Schur-convexity for Lehmer mean of $n$ variables

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Abstract

Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity for Lehmer mean of $n$ variables are investigated, and some mean value inequalities of $n$ variables are established. ©2016 All rights reserved.

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1. Introduction and preliminaries

Throughout the paper we denote the set of $n$-dimensional row vector on the real number field by $\mathbb{R}^n$. Also,

$$\mathbb{R}_+^n = \{ \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \ldots, n \}.$$ 

In particular, $\mathbb{R}_1^1$ and $\mathbb{R}_1^1$ denoted by $\mathbb{R}$ and $\mathbb{R}_+$ respectively.

For $x, y > 0$ and $p \in \mathbb{R}$, the Lehmer mean values $L_p(x, y)$ were introduced by Lehmer [13] as follows:

$$L_p(x, y) = \frac{x^p + y^p}{x^{p-1} + y^{p-1}}.$$ 

Many mean values are special cases of the Lehmer mean values, for example

$$A(x, y) = \frac{x + y}{2} = L_1(x, y)$$

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is the arithmetic mean,

\[ G(x, y) = \sqrt{xy} = L_{\frac{1}{2}}(x, y) \]

is the geometric mean,

\[ H(x, y) = \frac{2xy}{x + y} = L_0(x, y) \]

is the harmonic mean,

\[ \bar{H}(x, y) = \frac{x^2 + y^2}{x + y} = L_2(x, y) \]

is the anti-harmonic mean.

Investigation of the elementary properties and inequalities for \( L_p(x, y) \) has attracted the attention of a considerable number of mathematicians (see [1–3, 10–12, 14, 21, 23, 26, 28–31]).

In 2009, Gu and Shi [11] discussed the Schur convexity and Schur geometric convexity of the Lehmer means \( L_p(x, y) \) with respect to \((x, y) \in \mathbb{R}^2_+ \) for fixed \( p \). Subsequently, Xia and Chu [36] researched the Schur harmonic convexity of the Lehmer means \( L_p(x, y) \) with respect to \((x, y) \in \mathbb{R}^2_+ \) for fixed \( p \).

Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+ \). For Schur-convexity and Schur-geometric convexity of \( n \) variables Lehmer mean,

\[ L_p(x) = L_p(x_1, x_2, \ldots, x_n) = \frac{\sum_{i=1}^{n} x_i^p}{\sum_{i=1}^{n} x_i^{p-1}}, \]

Gu and Shi [11] obtained the following results.

**Theorem 1.1.** Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+ \) and \( p \in \mathbb{R} \). If \( 1 \leq p \leq 2 \), then \( L_p(x) \) is Schur-convex with \( x \in \mathbb{R}^n_+ \), if \( 0 \leq p \leq 1 \), then \( L_p(x) \) is Schur-concave with \( x \in \mathbb{R}^n_+ \).

Furthermore, Gu and Shi [11] proposed the following conjecture.

**Conjecture 1.2.** If \( p \geq 2 \), then \( L_p(x) \) is Schur-convex with \( x \in \mathbb{R}^n_+ \), if \( p \leq 0 \), then \( L_p(x) \) is Schur-concave with \( x \in \mathbb{R}^n_+ \).

We first point out that this conjecture does not hold.

In fact, for \( n = 3, p = 3 \), by computing, we have

\[ \Delta := (x_1 - x_2) \left( \frac{\partial L_3(x)}{\partial x_1} - \frac{\partial L_3(x)}{\partial x_2} \right) = \frac{(x_1 - x_2)^2 \lambda(x)}{(x_1^3 + x_2^3 + x_3^3)^2}, \]

where

\[ \lambda(x) = \lambda(x_1, x_2, x_3) = 3(x_1 + x_2)(x_1^2 + x_2^2 + x_3^2) - 2(x_1^3 + x_2^3 + x_3^3), \]

if \( x = (1, 3, 7) \), then \( \lambda(x) = -34 \), so that \( \Delta < 0 \), but by taking \( y = (1, 2, 3) \), then \( \lambda(y) = 54 \), so that \( \Delta > 0 \). According to Lemma 2.4 in second section, we assert that the Schur-convexity of \( L_3(x_1, x_2, x_3) \) is not determined on the whole \( \mathbb{R}_+^3 \).

It can easily be shown that \( L_{-2}(x_1, x_2, x_3) = \frac{1}{L_3(x_1^{-1}, x_2^{-1}, x_3^{-1})} \), since the Schur-convexity of \( L_3(x_1, x_2, x_3) \) is not determined on the whole \( \mathbb{R}_+^3 \), \( L_{-2}(x_1, x_2, x_3) \) so does.

In this paper, we study Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity of \( L_p(x) \) on certain subsets of \( \mathbb{R}_+^n \). As consequences, some interesting inequalities are obtained.

Our main results are as follows:

**Theorem 1.3.** Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n \), \( n \geq 2 \) and \( p \in \mathbb{R} \).

(I) If \( p \geq 2 \), then for any \( a > 0 \), \( L_p(x) \) is Schur-convex with \( x \in \left[ \frac{(p-2)a}{p}, \frac{a}{p} \right]^n \).

(II) If \( p < 0 \), then for any \( a > 0 \), \( L_p(x) \) is Schur-concave with \( x \in \left[ \frac{a}{p}, \frac{(p-2)a}{p} \right]^n \).
Theorem 1.4. Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+ \), \( n \geq 2 \) and \( p \in \mathbb{R} \).

1. If \( p < \frac{1}{2} \) and \( p \neq 0 \), then for any \( a > 0 \), \( L_p(x) \) is Schur-geometrically concave with \( x \in [a, (p-1)^2a]^n \).

2. If \( p > \frac{1}{2} \), then for any \( a > 0 \), \( L_p(x) \) is Schur-geometrically convex with \( x \in [(p-1)^2a, a]^n \).

3. If \( p = 0 \), then \( L_p(x) \) is Schur-geometrically convex with \( x \in \mathbb{R}^n_+ \).

Theorem 1.5. Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+, n \geq 2 \) and \( p \in \mathbb{R} \).

1. If \( 0 \leq p \leq 1 \), then \( L_p(x) \) is Schur-harmonically convex with \( x \in \mathbb{R}^n_+ \), if \( -1 \leq p \leq 0 \), then \( L_p(x) \) is Schur-harmonically concave with \( x \in \mathbb{R}^n_+ \).

2. If \( p > 1 \), then for any \( a > 0 \), \( L_p(x) \) is Schur-harmonically convex with \( x \in \mathbb{R}^n_+ \).

3. If \( p < -1 \), then for any \( a > 0 \), \( L_p(x) \) is Schur-harmonically concave with \( x \in [a, (p-1)^2a]^n \).

2. Definitions and lemmas

We need the following definitions and lemmas.

Definition 2.1 ([17, 27]). Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \).

1. \( x \) is said to be majorized by \( y \) (in symbols \( x \prec y \)), if \( \sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i \), for \( k = 1, 2, \ldots, n-1 \) and \( \sum_{i=1}^n x_i = \sum_{i=1}^n y_i \), where \( x_1 \geq \cdots \geq x_n \) and \( y_1 \geq \cdots \geq y_n \) are rearrangements of \( x \) and \( y \) in a descending order.

2. \( \Omega \subset \mathbb{R}^n \) is called a convex set, if \( (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \ldots, \alpha x_n + \beta y_n) \in \Omega \), for any \( x \) and \( y \in \Omega \), where \( \alpha, \beta \in [0, 1] \) with \( \alpha + \beta = 1 \).

3. Let \( \Omega \subset \mathbb{R}^n \), \( \varphi : \Omega \rightarrow \mathbb{R} \) is said to be a Schur-convex function on \( \Omega \), if \( x \prec y \) on \( \Omega \) implies \( \varphi(x) \leq \varphi(y) \). \( \varphi \) is said to be a Schur-concave function on \( \Omega \), and only if \( -\varphi \) is Schur-convex function.

Definition 2.2 ([20, 44]). Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n_+ \).

1. \( \Omega \subset \mathbb{R}^n_+ \) is called a geometrically convex set, if \( (x_1^\alpha y_1^\beta, x_2^\alpha y_2^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega \), for any \( x \) and \( y \in \Omega \), where \( \alpha, \beta \in [0, 1] \) with \( \alpha + \beta = 1 \).

2. Let \( \Omega \subset \mathbb{R}^n_+, \varphi : \Omega \rightarrow \mathbb{R}_+ \) is said to be a Schur-geometrically convex function on \( \Omega \), if \( \ln x_1, \ln x_2, \ldots, \ln x_n \prec (\ln y_1, \ln y_2, \ldots, \ln y_n) \) on \( \Omega \) implies \( \varphi(x) \leq \varphi(y) \). \( \varphi \) is said to be a Schur-geometrically concave function on \( \Omega \), if and only if \( -\varphi \) is Schur-geometrically convex function.

Definition 2.3 ([4, 18]). Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n_+ \).

1. A set \( \Omega \subset \mathbb{R}^n_+ \) is said to be a harmonically convex set, if

\[
\left( \frac{x_1 y_1}{\lambda x_1 + (1-\lambda)y_1}, \frac{x_2 y_2}{\lambda x_2 + (1-\lambda)y_2}, \ldots, \frac{x_n y_n}{\lambda x_n + (1-\lambda)y_n} \right) \in \Omega,
\]

for every \( x, y \in \Omega \) and \( \lambda \in [0, 1] \).

2. A function \( \varphi : \Omega \rightarrow \mathbb{R}_+ \) is said to be a Schur-harmonically convex function on \( \Omega \), if \( \left( \frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n} \right) \prec (\frac{1}{y_1}, \frac{1}{y_2}, \ldots, \frac{1}{y_n}) \) implies \( \varphi(x) \leq \varphi(y) \). A function \( \varphi \) is said to be a Schur-harmonically concave function on \( \Omega \), if and only if \( -\varphi \) is a Schur-harmonically convex function.

Lemma 2.4 ([17, 27]). Let \( \Omega \subset \mathbb{R}^n \) is convex set, and has a nonempty interior set \( \Omega^0 \). Let \( \varphi : \Omega \rightarrow \mathbb{R} \) be continuous on \( \Omega \) and differentiable in \( \Omega^0 \). Then \( \varphi \) is the Schur – convex (or Schur – concave, resp.) function, if and only if it is symmetric on \( \Omega \) and if

\[
(x_1 - x_2) \left( \frac{\partial \varphi(x)}{\partial x_1} - \frac{\partial \varphi(x)}{\partial x_2} \right) \geq 0, \quad (or \ \leq 0 \ resp.),
\]

holds for any \( x = (x_1, x_2, \cdots, x_n) \in \Omega^0 \).
Remark 2.5 ([9, 19]). It is easy to see that the condition (2.1) is equivalent to

$$\frac{\partial \varphi(x)}{\partial x_i} \leq \frac{\partial \varphi(x)}{\partial x_{i+1}}, \quad (\text{or } \geq \text{ resp.}), \quad i = 1, \ldots, n - 1, \quad \text{for all } x \in D \cap \Omega,$$

where $D = \{ x : x_1 \leq x_2 \leq \cdots \leq x_n \}$.

The condition (2.1) is also equivalent to

$$\frac{\partial \varphi(x)}{\partial x_i} \geq \frac{\partial \varphi(x)}{\partial x_{i+1}}, \quad (\text{or } \leq \text{ resp.}), \quad i = 1, \ldots, n - 1, \quad \text{for all } x \in E \cap \Omega,$$

where $E = \{ x : x_1 \geq x_2 \geq \cdots \geq x_n \}$.

Lemma 2.6 ([20, 41]). Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric geometrically convex set with a nonempty interior $\Omega^0$. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on $\Omega$ and differentiable on $\Omega^0$. Then $\varphi$ is a Schur-geometrically convex (or Schur-geometrically concave, resp.) function, if and only if $\varphi$ is symmetric on $\Omega$ and

$$(x_1 - x_2) \left( x_1 \frac{\partial \varphi(x)}{\partial x_1} - x_2 \frac{\partial \varphi(x)}{\partial x_2} \right) \geq 0, \quad (\text{or } \leq 0 \text{ resp.}), \quad (2.2)$$

holds for any $x = (x_1, x_2, \ldots, x_n) \in \Omega^0$.

Remark 2.7. It is easy to see that the condition (2.2) is equivalent to

$$x_i \frac{\partial \varphi(x)}{\partial x_i} \leq x_{i+1} \frac{\partial \varphi(x)}{\partial x_{i+1}}, \quad (\text{or } \geq \text{ resp.}), \quad i = 1, \ldots, n - 1, \quad \text{for all } x \in D \cap \Omega,$$

where $D = \{ x : x_1 \leq x_2 \leq \cdots \leq x_n \}$.

The condition (2.2) is also equivalent to

$$x_i \frac{\partial \varphi(x)}{\partial x_i} \geq x_{i+1} \frac{\partial \varphi(x)}{\partial x_{i+1}}, \quad (\text{or } \leq \text{ resp.}), \quad i = 1, \ldots, n - 1, \quad \text{for all } x \in E \cap \Omega,$$

where $E = \{ x : x_1 \geq x_2 \geq \cdots \geq x_n \}$.

Lemma 2.8 ([4, 18]). Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric harmonically convex set with a nonempty interior $\Omega^0$. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on $\Omega$ and differentiable on $\Omega^0$. Then $\varphi$ is a Schur-harmonically convex (or Schur-harmonically concave, resp.) function, if and only if $\varphi$ is symmetric on $\Omega$ and

$$(x_1 - x_2) \left( x_1^2 \frac{\partial \varphi(x)}{\partial x_1} - x_2^2 \frac{\partial \varphi(x)}{\partial x_2} \right) \geq 0, \quad (\text{or } \leq 0 \text{ resp.}), \quad (2.3)$$

holds for any $x = (x_1, x_2, \ldots, x_n) \in \Omega^0$.

Remark 2.9. It is easy to see that the condition (2.3) is equivalent to

$$x_i^2 \frac{\partial \varphi(x)}{\partial x_i} \leq x_{i+1}^2 \frac{\partial \varphi(x)}{\partial x_{i+1}}, \quad (\text{or } \geq \text{ resp.}), \quad i = 1, \ldots, n - 1, \quad \text{for all } x \in D \cap \Omega,$$

where $D = \{ x : x_1 \leq x_2 \leq \cdots \leq x_n \}$.

The condition (2.3) is also equivalent to

$$x_i^2 \frac{\partial \varphi(x)}{\partial x_i} \geq x_{i+1}^2 \frac{\partial \varphi(x)}{\partial x_{i+1}}, \quad (\text{or } \leq \text{ resp.}), \quad i = 1, \ldots, n - 1, \quad \text{for all } x \in E \cap \Omega,$$

where $E = \{ x : x_1 \geq x_2 \geq \cdots \geq x_n \}$. 
Lemma 2.10. Let \( x_1 \geq x_2 \geq \cdots \geq x_n > 0, \ m \in \mathbb{R} \). Then
\[
x_1 \geq \frac{x_1^m + x_2^m + \cdots + x_n^m}{x_1^{m-1} + x_2^{m-1} + \cdots + x_n^{m-1}} \geq x_n.
\]

Proof.
\[
x_1(x_1^{m-1} + x_2^{m-1} + \cdots + x_n^{m-1}) - (x_1^m + x_2^m + \cdots + x_n^m)
= x_1^{m-1}(x_1 - x_1) + x_2^{m-1}(x_1 - x_2) + \cdots + x_n^{m-1}(x_1 - x_n) \geq 0,
\]
\[
x_n(x_1^{m-1} + x_2^{m-1} + \cdots + x_n^{m-1}) - (x_1^m + x_2^m + \cdots + x_n^m)
= x_n^{m-1}(x_n - x_1) + x_2^{m-1}(x_n - x_2) + \cdots + x_n^{m-1}(x_n - x_n) \leq 0.
\]

We have thus proved the Lemma 2.10.

Lemma 2.11 (17). Let \( \mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n \) and \( A_n(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^{n} x_i \). Then
\[
\mathbf{u} = \left( A_n(\mathbf{x}), A_n(\mathbf{x}), \cdots, A_n(\mathbf{x}) \right) \prec (x_1, x_2, \cdots, x_n) = \mathbf{x}.
\]

3. Proofs of theorems

3.1. Proof of Theorem 1.3

Proof. Straightforward computation gives
\[
\frac{\partial L_p(\mathbf{x})}{\partial x_i} = \frac{p a_i^{p-1} \sum_{j=1}^{n} x_j^{p-1} - (p-1)x_i^{p-2} \sum_{j=1}^{n} x_j^p}{(\sum_{j=1}^{n} x_j^{p-1})^2}, \quad i = 1, 2, \ldots, n,
\]
and then
\[
\frac{\partial L_p(\mathbf{x})}{\partial x_i} - \frac{\partial L_p(\mathbf{x})}{\partial x_{i+1}} = \frac{f_i(\mathbf{x})}{(\sum_{j=1}^{n} x_j^{p-1})^2}, \quad i = 1, 2, \ldots, n - 1,
\]
where
\[
f_i(\mathbf{x}) = p(x_i^{p-1} - x_{i+1}^{p-1}) \sum_{j=1}^{n} x_j^{p-1} - (p-1)(x_i^{p-2} - x_{i+1}^{p-2}) \sum_{j=1}^{n} x_j^p.
\]

It is clear that \( L_p(\mathbf{x}) \) is symmetric with \( \mathbf{x} \in \mathbb{R}_+^n \). Without loss of generality, we may assume that \( x_1 \geq x_2 \geq \cdots \geq x_n > 0 \).

For any \( a > 0 \), according to the integral mean value theorem, there is a \( \xi \) which lies between \( x_i \) and \( x_{i+1} \), such that
\[
p(x_i^{p-1} - x_{i+1}^{p-1}) - (p-1)(x_i^{p-2} - x_{i+1}^{p-2}) = (p-1)p \int_{x_{i+1}}^{x_i} x^{p-2}dx - a(p-2)(p-1) \int_{x_{i+1}}^{x_i} x^{p-3}dx
= (p-1) \int_{x_{i+1}}^{x_i} [px^{p-2} - a(p-2)x^{p-3}]dx
= (p-1)(p\xi^{p-2} - a(p-2)\xi^{p-3})(x_i - x_{i+1})
= (p-1)\xi^{p-3}\left( \xi - \frac{(p-2)a}{p} \right) (x_i - x_{i+1}).
\]
Proof of (I): When \( p \geq 2 \) and \( a \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq \frac{(p-2)a}{p} > 0 \), from (3.2), we have
\[
p(x_i^{p-1} - x_{i+1}^{p-1}) - a(p-1)(x_i^{p-2} - x_{i+1}^{p-2}) \geq 0,
\]
that is,
\[
\frac{p(x_i^{p-1} - x_{i+1}^{p-1})}{(p-1)(x_i^{p-2} - x_{i+1}^{p-2})} \geq a,
\]
and then from Lemma 2.10 it follows that
\[
\frac{p(x_i^{p-1} - x_{i+1}^{p-1})}{(p-1)(x_i^{p-2} - x_{i+1}^{p-2})} \geq x_1 \geq \sum_{j=1}^{n} x_j^{p-1},
\]
namely, \( f_i(x) \geq 0 \), and then \( \frac{\partial L_p(x)}{\partial x_i} \geq \frac{\partial L_p(x)}{\partial x_{i+1}} \). By Lemma 2.4 and Remark 2.5 it follows that \( L_p(x) \) is Schur-convex with \( x \in \left[ \frac{p-2}{p}a, a \right]^n \).

Proof of (II): When \( p < 0 \) and \( \frac{(p-2)a}{p} \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq a > 0 \), from (3.2), we have
\[
p(x_i^{p-1} - x_{i+1}^{p-1}) - a(p-1)(x_i^{p-2} - x_{i+1}^{p-2}) \leq 0,
\]
that is,
\[
\frac{p(x_i^{p-1} - x_{i+1}^{p-1})}{(p-1)(x_i^{p-2} - x_{i+1}^{p-2})} \leq a,
\]
and then from Lemma 2.10 it follows that
\[
\frac{p(x_i^{p-1} - x_{i+1}^{p-1})}{(p-1)(x_i^{p-2} - x_{i+1}^{p-2})} \leq x_n \leq \sum_{j=1}^{n} x_j^{p-1},
\]
namely, \( f_i(x) \leq 0 \), and then \( \frac{\partial L_p(x)}{\partial x_i} \leq \frac{\partial L_p(x)}{\partial x_{i+1}} \). By Lemma 2.4 and Remark 2.5 it follows that \( L_p(x) \) is Schur-concave with \( x \in \left[ a, \frac{p-2}{p}a \right]^n \).

The proof of Theorem 1.3 is complete. \( \square \)

3.2. Proof of Theorem 1.4

Proof. From (3.1), we have
\[
x_i \frac{\partial L_p(x)}{\partial x_i} - x_{i+1} \frac{\partial L_p(x)}{\partial x_{i+1}} = \frac{g_i(x)}{(\sum_{i=1}^{n} x_j^{p-1})^2}, \quad i = 1, 2, \ldots, n-1,
\]
where
\[
g_i(x) = p(x_i^{p} - x_{i+1}^{p}) \sum_{j=1}^{n} x_j^{p-1} - (p-1)(x_i^{p-1} - x_{i+1}^{p-1}) \sum_{j=1}^{n} x_j^{p}.
\]

It is clear that \( L_p(x) \) is symmetric with \( x \in \mathbb{R}_+^n \). Without loss of generality, we may assume that \( x_1 \geq x_2 \geq \cdots \geq x_n > 0 \).

For any \( a > 0 \), according to the integral mean value theorem, there is a \( \xi \) which lies between \( x_i \) and \( x_{i+1} \), such that
\[
p(x_i^{p} - x_{i+1}^{p}) - a(p-1)(x_i^{p-1} - x_{i+1}^{p-1}) = p^2 \int_{x_{i+1}}^{x_i} x^{p-1} dx - a(p-1)^2 \int_{x_{i+1}}^{x_i} x^{p-2} dx
\]
\[
= \int_{x_{i+1}}^{x_i} [p^2 x^{p-1} - a(p-1)^2 x^{p-2}] dx \quad \text{(3.3)}
\]
\[ \begin{align*}
&= [p^2 \xi^{p-1} - a(p - 1)^2 \xi^{p-2}](x_i - x_{i+1}) \\
&= p^2 \xi^{p-2} \left[ \xi - (\frac{p-1}{p})^2 a \right] (x_i - x_{i+1}).
\end{align*} \]

Proof of (I): When \( p \geq \frac{1}{2} \) and \( a \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq (\frac{p-1}{p})^2 a > 0 \), from (3.3), we have
\[
p(x_i^p - x_{i+1}^p) - a(p - 1)(x_i^{p-1} - x_{i+1}^{p-1}) \geq 0,
\]
that is,
\[
\frac{p(x_i^p - x_{i+1}^p)}{(p - 1)(x_i^{p-1} - x_{i+1}^{p-1})} \geq a,
\]
and then from Lemma 2.10 it follows that
\[
\frac{p(x_i^p - x_{i+1}^p)}{(p - 1)(x_i^{p-1} - x_{i+1}^{p-1})} \geq x_1 \geq \sum_{j=1}^n x_j^p,
\]
amely, \( g_i(x) \geq 0 \), and then \( x_i \frac{\partial L_p(x)}{\partial x_i} \geq x_{i+1} \frac{\partial L_p(x)}{\partial x_{i+1}} \). By Lemma 2.6 and Remark 2.7 it follows that \( L_p(x) \) is Schur-geometrically convex with \( x \in \left(\frac{p-1}{p}\right)^2 a, a\right)^n \).

Proof of (II): When \( p < \frac{1}{2} \) and \((\frac{p-1}{p})^2 a < x_1 \geq x_2 \geq \cdots \geq x_n \geq a > 0 \), from (3.3), we have
\[
p(x_i^p - x_{i+1}^p) - a(p - 1)(x_i^{p-1} - x_{i+1}^{p-1}) \leq 0,
\]
that is,
\[
\frac{p(x_i^p - x_{i+1}^p)}{(p - 1)(x_i^{p-1} - x_{i+1}^{p-1})} \leq a,
\]
and then from Lemma 2.10 it follows that
\[
\frac{p(x_i^p - x_{i+1}^p)}{(p - 1)(x_i^{p-1} - x_{i+1}^{p-1})} \leq x_n \leq \sum_{j=1}^n x_j^p,
\]
amely, \( g_i(x) \leq 0 \), and then \( x_i \frac{\partial L_p(x)}{\partial x_i} \leq x_{i+1} \frac{\partial L_p(x)}{\partial x_{i+1}} \). By Lemma 2.6 and Remark 2.7 it follows that \( L_p(x) \) is Schur-geometrically concave with \( x \in (a, (\frac{p-1}{p})^2 a)^n \).

Proof of (III): When \( p = 0 \), \( g_i(x) \leq 0 \), it follows that \( L_p(x) \) is Schur-geometrically concave with \( x \in \mathbb{R}_+^n \).

The proof of Theorem 1.4 is complete. \( \square \)

3.3. Proof of Theorem 1.5

Proof. From (3.1), we have
\[
x_i^2 \frac{\partial L_p(x)}{\partial x_i} - x_{i+1}^2 \frac{\partial L_p(x)}{\partial x_{i+1}} = \frac{h_i(x)}{(\sum_{j=1}^n x_j^{p-1})^2}, \quad i = 1, 2, \ldots, n - 1,
\]

where
\[
h_i(x) = p(x_i^{p+1} - x_{i+1}^{p+1}) \sum_{j=1}^n x_j^{p-1} - (p - 1)(x_i^p - x_{i+1}^p) \sum_{j=1}^n x_j^p.
\]
It is clear that \( L_p(x) \) is symmetric with \( x \in \mathbb{R}_+^n \). Without loss of generality, we may assume that \( x_1 \geq x_2 \geq \cdots \geq x_n > 0 \).
Proof of (I): According to the integral mean value theorem, there is a $\xi$ which lies between $x_i$ and $x_{i+1}$, such that

$$\begin{align*}
h_i(x) &= \sum_{j=1}^{n} x_j^{p-1} \left[ p(x_i^{p-1} - x_{i+1}^{p-1}) - (p-1)(x_i^p - x_{i+1}^p) \sum_{j=1}^{n} x_j^p \right] \\
&= \sum_{j=1}^{n} x_j^{p-1} \left[ (p+1)p \int_{x_i}^{x_i} x^p dx - p(p-1) \sum_{j=1}^{n} x_j^p \int_{x_i}^{x_{i+1}} x^{p-1} dx \right] \\
&= \sum_{j=1}^{n} x_j^{p-1} \left[ (p+1)p \int_{x_i}^{x_{i+1}} (p+1)x^p - (p-1) \sum_{j=1}^{n} x_j^p x^{p-1} \right] dx \\
&= \sum_{j=1}^{n} x_j^{p-1} \left[ (p+1)\xi^p - (p-1) \sum_{j=1}^{n} x_j^p \xi^{p-1} \right] (x_i - x_{i+1}) \\
&= \sum_{j=1}^{n} x_j^{p-1}(p+1)\xi^p - \left[ \frac{p-1}{p+1} \sum_{j=1}^{n} x_j^p \right] (x_i - x_{i+1}).
\end{align*}$$

(3.5)

Notice that for $-1 < p \leq 1$, $\xi - \frac{p-1}{p+1} \sum_{j=1}^{n} x_j^p \geq 0$.

When $0 < p \leq 1$, from (3.5), we have $h_i(x) \geq 0$, and then $x_i^2 \frac{\partial L_p(x)}{\partial x_i} \geq x_{i+1}^2 \frac{\partial L_p(x)}{\partial x_{i+1}}$. By Lemma 2.8 and Remark 2.9, it follows that $L_p(x)$ is Schur-harmonically convex with $x \in \mathbb{R}^n$. When $-1 < p \leq 0$, $h_i(x) \leq 0$, and then $x_i^2 \frac{\partial L_p(x)}{\partial x_i} \leq x_{i+1}^2 \frac{\partial L_p(x)}{\partial x_{i+1}}$. By Lemma 2.8 and Remark 2.9, it follows that $L_p(x)$ is Schur-harmonically concave with $x \in \mathbb{R}^n$.

When $p = -1$, $h_i(x) = 2 \sum_{j=1}^{n} x_j^{-1}(x_i^{-1} - x_{i+1}^{-1}) \leq 0$, it follows that $L_p(x)$ is Schur-harmonically concave with $x \in \mathbb{R}^n$.

Proof of (II): For any $a > 0$, according to the integral mean value theorem, there is a $\xi$ which lies between $x_i$ and $x_{i+1}$, such that

$$p(x_i^{p+1} - x_{i+1}^{p+1}) - a(p-1)(x_i^p - x_{i+1}^p) = p(p+1) \int_{x_i}^{x_{i+1}} x^p dx - a(p-1) \int_{x_i}^{x_{i+1}} x^{p-1} dx$$

$$= p \int_{x_i}^{x_{i+1}} ((p+1)x^p - a(p-1)x^{p-1}) dx$$

$$= p[(p+1)\xi^p - a(p-1)\xi^{p-1}](x_i - x_{i+1})$$

$$= p(p+1)\xi^{p-1} \left[ \xi - \frac{(p-1)a}{p+1} \right] (x_i - x_{i+1}).$$

(3.6)

When $p \geq 1$ and $a \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq \frac{p-1}{p+1}a > 0$, from (3.6) we have

$$p(x_i^{p+1} - x_{i+1}^{p+1}) - a(p-1)(x_i^p - x_{i+1}^p) \geq 0,$$

that is,

$$\frac{p(x_i^{p+1} - x_{i+1}^{p+1})}{(p-1)(x_i^p - x_{i+1}^p)} \geq a,$$

and then from Lemma 2.10 it follows that

$$\frac{p(x_i^{p+1} - x_{i+1}^{p+1})}{(p-1)(x_i^p - x_{i+1}^p)} \geq x_1 \geq \sum_{j=1}^{n} x_j^p \sum_{j=1}^{n} x_j^{p-1},$$

namely, $h_i(x) \geq 0$, and then $x_i^2 \frac{\partial L_p(x)}{\partial x_i} \geq x_{i+1}^2 \frac{\partial L_p(x)}{\partial x_{i+1}}$. By Lemma 2.8 and Remark 2.9, it follows that $L_p(x)$ is Schur-harmonically convex with $x \in \left[ \frac{p-1}{p+1}a, a \right]$. 
Proof of (III): When \( p < -1 \) and \( \frac{p-1}{p+1} a \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq a > 0 \), from (3.6), we have

\[
p(x_i^{p+1} - x_{i+1}^{p+1}) - a(p - 1)(x_i^{p} - x_{i+1}^{p}) \leq 0,
\]

that is,

\[
p(x_i^{p+1} - x_{i+1}^{p+1}) \leq \frac{a}{(p-1)(x_i^{p} - x_{i+1}^{p})},
\]

and then from Lemma \((2.10)\), it follows that

\[
\frac{p(x_i^{p+1} - x_{i+1}^{p+1})}{(p-1)(x_i^{p} - x_{i+1}^{p})} \leq a,
\]

namely, \( h_i(x) \leq 0 \), and then \( x_1^2 \frac{\partial L_p(x)}{\partial x_1} \leq x_2^2 \frac{\partial L_p(x)}{\partial x_2} \). By Lemma \((2.8)\) and Remark \((2.9)\) it follows that \( L_p(x) \) is Schur-harmonically concave with \( x \in [a, \frac{p-1}{p+1} a] \).

The proof of Theorem \((1.5)\) is complete.

4. Applications

**Theorem 4.1.** For any \( a > 0 \), if \( p \geq 2 \) and \( x = (x_1, x_2, \ldots, x_n) \in \left[ \frac{p-2}{p} a, a \right]^n \), then we have

\[
A_n(x) \geq L_p(x).
\] (4.1)

If \( p < 0 \) and \( x \in \left[ a, \frac{p-2}{p} a \right]^n \), then the inequality (4.1) is reversed.

**Proof.** If \( p \geq 2 \) and \( x \in \left[ \frac{p-2}{p} a, a \right]^n \), then by Theorem \((1.3)\), from Lemma \((2.11)\) we have

\[
L_p(u) \geq L_p(x),
\]

rearranging gives (4.1), if \( p < 0 \) and \( x \in \left[ a, \frac{p-2}{p} a \right]^n \), then the inequality (4.1) is reversed.

The proof is complete.

**Theorem 4.2.** For any \( a > 0 \), if \( p > \frac{1}{2} \) and \( x = (x_1, x_2, \ldots, x_n) \in \left[ \left( \frac{p-1}{p} \right)^2 a, a \right]^n \), then we have

\[
G_n(x) \leq L_p(x).
\] (4.2)

where \( G_n(x) = \sqrt[n]{x_1 x_2 \cdots x_n} \) is the geometric mean of \( x \). If \( p < \frac{1}{2}, p \neq 0 \) and \( x \in \left[ a, \left( \frac{p-1}{p} \right)^2 a \right]^n \), then the inequality (4.2) is reversed.

**Proof.** By Lemma \((2.11)\) we have

\[
\left( \log G_n(x), \cdots, \log G_n(x) \right) \prec (\log x_1, \log x_2, \cdots, \log x_n),
\]

if \( p > \frac{1}{2} \) and \( x \in \left[ \left( \frac{p-1}{p} \right)^2 a, a \right]^n \), by Theorem \((1.4)\) it follows

\[
L_p \left( G_n(x), \cdots, G_n(x) \right) \leq L_p \left( x_1, x_2, \cdots, x_n \right),
\]

rearranging gives (4.2). If \( p < \frac{1}{2}, p \neq 0 \) and \( x \in \left[ a, \left( \frac{p-1}{p} \right)^2 a \right]^n \), then the inequality (4.2) is reversed.

The proof is complete.
Theorem 4.3. For any $a > 0$, if $p > 1$ and $x \in \left[\frac{p-1}{p+1}a, a\right]_n$, then we have
\[
H_n(x) \leq L_p(x),
\] (4.3)
where $H_n(x) = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$ is the harmonic mean of $x$. If $p < -1$ and $x \in \left[a, \frac{p-1}{p+1}a\right]_n$, then the inequality (4.3) is reversed.

Proof. By Lemma 2.11, we have
\[
\left(\frac{1}{H_n(x)}, \cdots, \frac{1}{H_n(x)}\right) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right).
\]
If $p > 1$ and $x \in \left[\frac{p-1}{p+1}a, a\right]_n$, by Theorem 1.5, it follows
\[
L_p\left(\underbrace{H_n(x), \cdots, H_n(x)}_{n}\right) \leq L_p\left(x_1, x_2, \cdots, x_n\right),
\]
rearranging gives (4.3), if $p < -1$ and $x \in \left[a, \frac{p-1}{p+1}a\right]_n$, then the inequality (4.3) is reversed.

The proof is complete.

In recent years, the study on the properties of the mean by using theory of majorization is unusually active, interested readers may refer to the literature [5–9, 15, 16, 19, 22, 24, 25, 32–35, 37–43, 45].

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References