means that there exists a positive number
\[
t_* = \sup \{t > 0 : z(t) \in G_\mu(\vartheta, \rho_2)\}.
\]
Claim 2. Let us take $t_\ast > T$. Since $z(t_\ast) \in \partial G_\mu(\vartheta, \rho_2)$, we have $V(\zeta(t_\ast), \mu) = \vartheta$ or $\|\xi(t_\ast)\| = \rho_2$. From (2.8) it follows that $V(\zeta(t_\ast), \mu) < \vartheta$. Therefore $\|\xi(t_\ast)\| = \rho_2$ and from $\|\xi(0)\| \leq \rho_0$ we obtain

$$\|\xi(t_\ast)\| - \|\xi(0)\| \geq \rho_2 - \rho_0 = (\rho_2 - \rho_1)/2.$$  

Let $\varphi(t)$, $\rho(t)$ be polar coordinates of $\xi(t)$. Then

$$\rho(t_\ast) - \rho(0) \geq (\rho_2 - \rho_1)/2. $$

Therefore,

$$\max_{t \in [0, t_\ast]} \|\rho'(t)\| \geq (\rho_2 - \rho_1)/2t_\ast. \quad (2.14)$$

On the other hand, since

$$\|\rho'(t)\| = \frac{\langle qy, \xi(t) \rangle}{\|\xi(t)\|}, \quad y \in F(t, \zeta(t), \xi(t), \mu),$$

and $z(t) \in G_\mu(\vartheta, \rho_2)$ for $t \in (0, t_\ast)$, from (2.7) it follows the estimate

$$\max_{t \in [0, t_\ast]} \|\rho'(t)\| < (\rho_2 - \rho_1)/2T. \quad (2.15)$$

Comparing (2.14) and (2.15) we see that $t_\ast > T$. Therefore, any trajectory $z(\cdot)$, starting at $\partial G_\mu(\vartheta, \rho_0)$ for $t \in (0, T]$, satisfies the estimate $z(t) \in G_\mu(\vartheta, \rho_2)$.

Claim 2. Let us take $z(0) \in \overline{G_\zeta(\vartheta)} \times \partial G_\xi(\nu_0)$ and

$$\rho(0) = \rho_0 = (\rho_1 + \rho_2)/2.$$ 

Since $z(t) \in G_\mu(\vartheta, \rho_2)$ for $t \in (0, T]$, we have the estimate

$$\max_{t \in [0, t_\ast]} |\rho'(t)| < (\rho_2 - \rho_1)/2T.$$ 

Therefore,

$$\rho(t) > \rho(0) - (\rho_2 - \rho_1)t/2T, \quad t \in (0, T]$$

and we obtain

$$\rho(t) > \rho_1, \quad t \in (0, T].$$

Then

$$z(t) \in \Omega_\mu(\vartheta, \rho_1, \rho_2), \quad t \in (0, T].$$

Let us denote $\omega(t, \mu) = W(\xi(t), \mu).$ The map $\nabla W : \mathbb{R}^2 \times \Lambda \rightarrow \mathbb{R}^2$ is defined as

$$\nabla W(\xi, \mu) = \frac{\partial W(\xi, \mu)}{\partial \xi}.$$ 

Then for each $\mu \in \Lambda$

$$\omega'(t, \mu) = \langle \nabla W(\xi(t), \mu), qy \rangle,$$

where $y \in F(t, z, \mu)$ and

$$\beta(t, \mu) < \omega'(t, \mu) < \alpha(t, \mu).$$

Now by using the integral representation of the function $\omega(t, \mu)$ we obtain

$$\int_0^T \beta(\tau, \mu) d\tau < \omega(T, \mu) - \omega(0, \mu) < \int_0^T \alpha(\tau, \mu) d\tau. \quad (2.16)$$

From (2.9) it follows that

$$2\pi(N_\mu - 1) < \omega(T, \mu) - \omega(0, \mu) < 2\pi N_\mu.$$ 

Then we have that $\xi(T) \neq \xi(0)$ for $\mu : r + \varepsilon/2 \leq |\mu - \mu_0| < r + \varepsilon$, $1/2 < \lambda < 1$ and $z_0 \in \partial G_\mu(\vartheta, \rho_0)$. 

Claim 3. Let us define a map $f_V : \mathbb{R}^{n-2} \times \Lambda \to \mathbb{R}^{n-2}$ by
\[
 f_V(\zeta, \mu) = \begin{cases} 
 \nabla V(\zeta, \mu), & \text{if } \|\nabla V(\zeta, \mu)\| \leq 1, \\
 \nabla V(\zeta, \mu) - \left( \frac{\|\nabla V(\zeta, \mu)\|}{\|\nabla V(\zeta, \mu)\|} \right), & \text{if } \|\nabla V(\zeta, \mu)\| > 1,
\end{cases}
\]
and a map $P : [0, T] \times \mathbb{R}^{n-2} \times \Lambda \to \mathbb{R}^{n-2}$ by the formula
\[
P(t, \zeta_0, \mu) = \zeta(t) - \zeta_0,
\]
where $\zeta : [0, T] \to \mathbb{R}^{n-2}$ is the unique solution to the problem $\zeta'(t) = f_V(\zeta, \mu)$, $\zeta(0) = \zeta_0$. It is clearly a well-defined single-valued (continuous) map since $f_V$ is bounded and it satisfies the Lipschitz condition with respect to the second variable.

Let $\Phi : [0, T] \times \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$ be given by
\[
\Phi(t, z, \mu) = \{ z(t) - z_0 \mid z'(s) \in F(s, z(s), \mu) \text{ a.e. } s \in [0, T], z(0) = z_0, pz(0) = \zeta_0 \}.
\]
Assume that there is $\varepsilon : 0 < \varepsilon \leq \eta/2$ such that, for all $\mu : r + \varepsilon/2 \leq |\mu - \mu_0| < r + \varepsilon$ and $z_0 \in \overline{G_{\mu}(\vartheta, \rho_0)}$
\[
0 \notin \Phi(T, z_0, \mu).
\]
(2.17)

If condition (2.17) is not satisfied, then there are sequences $\mu_n \to \mu \in \Lambda$ and $y_n \to 0$ in $\mathbb{R}^n$ such that $\mu_n \neq \mu_m, y_n \neq y_m$ for $n \neq m, |\mu_n - \mu_0| = r$ and
\[
0 \in \Phi(T, y_n, \mu_n).
\]
Hence there is a sequence of solutions $z_n : [0, T] \to \mathbb{R}^n$ to problem (2.1) for $\mu = \mu_n$ such that $z_n(0) = z_n(T) = z_n$. Notice that the Gronwall inequality implies $z_n \to \bar{z}$ in $C([0, T]; \mathbb{R}^n)$. Then $\bar{z}$ is the solution of problem (2.1) for $\mu = \bar{\mu}$ such that $\bar{z}(0) = \bar{z}(T) = 0$.

Claim 4. For each $\mu \in \Lambda : r + \varepsilon/2 \leq |\mu - \mu_0| < r + \varepsilon, z_0 \in \partial G_\zeta(\vartheta) \times \overline{G_\zeta(\rho_0)}$ and $t \in [0, T]$, we shall show that
\[
0 \neq P(t, \zeta_0, \mu).
\]
Indeed, take the solution $\zeta(t)$ to the problem $\zeta'(t) = f_V(\zeta(t), \mu), \zeta(0) \in \partial G_\zeta(\vartheta)$. Then
\[
V(\zeta(t), \mu) - V(\zeta(0), \mu) = \int_0^t \langle \nabla V(\zeta(s), \mu), \zeta'(s) \rangle ds \\
= \int_0^t \langle \nabla V(\zeta(s), \mu), f_V(\zeta(s), \mu) \rangle ds > 0.
\]
Thus $\zeta(t) \neq \zeta(0) = \zeta_0$ and
\[
0 \neq P(t, \zeta_0, \mu).
\]

Claim 5. For $t \in [0, T]$, consider a map $h_t : \mathbb{R}^{n-2} \times \Lambda \times [0, 1] \to \mathbb{R}^{n-2}$,
\[
h_t(\zeta_0, \mu, \lambda) = (1 - \lambda)\nabla V(\zeta_0, \mu) + \lambda P(t, \zeta_0, \mu).
\]
We shall show that there is $\tau \in [0, T]$ such that, for $\mu \in \Lambda : r + \varepsilon/2 \leq |\mu - \mu_0| < r + \varepsilon, \zeta(0) \in \partial G_\zeta(\vartheta)$ and $\lambda \in [0, 1],$
\[
0 \neq h_\tau(\zeta_0, \mu, \lambda).
\]
Indeed, the continuity of $f_V, V,$ and (2.10) imply that there is $\tau > 0$ such that for $\zeta(0) \in \partial G_\zeta(\vartheta), \zeta'_0 \in \mathbb{R}^{n-2}$ such that $|V(\zeta_0, \mu) - V(\zeta'_0, \mu)| \leq \tau$, we have
\[
\langle \nabla V(\zeta_0, \mu), f_V(\zeta'_0, \mu) \rangle > 0.
\]
Now suppose that for $\mu \in \Lambda : r + \varepsilon/2 \leq |\mu - \mu_0| < r + \varepsilon$; $\zeta(0) \in \partial G_\zeta(\vartheta)$ and $\lambda \in [0, 1]$

\[ h_\tau(\zeta_0, \mu, \lambda) = 0. \]

Then from \[(2.10)\] and Claim 4 for each $\lambda \in (0, 1)$ we obtain

\[
\zeta(\tau) - \zeta_0 = \frac{\lambda - 1}{\lambda} \nabla V(\zeta_0, \mu),
\]

where $\zeta' = f_V(\zeta, \mu)$ on $[0, T]$ and $\zeta(0) = \zeta_0$. Since $|f_V| \leq 1$, it is clear that, for all $\theta \in [0, \tau]$, $|V(\zeta(\theta), \mu) - V(\zeta_0, \mu)| \leq \tau$. Thus

\[
0 > \langle \zeta(\tau) - \zeta_0, \nabla V(\zeta_0, \mu) \rangle = \int_0^\tau (f_V(\zeta(\theta), \mu), \nabla V(\zeta_0, \mu)) d\theta > 0,
\]

a contradiction.

**Claim 6.** Now let

\[
k(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in [0, \frac{1}{2}), \\
2 - 2\lambda, & \text{if } \lambda \in [\frac{1}{2}, 1]; 
\end{cases}
\]

and

\[
t(\lambda) = \begin{cases} 
2(T - \tau)\lambda + \tau, & \text{if } \lambda \in [0, \frac{1}{2}), \\
T, & \text{if } \lambda \in [\frac{1}{2}, 1]. 
\end{cases}
\]

Let us consider a multimap $\Psi' : \mathbb{R}^n \times \Lambda \times [0, 1] \to \mathbb{R}^n$, given by

\[
\Psi'(z_0, \mu, \lambda) = \{ z(t(\lambda)) - z_0 \mid z'(\theta) \in k(\lambda)f_V(pz(\theta), \mu) + (1 - k(\lambda))F(\theta, z(\theta), \mu) \},
\]

where $z(0) = z_0$, $pz(0) = \zeta_0$. It is clear that

\[
\Psi'(z_0, \mu, \lambda) = \begin{cases} 
h_\tau(\zeta_0, \mu, 1) = P(\tau, \zeta_0, \mu), & \text{if } \lambda = 0, \\
\Phi(T, z_0, \mu) := \Phi_T, & \text{if } \lambda = 1. 
\end{cases}
\]

In view of condition \[(2.17)\] and Claim 5, if $\mu : r + \varepsilon/2 \leq |\mu - \mu_0| < r + \varepsilon$ and $z_0 \in G_\mu(\vartheta, \rho_0)$, then

\[
0 \notin \Psi'(z_0, \mu, \lambda), \quad \lambda = 0, 1.
\]

Now we show that also for $z_0 \in \partial G_\mu(\vartheta, \rho_0)$

\[
0 \notin \Psi'(z_0, \mu, \lambda) \quad \text{for all } \lambda \in (0, 1),
\]

i.e., each $\psi' \in \Psi'(z_0, \mu, \lambda)$ is non-zero. By the definition of $\Psi'$,

\[
\psi' = z(t(\lambda)) - z_0,
\]

where the function $z : [0, T] \to \mathbb{R}^n$ is such that $z(0) = z_0$, $pz(0) = \zeta_0$ and

\[
z'(\theta) \in k(\lambda)f_V(pz(\theta), \mu) + (1 - k(\lambda))F(\theta, z(\theta), \mu),
\]

i.e., $z'(\theta) = k(\lambda)f_V(pz(\theta), \mu) + (1 - k(\lambda))y(\theta)$, where $y(\theta) \in F(\theta, z(\theta), \mu)$. Then, for $0 < \lambda \leq 1/2$ by Claim 4 we obtain

\[
\psi' = P(t(\lambda), \zeta_0, \mu) \neq 0.
\]

If $1/2 < \lambda < 1$, then

\[
V(p(\psi' + z_0), \mu) - V(pz_0, \mu) \geq k(\lambda) \int_0^T \langle \nabla V(pz(\theta), \mu), f_V(pz(\theta), \mu) \rangle d\theta > 0.
\]
for $z_0 \in \partial G_\zeta(\vartheta) \times \partial G_\zeta(\rho_0)$. Hence $p\psi' \neq 0$.

Now we shall show that for $z_0 \in G_\zeta(\vartheta) \times \partial G_\zeta(\rho_0)$ and $\mu : r + \varepsilon / 2 \leq |\mu - \mu_0| < r + \varepsilon$, $1 / 2 < \lambda < 1$, $q\psi' \neq 0$.

We have

$$\omega(T, \mu) - \omega(0, \mu) = \int_0^T \omega'(\tau, \mu) d\tau = \int_0^T (\nabla W(\xi(\tau), \mu), q\tilde{y}) d\tau = (1 - k(\lambda)) \int_0^T (\nabla W(\xi(\tau), \mu), qy) d\tau,$$

where $\tilde{y} \in k(\lambda)f(\tau(\mu) + (1 - k(\lambda))F(\tau, z(\tau), \mu))$, $y \in F(\tau, z(\tau), \mu)$. Then by (2.16) we have

$$(1 - k(\lambda))2\pi(N_\mu - 1) < \omega(T, \mu) - \omega(0, \mu) < (1 - k(\lambda))2\pi N_\mu.$$

It follows that $\xi(T) \neq \xi(0)$, i.e., $q\psi' \neq 0$. $\mu : r + \varepsilon / 2 \leq |\mu - \mu_0| < r + \varepsilon$, $1 / 2 < \lambda < 1$ and $z_0 \in G_\zeta(\vartheta) \times \partial G_\zeta(\rho_0)$.

Claim 7. Finally, we consider a multimap $\Psi : \mathbb{R}^n \times \Lambda \times [0, 1] \to \mathbb{R}^n$ given by

$$\Psi(z_0, \mu, \lambda) = \begin{cases} h_\tau(pz_0, \mu, 2\lambda), & \text{if } \lambda \in [0, 1/2], \\ \Psi'(z_0, \mu, 2\lambda - 1), & \text{if } \lambda \in (1/2, 1]. \end{cases}$$

In view of assumptions (2.10), (2.17), and Claims 5 and 6 for $\mu : r + \varepsilon / 2 \leq |\mu - \mu_0| < r + \varepsilon$ and $z_0 \in G_\mu(\vartheta, \rho_0)$ we obtain

$$0 \notin \Psi(z_0, \mu, \lambda), \quad \lambda = 0, 1.$$

For $\mu : r + \varepsilon / 2 \leq |\mu - \mu_0| < r + \varepsilon$ and $z_0 \in \partial G_\mu(\vartheta, \rho_0)$ we also have

$$0 \notin \Psi(z_0, \mu, \lambda), \quad \lambda \in [0, 1].$$

Then

$$\Bi(\Phi_T) = \Bi(\nabla V) \neq 0.$$

The assertion follows from Lemma 114.

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