Some fixed point results of multi-valued nonlinear $F$-contractions without the Hausdorff metric

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Abstract

Fixed point results for several multi-valued nonlinear $F$-contractions without the Hausdorff metric are given and three examples are included. The results obtained in this paper differ from the corresponding results in the literature. ©2016 All rights reserved.

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1. Introduction and preliminaries

Throughout this article, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^{+} = [0, +\infty)$, $\mathbb{N}_{0} = \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers. Let $(X, d)$ be a metric space, $CL(X)$, $CB(X)$ and $C(X)$ denote the families of all nonempty closed, all nonempty bounded closed and all nonempty compact subsets of $X$, respectively. For $T : X \to CL(X)$, $A, B \in X$ and $x \in X$, put

$$d(x, B) = \inf\{d(x, y), y \in B\}, \quad f(x) = d(x, Tx),$$

$$H(A, B) = \begin{cases} \max \{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \}, & \text{if the maximum exists,} \\ +\infty, & \text{otherwise.} \end{cases}$$

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Such a mapping $H$ is called a \textit{generalized Hausdorff metric} induced by $d$ in $CL(X)$. A sequence \( \{x_n\}_{n \in \mathbb{N}_0} \subseteq X \) is said to be an \textit{orbit} of $T$ if $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}_0$. A function $h : X \to \mathbb{R}^+$ is said to be $T$-\textit{orbitally lower semi-continuous} at $z \in X$ if $h(z) \leq \liminf_{n\to\infty} h(x_n)$ for any orbit \( \{x_n\}_{n \in \mathbb{N}_0} \subseteq X \) of $T$ with $\lim_{n\to\infty} x_n = z$.

It is well-known that the Banach contraction principle has a lot of generalizations and applications, (see [2–6, 7–9, 10, 12, 17–19, 25]). In 1969, Nadler [19] obtained the following fixed point theorem for the multi-valued contraction mappings.

\textbf{Theorem 1.1} ([19]). Let $(X, d)$ be a complete metric space and $T$ a mapping from $X$ to $CB(X)$ such that
\begin{equation}
H(Tx, Ty) \leq cd(x, y), \quad \forall x, y \in X,
\end{equation}
where $c \in [0, 1)$ is a constant. Then $T$ has a fixed point.

Later, many researchers generalized Theorem 1.1 in various directions (see [1, 3–6, 9, 10, 13, 14, 16, 18–24]). In 1972, Reich [22] extended Theorem 1.1 and proved the following fixed point theorem for the multi-valued contraction mapping which maps points into compact sets.

\textbf{Theorem 1.2} ([22]). Let $(X, d)$ be a complete metric space and $T : X \to C(X)$ satisfies
\begin{equation}
H(Tx, Ty) \leq \varphi(d(x, y))d(x, y), \quad \forall x, y \in X,
\end{equation}
where
\begin{equation}
\varphi : (0, +\infty) \to [0, 1) \quad \text{with} \quad \limsup_{r \to t^+} \varphi(r) < 1, \quad \forall t \in (0, +\infty).
\end{equation}
Then $T$ has a fixed point.

In 1989, Mizoguchi and Takahashi [18] responded to the conjecture which has been asked whether Reich’s theorem [22] can be extended to multi-valued mappings whose range consists of bounded and closed sets and proved the following result.

\textbf{Theorem 1.3} ([18]). Let $(X, d)$ be a complete metric space and $T : X \to CB(X)$ satisfy that
\begin{equation}
H(Tx, Ty) \leq \varphi(d(x, y))d(x, y), \quad \forall x, y \in X \text{ with } x \neq y,
\end{equation}
where
\begin{equation}
\varphi : (0, +\infty) \to [0, 1) \quad \text{with} \quad \limsup_{r \to t^+} \varphi(r) < 1, \quad \forall t \in \mathbb{R}^+.
\end{equation}
Then $T$ has a fixed point.

In 2006, Feng and Liu [10] generalized Theorem 1.1 to a new type of multi-valued nonlinear contraction mapping without using the Hausdorff metric. Ćirić [5, 6], and Klim and Wardowski [14] extended the result of Feng and Liu [10] and showed the existence of fixed points for some new set-valued contraction mappings. Pathak and Shahzad [21] introduced a new concept of generalized contraction of set-valued mappings and got fixed point theorems for such mappings.

In 2012, Wardowski [25] introduced the concept of $F$-contractions for single-valued mappings and proved a fixed point theorem for the $F$-contraction, which is a generalization of the Banach contraction principle.

\textbf{Definition 1.4} ([25]). Let $F : (0, +\infty) \to \mathbb{R}$ be a mapping satisfying:

(F1) $F$ is strictly increasing;

(F2) for each sequence \( \{\alpha_n\}_{n \in \mathbb{N}} \) of positive numbers $\lim_{n \to +\infty} \alpha_n = 0$ if and only if $\lim_{n \to +\infty} F(\alpha_n) = -\infty$;

(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$. 

\begin{thebibliography}{99}

Let \( F \) be the family of all functions \( F \) that satisfy (F1)-(F3).

**Definition 1.5** ([25]). Let \((X, d)\) be a metric space. A self-mapping \( T \) on \( X \) is called an \( F \)-contraction if there exist \( F \in \mathcal{F} \) and \( \tau > 0 \) such that

\[
\tau + F(d(Tx, Ty)) \leq F(d(x, y)), \quad \forall x, y \in X \quad \text{with} \quad d(Tx, Ty) > 0.
\]

**Theorem 2.1.** Let \((X, d)\) be a complete metric space and let \( T : X \to X \) be an \( F \)-contraction. Then \( T \) has a unique fixed point \( u \in X \) and for every \( x_0 \in X \) a sequence \( \{T^n x_0\}_{n \in \mathbb{N}} \) is convergent to \( u \).

Recently, the researchers have been attracted to study new classes of \( F \)-contractions and to prove the existence of fixed point theorems for these \( F \)-contractions. Three examples are included.

The purpose of this paper is to introduce some new multi-valued nonlinear \( F \)-contractions without using the Hausdorff metric and to establish the existence and iterative approximations of fixed points for these multi-valued nonlinear \( F \)-contractions in complete metric spaces. Three examples are included.

### 2. Main results

In this section, we establish four fixed point theorems for the multi-valued nonlinear \( F \)-contractions (a1), (a3), (a4), and (a6) in complete metric spaces.

**Theorem 2.1.** Let \((X, d)\) be a complete metric space, \( T : X \to CL(X) \) be a multi-valued mapping such that

(a1) for any \( x \in X - Tx \) there is \( y \in Tx - Ty \) with

\[
F(d(x, y)) \leq F(f(x)) + \tau, \quad F(f(y)) + \tau + \eta(f(x)) \leq F(d(x, y)),
\]

where \( F \in \mathcal{F} \), \( \tau > 0 \) and \( \eta : (0, +\infty) \to (0, +\infty) \) satisfies that

(a2) \( \liminf_{s \to +} \eta(s) > 0 \), \( \forall t \in \mathbb{R}^+ \).

Then, for each \( x_0 \in X \) there exists an orbit \( \{x_n\}_{n \in \mathbb{N}} \) of \( T \) and \( z \in X \) such that \( \lim_{n \to \infty} x_n = z \). Furthermore, \( z \) is a fixed point of \( T \) in \( X \) if and only if the function \( f \) is \( T \)-orbitally lower semi-continuous at \( z \).

**Proof.** Let \( x_0 \in X \) be an arbitrary point with \( x_0 \notin Tx_0 \). It follows from (a1) that there exists \( x_1 \in Tx_0 - Tx_1 \) satisfying

\[
F(d(x_0, x_1)) \leq F(f(x_0)) + \tau, \quad F(f(x_1)) + \tau + \eta(f(x_0)) \leq F(d(x_0, x_1)). \tag{2.1}
\]

In light of (2.1) and \( \eta(f(x_0)) > 0 \), we deduce that

\[
F(f(x_1)) \leq F(d(x_0, x_1)) - \tau - \eta(f(x_0)) \leq F(f(x_0)) + \tau - \eta(f(x_0)) = F(f(x_0)) - \eta(f(x_0)) < F(f(x_0)).
\]

In terms of (a1) there exists \( x_2 \in Tx_1 - Tx_2 \) with

\[
F(d(x_1, x_2)) \leq F(f(x_1)) + \tau, \quad F(f(x_2)) + \tau + \eta(f(x_1)) \leq F(d(x_1, x_2)),
\]

which together with (2.1), \( \eta(f(x_0)) > 0 \) and \( \eta(f(x_1)) > 0 \) mean that

\[
F(f(x_2)) \leq F(d(x_1, x_2)) - \tau - \eta(f(x_1)) \leq F(f(x_1)) + \tau - \eta(f(x_1)) = F(f(x_1)) - \eta(f(x_1)) < F(f(x_1)),
\]
\[ F(d(x_1, x_2)) \leq F(f(x_1)) + \tau \]
\[ \leq F(d(x_0, x_1)) - \tau - \eta(f(x_0)) + \tau \]
\[ = F(d(x_0, x_1)) - \eta(f(x_0)) \]
\[ < F(d(x_0, x_1)). \]

Repeating this process, we obtain an orbit \( \{x_n\}_{n \in \mathbb{N}_0} \subset X \) of \( T \) satisfying
\[ F(d(x_n, x_{n+1})) \leq F(f(x_n)) + \tau, \]
\[ F(f(x_{n+1})) + \tau + \eta(f(x_n)) \leq F(d(x_n, x_{n+1})), \quad x_{n+1} \in Tx_n - Tx_{n+1}, \quad \forall n \in \mathbb{N}_0. \tag{2.2} \]

In view of (2.2) and \( \eta(f(x_{n-1})) > 0 \) for each \( n \in \mathbb{N} \), we have
\[ F(f(x_{n})) \leq F(d(x_{n-1}, x_n)) - \tau - \eta(f(x_{n-1})) \]
\[ \leq F(f(x_{n-1})) + \tau - \tau - \eta(f(x_{n-1})) \]
\[ = F(f(x_{n-1})) - \eta(f(x_{n-1})) \]
\[ < F(f(x_{n-1})), \quad \forall n \in \mathbb{N}. \tag{2.3} \]

It follows from (2.3) and (F1) that
\[ 0 < f(x_n) < f(x_{n-1}), \quad \forall n \in \mathbb{N}. \tag{2.4} \]

Note that (2.4) implies that there exists a constant \( a \in \mathbb{R}^+ \) with
\[ \lim_{n \to \infty} f(x_n) = a. \tag{2.5} \]

By virtue of (a2) there exists a constant \( b > 0 \) satisfying
\[ \lim \inf_{s \to a^+} \eta(s) = 2b, \]
which means that for \( \varepsilon = b \), there exists \( \delta > 0 \) satisfying
\[ \eta(s) - 2b > -\varepsilon, \quad \forall s \in (a, a + \delta), \]
that is,
\[ \eta(s) > b, \quad \forall s \in (a, a + \delta). \tag{2.6} \]

Clearly, (2.4)-(2.6) ensure that there exists \( n_0 \in \mathbb{N} \) satisfying
\[ a < f(x_n) < a + \delta, \quad \eta(f(x_n)) > b, \quad \forall n \geq n_0. \tag{2.7} \]

Making use of (2.3) and (2.7), we arrive at
\[ F(f(x_n)) \leq F(f(x_{n-1})) - \eta(f(x_{n-1})) \]
\[ \leq F(f(x_{n-2})) - \eta(f(x_{n-2})) - \eta(f(x_{n-1})) \]
\[ \vdots \]
\[ \leq F(f(x_{n_0})) - \eta(f(x_{n_0})) - \eta(f(x_{n_0+1})) - \cdots - \eta(f(x_{n-1})) \]
\[ \leq F(f(x_{n_0})) - (n - n_0)b, \]
which implies that
\[ \lim_{n \to \infty} F(f(x_n)) = -\infty. \tag{2.8} \]

By means of (F2), (2.5) and (2.8), we conclude immediately that
\[ a = \lim_{n \to \infty} f(x_n) = 0. \tag{2.9} \]
Using (2.2) and (2.7), we infer that
\[
F(d(x_n, x_{n+1})) \leq F(f(x_n)) + \tau
\]
\[
\leq F(d(x_{n-1}, x_n)) - \tau - \eta(f(x_{n-1})) + \tau
\]
\[
= F(d(x_{n-1}, x_n)) - \eta(f(x_{n-1}))
\]
\[
\leq F(d(x_{n-2}, x_{n-1})) - \eta(f(x_{n-2})) - \eta(f(x_{n-1}))
\]
\[
\vdots
\]
\[
\leq F(d(x_{n_0}, x_{n_0+1})) - \eta(f(x_{n_0})) - \eta(f(x_{n_0+1})) - \cdots - \eta(f(x_{n-1}))
\]
\[
\leq F(d(x_{n_0}, x_{n_0+1})) - (n - n_0) \beta
\]
\[
\to -\infty \text{ as } n \to \infty.
\]
(2.10)

That is,
\[
\lim_{n \to \infty} F(d(x_n, x_{n+1})) = -\infty.
\]

It follows from (2.10) and (F2) that
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.
\]
(2.11)

It is clear that (F3) and (2.11) ensure that there exists \(k \in (0, 1)\) such that
\[
\lim_{n \to \infty} [d^k(x_n, x_{n+1})F(d(x_n, x_{n+1}))] = 0.
\]
(2.12)

Using (2.10)-(2.12), we derive that
\[
0 \leq \limsup_{n \to \infty} \left\{ (n - n_0) \beta d^k(x_n, x_{n+1}) \right\}
\]
\[
\leq \limsup_{n \to \infty} \left\{ (F(d(x_{n_0}, x_{n_0+1})) - F(d(x_n, x_{n+1})))d^k(x_n, x_{n+1}) \right\}
\]
\[
= 0,
\]
which yields that
\[
\lim_{n \to \infty} (n - n_0) \beta d^k(x_n, x_{n+1}) = 0,
\]
that is,
\[
\lim_{n \to \infty} nd^k(x_n, x_{n+1}) = 0.
\]
(2.13)

It follows from (2.13) that there exists \(n_1 \geq n_0\) satisfying
\[
nd^k(x_n, x_{n+1}) \leq 1, \quad \forall n \geq n_1,
\]
that is,
\[
d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}, \quad \forall n \geq n_1,
\]
which gives that
\[
d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)
\]
\[
\leq \sum_{i=n}^{m-1} d(x_i, x_{i+1})
\]
\[
\leq \sum_{i=n}^{\infty} d(x_i, x_{i+1})
\]
\[
\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}, \quad \forall m > n \geq n_1,
\]
which together with the convergence of the series \(\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}\) means that \(\{x_n\}_{n \in \mathbb{N}_0}\) is a Cauchy sequence. Since
(X, d) is a complete metric space, there exists a point z ∈ X such that
\[
\lim_{n \to \infty} x_n = z.
\] (2.14)

Suppose that f is T-orbitally lower semi-continuous at z. It follows from (2.9) and (2.14) that
\[
d(z, Tz) = f(z) \leq \liminf_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(x_n) = 0,
\]
that is, z ∈ X is a fixed point of T.

Conversely, suppose that z ∈ X is a fixed point of T. For each orbit \( \{y_n\}_{n \in \mathbb{N}_0} \) of T with \( \lim_{n \to \infty} y_n = z \), we deduce that
\[
f(z) = d(z, Tz) = 0 \leq \liminf_{n \to \infty} f(y_n),
\]
which implies that f is T-orbitally lower semi-continuous in z. This completes the proof. □

**Theorem 2.2.** Let (X, d) be a complete metric space, T : X → CL(X) be a multi-valued mapping such that
(a3) for any x ∈ X − Tx there is y ∈ Tx − Ty with
\[
F(d(x, y)) \leq F(f(x)) + \tau, \quad F(f(y)) + \tau + \eta(d(x, y)) \leq F(d(x, y)),
\]
where F ∈ F, \( \tau > 0 \) and \( \eta : (0, +\infty) \to (0, +\infty) \) satisfies (a2).

Then, for each \( x_0 \in X \) there exists an orbit \( \{x_n\}_{n \in \mathbb{N}_0} \) of T and \( z \in X \) such that \( \lim_{n \to \infty} x_n = z \). Furthermore, z is a fixed point of T in X if and only if the function f is T-orbitally lower semi-continuous at z.

**Proof.** Let \( x_0 \in X \) be an arbitrary point with \( x_0 \not\in Tx_0 \). It follows from (a2) that there exists \( x_1 \in Tx_0 − Tx_1 \) satisfying
\[
F(d(x_0, x_1)) \leq F(f(x_0)) + \tau, \quad F(f(x_1)) + \tau + \eta(d(x_0, x_1)) \leq F(d(x_0, x_1)).
\] (2.15)

In view of (a3), there exists \( x_2 \in Tx_1 − Tx_2 \) with
\[
F(d(x_1, x_2)) \leq F(f(x_1)) + \tau, \quad F(f(x_2)) + \tau + \eta(d(x_1, x_2)) \leq F(d(x_1, x_2)),
\]
which together with (2.15) and \( \eta(d(x_0, x_1)) > 0 \) we have
\[
F(d(x_1, x_2)) \leq F(f(x_1)) + \tau
\leq F(d(x_0, x_1)) - \tau - \eta(d(x_0, x_1)) + \tau
\leq F(d(x_0, x_1)) - \eta(d(x_0, x_1))
\leq F(d(x_0, x_1)).
\]

Repeating this process, we obtain an orbit \( \{x_n\}_{n \in \mathbb{N}_0} \subset X \) of T satisfying
\[
F(d(x_n, x_{n+1})) \leq F(f(x_n)) + \tau,
F(f(x_{n+1}) + \tau + \eta(d(x_n, x_{n+1})) \leq F(d(x_n, x_{n+1})), \quad x_{n+1} \in Tx_n − Tx_{n+1}, \quad \forall n \in \mathbb{N}_0.
\] (2.16)

In light of (2.16) and \( \eta(d(x_{n-1}, x_n)) > 0 \) for each \( n \in \mathbb{N} \), we deduce that
\[
F(d(x_n, x_{n+1})) \leq F(f(x_n)) + \tau
\leq F(d(x_{n-1}, x_n)) - \tau - \eta(d(x_{n-1}, x_n)) + \tau
\leq F(d(x_{n-1}, x_n)) - \eta(d(x_{n-1}, x_n))
\leq F(d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N},
\] (2.17)
which together with \((F1)\) implies that

\[
0 < d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}.
\]  

(2.18)

Consequently, \((2.18)\) means that the sequence \(\{d(x_n, x_{n+1})\}_{n \in \mathbb{N}_0}\) converges to a constant \(a \in \mathbb{R}^+\), that is,

\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = a.
\]  

(2.19)

As in the proof of Theorem 2.1 we conclude that (2.6) holds. It follows from (2.6), (2.18) and (2.19) that there exists \(n_0 \in \mathbb{N}\) satisfying

\[
a < d(x_n, x_{n+1}) < a + \delta, \quad \eta(d(x_n, x_{n+1})) > b, \quad \forall n \geq n_0.
\]  

(2.20)

Using (2.17) and (2.20), we obtain that

\[
F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \eta(d(x_{n-1}, x_n))
\leq F(d(x_{n-2}, x_{n-1})) - \eta(d(x_{n-2}, x_{n-1})) - \eta(d(x_{n-1}, x_n))
\vdots
\leq F(d(x_{n_0}, x_{n_0+1})) - \eta(d(x_{n_0}, x_{n_0+1})) - \eta(d(x_{n_0+1}, x_{n_0+2})) - \cdots - \eta(d(x_{n-1}, x_n))
\leq F(d(x_{n_0}, x_{n_0+1})) - (n - n_0)b
\to -\infty
\text{ as } n \to \infty,
\]

which implies (2.11). The rest of the proof is similar to that of Theorem 2.1 and is omitted. This completes the proof. \(\square\)

**Theorem 2.3.** Let \((X, d)\) be a complete metric space, \(T : X \to CL(X)\) be a multi-valued mapping such that

(a4) for any \(x \in X - Tx\) there is \(y \in Tx - Ty\) with

\[
F(d(x, y)) \leq F(f(x)) + \frac{1}{2}\eta(f(x)), \quad F(f(y)) + \eta(f(x)) \leq F(d(x, y)),
\]

where \(F \in \mathcal{F}\), \(\eta : (0, +\infty) \to (0, +\infty)\) satisfies (a2) and

(a5) \(\limsup_{s \to 0^+} \eta(s) < +\infty\).

Then, for each \(x_0 \in X\) there exists an orbit \(\{x_n\}_{n \in \mathbb{N}_0}\) of \(T\) and \(z \in X\) such that \(\lim_{n \to \infty} x_n = z\). Furthermore, \(z\) is a fixed point of \(T\) in \(X\) if and only if the function \(f\) is \(T\)-orbitally lower semi-continuous at \(z\).

**Proof.** Let \(x_0 \in X\) be an arbitrary point with \(x_0 \notin Tx_0\). It follows from (a4) that there exists \(x_1 \in Tx_0 - Tx_1\) satisfying

\[
F(d(x_0, x_1)) \leq F(f(x_0)) + \frac{1}{2}\eta(f(x_0)), \quad F(f(x_1)) + \eta(f(x_0)) \leq F(d(x_0, x_1)).
\]  

(2.21)

It follows from (2.21) and \(\eta(f(x_0)) > 0\) that

\[
F(f(x_1)) \leq F(d(x_0, x_1)) - \eta(f(x_0))
\leq F(f(x_0)) + \frac{1}{2}\eta(f(x_0)) - \eta(f(x_0))
= F(f(x_0)) - \frac{1}{2}\eta(f(x_0))
< F(f(x_0)).
\]
(a4) implies that there exists \( x_2 \in Tx_1 - Tx_2 \) with
\[
F(d(x_1, x_2)) \leq F(f(x_1)) + \frac{1}{2} \eta(f(x_1)), \quad F(f(x_2)) + \eta(f(x_1)) \leq F(d(x_1, x_2)),
\]
which together with (2.21) and \( \eta(f(x_1)) > 0 \) give that
\[
F(f(x_2)) \leq F(d(x_1, x_2)) - \eta(f(x_1)) \\
\leq F(f(x_1)) + \frac{1}{2} \eta(f(x_1)) - \eta(f(x_1)) \\
= F(f(x_1)) - \frac{1}{2} \eta(f(x_1)) \\
< F(f(x_1)),
\]
\[
F(d(x_1, x_2)) \leq F(f(x_1)) + \frac{1}{2} \eta(f(x_1)) \\
\leq F(d(x_0, x_1)) - \eta(f(x_0)) + \frac{1}{2} \eta(f(x_1)).
\]
Repeating this process, we obtain an orbit \( \{x_n\}_{n \in \mathbb{N}_0} \in X \) of \( T \) satisfying
\[
F(d(x_n, x_{n+1})) \leq F(f(x_n)) + \frac{1}{2} \eta(f(x_n)), \\
F(f(x_{n+1})) + \eta(f(x_n)) \leq F(d(x_n, x_{n+1})), \quad x_{n+1} \in Tx_n - Tx_{n+1}, \quad \forall n \in \mathbb{N}_0. \tag{2.22}
\]
In view of (2.22) and \( \eta(f(x_{n-1})) > 0 \) for each \( n \in \mathbb{N} \), we deduce that
\[
F(f(x_n)) \leq F(d(x_{n-1}, x_n)) - \eta(f(x_{n-1})) \\
\leq F(f(x_{n-1})) + \frac{1}{2} \eta(f(x_{n-1})) - \eta(f(x_{n-1})) \\
\leq F(f(x_{n-1})) - \frac{1}{2} \eta(f(x_{n-1})) \\
< F(f(x_{n-1})), \quad \forall n \in \mathbb{N} \tag{2.23}
\]
and
\[
F(d(x_n, x_{n+1})) \leq F(f(x_n)) + \frac{1}{2} \eta(f(x_n)) \\
\leq F(d(x_{n-1}, x_n)) - \eta(f(x_{n-1})) + \frac{1}{2} \eta(f(x_n)), \quad \forall n \in \mathbb{N}. \tag{2.24}
\]
Similar to the arguments of Theorem 2.1 we conclude that (2.4)-(2.7) hold. In terms of (2.23) and (2.7), we arrive at
\[
F(f(x_n)) \leq F(f(x_{n-1})) - \frac{1}{2} \eta(f(x_{n-1})) \\
\leq F(f(x_{n-2})) - \frac{1}{2} \eta(f(x_{n-2})) - \frac{1}{2} \eta(f(x_{n-1})) \\
\vdots \\
\leq F(f(x_0)) - \frac{1}{2} \eta(f(x_0)) - \frac{1}{2} \eta(f(x_{n+1})) - \cdots - \frac{1}{2} \eta(f(x_{n-1})) \\
\leq F(f(x_0)) - \frac{1}{2} (n - n_0) b \\
\to -\infty \quad \text{as } n \to \infty,
\]
which together with (2.5) and (F2), we derive that (2.8) and (2.9) hold.
In light of (2.7) and (2.24), we get that
\[
F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \eta(f(x_{n-1})) + \frac{1}{2} \eta(f(x_n)) \\
\leq F(d(x_{n-2}, x_{n-1})) - \eta(f(x_{n-2})) + \frac{1}{2} \eta(f(x_{n-1})) + \frac{1}{2} \eta(f(x_n)) \\
\vdots \\
\leq F(d(x_{n_0}, x_{n_0+1})) - \eta(f(x_{n_0})) + \frac{1}{2} \eta(f(x_{n_0+1})) - \cdots - \frac{1}{2} \eta(f(x_{n_1})) + \frac{1}{2} \eta(f(x_n)) \\
\leq F(d(x_{n_0}, x_{n_0+1})) - \frac{1}{2} (n - n_0 - 1)b + \frac{1}{2} \eta(f(x_n)), \quad \forall n \geq n_0.
\]
Taking upper limit in (2.25) and using (2.7), (2.9) and (a5), we get that
\[
\limsup_{n \to \infty} F(d(x_n, x_{n+1})) \leq \limsup_{n \to \infty} \left[ F(d(x_{n_0}, x_{n_0+1})) - \frac{1}{2} (n - n_0 - 1)b + \frac{1}{2} \eta(f(x_n)) \right] \\
\leq \limsup_{n \to \infty} \left[ F(d(x_{n_0}, x_{n_0+1})) - \frac{1}{2} (n - n_0 - 1)b \right] + \frac{1}{2} \limsup_{n \to \infty} \eta(f(x_n)) \\
= -\infty,
\]
that is, (2.11) holds. Similarly, we know that (2.12) holds.

It follows from (a5), (2.11), (2.12), and (2.25) that
\[
0 \leq \limsup_{n \to \infty} \left[ \frac{1}{2} (n - n_0 - 1)b d^k(x_n, x_{n+1}) \right] \\
\leq \limsup_{n \to \infty} \left\{ \left( F(d(x_{n_0}, x_{n_0+1})) - F(d(x_n, x_{n+1})) + \frac{1}{2} \eta(f(x_n)) \right) d^k(x_n, x_{n+1}) \right\} \\
\leq \limsup_{n \to \infty} \left\{ F(d(x_{n_0}, x_{n_0+1})) - F(d(x_n, x_{n+1})) \right\} d^k(x_n, x_{n+1}) \\
+ \frac{1}{2} \limsup_{n \to \infty} \eta(f(x_n)) d^k(x_n, x_{n+1}) \\
\leq 0 + \frac{1}{2} \limsup_{n \to \infty} \eta(f(x_n)) \cdot \limsup_{n \to \infty} d^k(x_n, x_{n+1}) \\
= 0,
\]
which means that
\[
\limsup_{n \to \infty} [(n - n_0 - 1)b d^k(x_n, x_{n+1})] = 0,
\]
which yields (2.13). The rest of the proof is similar to that of Theorem 2.1 and is omitted. This completes the proof. \qed

**Theorem 2.4.** Let \((X, d)\) be a complete metric space, \(T : X \to CL(X)\) be a multi-valued mapping such that

(a6) for any \(x \in X - Tx\) there is \(y \in Tx - Ty\) with
\[
F(d(x, y)) \leq F(f(x)) + \frac{1}{2} \eta(d(x, y)), \quad F(f(y)) + \eta(d(x, y)) \leq F(d(x, y)),
\]
where \(F \in \mathcal{F}, \eta : (0, +\infty) \to (0, +\infty)\) satisfies

(a7) \(\eta\) is decreasing,

(a8) \(\lim_{s \to 0^+} \eta(s) > 0\).
Then, for each \( x_0 \in X \) there exists an orbit \( \{x_n\}_{n=0}^\infty \) of \( T \) and \( z \in X \) such that \( \lim_{n \to \infty} x_n = z \). Furthermore, \( z \) is a fixed point of \( T \) in \( X \) if and only if the function \( f \) is \( T \)-orbitally lower semi-continuous at \( z \).

**Proof.** Let \( x_0 \in X \) be an arbitrary point with \( x_0 \notin Tx_0 \). It follows from (a6) that there exists \( x_1 \in Tx_0 - Tx_1 \) satisfying

\[
F(d(x_0, x_1)) \leq F(f(x_0)) + \frac{1}{2} \eta(d(x_0, x_1)), \quad F(f(x_1)) + \eta(d(x_0, x_1)) \leq F(d(x_0, x_1)). \tag{2.26}
\]

In view of (2.26) and \( \eta(d(x_0, x_1)) > 0 \), we arrive at

\[
F(f(x_1)) \leq F(d(x_0, x_1)) - \eta(d(x_0, x_1))
\leq F(f(x_0)) + \frac{1}{2} \eta(d(x_0, x_1)) - \eta(d(x_0, x_1))
= F(f(x_0)) - \frac{1}{2} \eta(d(x_0, x_1))
< F(f(x_0)),
\]

(a6) implies that there exists \( x_2 \in Tx_1 - Tx_2 \) with

\[
F(d(x_1, x_2)) \leq F(f(x_1)) + \frac{1}{2} \eta(d(x_1, x_2)), \quad F(f(x_2)) + \eta(d(x_1, x_2)) \leq F(d(x_1, x_2)),
\]

which together with (2.26) and \( \eta(d(x_1, x_2)) > 0 \) show that

\[
F(f(x_2)) \leq F(d(x_1, x_2)) - \eta(d(x_1, x_2))
\leq F(f(x_1)) + \frac{1}{2} \eta(d(x_1, x_2)) - \eta(d(x_1, x_2))
= F(f(x_1)) - \frac{1}{2} \eta(d(x_1, x_2))
< F(f(x_1)),
\]

\[
F(d(x_1, x_2)) \leq F(f(x_1)) + \frac{1}{2} \eta(d(x_1, x_2))
\leq F(d(x_0, x_1)) - \eta(d(x_0, x_1)) + \frac{1}{2} \eta(d(x_1, x_2)).
\]

Repeating this process, we obtain an orbit \( \{x_n\}_{n=0}^\infty \subset X \) of \( T \) satisfying

\[
F(d(x_n, x_{n+1})) \leq F(f(x_n)) + \frac{1}{2} \eta(d(x_n, x_{n+1})), \\
F(f(x_{n+1})) + \eta(d(x_n, x_{n+1})) \leq F(d(x_n, x_{n+1})), \quad x_{n+1} \in Tx_n - Tx_{n+1}, \quad \forall n \in \mathbb{N}_0. \tag{2.27}
\]

Suppose that there exists some \( n_0 \in \mathbb{N} \) satisfying

\[
d(x_{n_0}, x_{n_0+1}) \geq d(x_{n_0-1}, x_{n_0}), \tag{2.28}
\]

which together with (a7) gives that

\[
\eta(d(x_{n_0}, x_{n_0+1})) \leq \eta(d(x_{n_0-1}, x_{n_0})). \tag{2.29}
\]

In terms of (2.27)-(2.29) and \( \eta(d(x_{n_0}, x_{n_0+1})) > 0 \), we deduce that

\[
F(d(x_{n_0-1}, x_{n_0})) \leq F(d(x_{n_0}, x_{n_0+1})) \leq F(f(x_{n_0})) + \frac{1}{2} \eta(d(x_{n_0}, x_{n_0+1}))
\]
Using (2.33) and (a7), we arrive at

\[ F(d(x_{n_0-1},x_{n_0})) - \eta(d(x_{n_0-1},x_{n_0})) + \frac{1}{2} \eta(d(x_{n_0},x_{n_0+1})) \leq F(d(x_{n_0},x_{n_0+1})) - \eta(d(x_{n_0},x_{n_0+1})) + \frac{1}{2} \eta(d(x_{n_0},x_{n_0+1})) \]

\[ = F(d(x_{n_0-1},x_{n_0})) - \frac{1}{2} \eta(d(x_{n_0},x_{n_0+1})) \]

\[ < F(d(x_{n_0-1},x_{n_0})), \]

which is contradiction. Therefore,

\[ 0 < d(x_n,x_{n+1}) < d(x_{n-1},x_n), \quad \forall n \in \mathbb{N}. \tag{2.30} \]

It is clear that (2.30) implies (2.19) for some \( a \in \mathbb{R} \). (a7), (a8), (2.19), and (2.30) imply that

\[ \lim_{n \to \infty} \eta(d(x_n,x_{n+1})) = 2b \tag{2.31} \]

for some \( b > 0 \). It is easy to see that (2.19), (2.30), and (2.31) ensure that there exists \( n_1 > n_0 \) satisfying

\[ a < d(x_n,x_{n+1}) < a + \delta, \quad \eta(d(x_n,x_{n+1})) > b, \quad \forall n \geq n_1. \tag{2.32} \]

It follows from (2.27), (2.30), and (2.32) that

\[
F(d(x_n,x_{n+1})) \leq F(f(x_n)) + \frac{1}{2} \eta(d(x_n,x_{n+1})) \\
\leq F(d(x_{n-1},x_n)) - \eta(d(x_{n-1},x_n)) + \frac{1}{2} \eta(d(x_n,x_{n+1})) \\
\leq F(d(x_{n-2},x_{n-1})) - \eta(d(x_{n-2},x_{n-1})) - \frac{1}{2} \eta(d(x_{n-1},x_n)) + \frac{1}{2} \eta(d(x_n,x_{n+1})) \\
\vdots \\
\leq F(d(x_{n_1},x_{n_1+1})) - \eta(d(x_{n_1},x_{n_{1+1}})) - \frac{1}{2} \eta(d(x_{n_1+1},x_{n_{1+2}})) - \cdots \\
- \frac{1}{2} \eta(d(x_{n_1},x_n)) + \frac{1}{2} \eta(d(x_n,x_{n+1})) \\
\leq F(d(x_{n_1},x_{n_1+1})) - \frac{1}{2} (n - n_1 - 1)b + \frac{1}{2} \eta(d(x_n,x_{n+1})), \quad \forall n \geq n_1.
\]

Using (2.33) and (a7), we arrive at

\[
\limsup_{n \to \infty} F(d(x_n,x_{n+1})) \leq \limsup_{n \to \infty} \left[ F(d(x_{n_1},x_{n_1+1})) - \frac{1}{2} (n - n_1 - 1)b + \frac{1}{2} \eta(d(x_n,x_{n+1})) \right] \\
\leq \limsup_{n \to \infty} \left[ F(d(x_{n_1},x_{n_1+1})) - \frac{1}{2} (n - n_1 - 1)b \right] + \frac{1}{2} \limsup_{n \to \infty} \eta(d(x_n,x_{n+1})) \\
= -\infty,
\]

that is,

\[ \lim_{n \to \infty} F(d(x_n,x_{n+1})) = -\infty. \]

In view of (F2) and (2.19), we get that

\[ a = \lim_{n \to \infty} d(x_n,x_{n+1}) = 0. \tag{2.34} \]

In view of (F3) and (2.33), ensure that there exists \( k \in (0,1) \) such that

\[ \lim_{n \to \infty} [d^k(x_n,x_{n+1})F(d(x_n,x_{n+1}))] = 0. \tag{2.35} \]
In light of (a7) and (2.33)-(2.35), we deduce that

\[
0 \leq \limsup_{n \to \infty} \left[ \frac{1}{2} (n - n_0 - 1) bd^k(x_n, x_{n+1}) \right]
\]

\[
\leq \limsup_{n \to \infty} \left[ \left( F(d(x_{n_1}, x_{n_1+1})) - F(d(x_n, x_{n+1})) + \frac{1}{2} \eta(d(x_n, x_{n+1})) \right) d^k(x_n, x_{n+1}) \right]
\]

\[
\leq \limsup_{n \to \infty} \left( F(d(x_{n_1}, x_{n_1+1})) - F(d(x_n, x_{n+1})) d^k(x_n, x_{n+1}) \right)
\]

\[
+ \limsup_{n \to \infty} \left[ \frac{1}{2} \eta(d(x_n, x_{n+1})) d^k(x_n, x_{n+1}) \right]
\]

\[
\leq 0 + \limsup_{n \to \infty} \frac{1}{2} \eta(d(x_n, x_{n+1}) \cdot \limsup_{n \to \infty} d^k(x_n, x_{n+1})
\]

\[
= 0,
\]

which connotes (2.13). The rest of the proof is similar to that of Theorem 2.1 and is omitted. This completes the proof. \(\square\)

### 3. Remarks and examples

**Remark 3.1.** The following examples show that Theorems 2.1-2.4 differ from Theorems 1.1-1.3.

**Example 3.2.** Let \(X = \mathbb{R}\) be endowed with the Euclidean metric \(d = | \cdot |\). Let \(\tau = \ln \frac{4}{5}\), \(T : X \to CL(X), F : (0, +\infty) \to \mathbb{R}\) and \(\eta : (0, +\infty) \to (0, +\infty)\) be defined by

\[
Tx = \begin{cases} (-\infty, 2x] \cup \left[ \frac{x}{3}, 0 \right], & x \in (-\infty, 0), \\ \left[ 0, \frac{x}{3} \right] \cup [3x, +\infty), & x \in [0, +\infty), \\ \end{cases}
\]

\[
F(t) = \ln t, \quad \eta(t) = \ln \frac{6}{5}, \quad \forall t \in (0, +\infty).
\]

It is easy to see that

\[
f(x) = d(x, Tx) = \begin{cases} -\frac{x}{2}, & x \in (-\infty, 0), \\ \frac{2x}{3}, & x \in [0, +\infty), \\ \end{cases}
\]

is continuous in \(X\),

\[
\liminf_{s \to t^+} \eta(s) = \liminf_{s \to t^+} \ln \frac{6}{5} > 0, \quad \forall t \in \mathbb{R}^+.
\]

Put \(x \in X - Tx\). In order to verify (a1) and (a3), we consider the following two possible cases:

**Case 1.** Let \(x \in (-\infty, 0) - Tx\). It follows that \(x \in \left( 2x, \frac{3x}{2} \right)\). Put

\[
y = \frac{x}{2} \in (-\infty, 2x] \cup \left[ \frac{x}{2}, 0 \right) - (-\infty, x] \cup \left[ \frac{x}{4}, 0 \right) = Tx - Ty.
\]

It follows that

\[
F(d(x, y)) = \ln \left| \frac{x}{2} \right| \leq \ln \left| \frac{x}{4} \right| + \ln \frac{4}{3} = F(f(x)) + \tau,
\]

and

\[
F(f(y)) + \tau + \eta(f(x)) = F(f(y)) + \tau + \eta(d(x, y))
\]

\[
= \ln \left| \frac{x}{4} \right| + \ln \frac{4}{3} + \ln \frac{6}{5}
\]

\[
= \ln \left| \frac{2x}{5} \right| \leq \ln \left| \frac{x}{2} \right|
\]

\[
= F(d(x, y)).
\]
**Case 2.** Let \( x \in [0, +\infty) - Tx \). It follows that \( x \in \left( \frac{x}{3}, 3x \right) \). Put
\[
y = \frac{x}{3} \in \left[ 0, \frac{x}{3} \right] \cup [3x, +\infty) - \left[ 0, \frac{x}{9} \right] \cup [x, +\infty) = Tx - Ty.
\]

It is clear that
\[
F(d(x, y)) = \ln \frac{2x}{3} \leq \ln \frac{2x}{3} + \ln \frac{4}{3} = F(f(x)) + \tau,
\]
and
\[
F(f(y)) + \tau + \eta(f(x)) = F(f(y)) + \tau + \eta(d(x, y))
\]
\[
= \ln \frac{2x}{9} + \ln \frac{4}{3} + \ln \frac{6}{5}
\]
\[
= \ln \frac{16x}{45} \leq \ln \frac{2x}{3}
\]
\[
= F(d(x, y)).
\]

That is, (a1) and (a3) hold. It follows from both of Theorems 2.1 and 2.2 that \( T \) has a fixed point in \( X \). However, the mapping \( T \) does not satisfy (1.1), (1.2) and (1.4) in Theorems 1.1-1.3, respectively. In fact, put \( x_0 = -1 \) and \( y_0 = 1 \). It is clear that
\[
H(Tx_0, Ty_0) = H\left( (-\infty, -2] \cup \left\{ -\frac{1}{2} \right\}, \left[ 0, \frac{1}{3} \right] \cup [3, +\infty) \right)
\]
\[
= +\infty \not\leq 2r = rd(x_0, y_0), \quad \forall r \in (0, 1),
\]
\[
H(Tx_0, Ty_0) = +\infty \not\leq 2\varphi(d(x_0, y_0)) = \varphi(d(x_0, y_0))d(x_0, y_0)
\]
for any mapping \( \varphi : (0, +\infty) \to [0, 1) \) with each of \([1, 3)\) and \([1, 10)\).

**Example 3.3.** Let \( X = \mathbb{R}^+ \) be endowed with the Euclidean metric \( d = | \cdot | \). Let \( T : X \to CL(X), F : (0, +\infty) \to \mathbb{R}, \eta : (0, +\infty) \to (0, +\infty) \) be defined by
\[
Tx = \begin{cases} 
\left[ 0, \frac{x^2}{2} \right], & x \in [0, 1], \\
\left[ 0, \frac{1}{x} \right], & x \in (1, +\infty),
\end{cases}
\]
\[
F(t) = \ln t, \quad \eta(t) = \ln \frac{4}{3}, \quad \forall t \in (0, +\infty).
\]

It is easy to see that
\[
f(x) = d(x, Tx) = \begin{cases} 
\frac{x - x^2}{2}, & x \in [0, 1], \\
\frac{x}{1}, & x \in (1, +\infty),
\end{cases}
\]
is lower semi-continuous in \( X \),
\[
\limsup_{s \to +\infty} \eta(s) = \ln \frac{4}{3} < +\infty, \quad \liminf_{s \to +\infty} \eta(s) = \ln \frac{4}{3} > 0, \quad \forall t \in \mathbb{R}^+.
\]

In order to verify (a4), we consider the following two possible cases:

**Case 1.** Let \( x \in [0, 1] \cap (X - Tx) \). It follows that \( x \in \left( \frac{x^2}{2}, 1 \right) \). Put \( y = \frac{x^2}{2} \in \left[ 0, \frac{x^2}{2} \right] - \left[ 0, \frac{x^4}{8} \right] = Tx - Ty \).

It follows that
\[
F(d(x, y)) = \ln \left( x - \frac{x^2}{2} \right) \leq \ln \left( x - \frac{x^2}{2} \right) + \frac{1}{2} \ln \frac{4}{3} = F(f(x)) + \frac{1}{2} \eta(f(x)),
\]
and
\[
F(f(y)) + \eta(f(x)) = \ln \left( \frac{x^2}{2} - \frac{x^4}{8} \right) + \ln \frac{4}{3}
\]
does not satisfy (1.1), (1.2) and (1.4) in Theorems 1.1-1.3, respectively. In fact, put
\[ F \colon (0, \infty) \to \mathbb{R} \]
\[ F(x) = \ln \left( x + \frac{x^2}{2} \right) + \ln \left( x - \frac{x^2}{2} \right) + \ln \frac{4}{3} \]
\[ \leq \ln \left( x - \frac{x^2}{2} \right) + \ln \frac{4}{3} \]
\[ = F(d(x, y)). \]

**Case 2.** Let \( x \in (1, +\infty) \cap (X - Tx) \). It follows that \( x \in (1, +\infty) \). Put \( y = \frac{x}{4} \in [0, \frac{1}{4}] - [0, \frac{1}{17}] = Tx - Ty \).

It is clear that
\[ F(d(x, y)) = \ln \left( x - \frac{1}{4} \right) \leq \ln \left( x - \frac{1}{4} \right) + \ln \frac{4}{3} = \frac{1}{2} \eta(f(x)), \]
\[ F(f(y)) + \eta(f(x)) = \ln \frac{7}{32} + \ln \frac{4}{3} = \ln \frac{7}{24} < \ln \frac{3}{4} \leq \ln \left( x - \frac{1}{4} \right) = F(d(x, y)). \]

That is, (a4) holds. It follows from Theorem 2.3 that
\[ T \] has a fixed point in \( X \). However, the mappings
\[ T \] does not satisfy (1.1), (1.2) and (1.4) in Theorems 1.1-1.3 respectively. In fact, put \( x_0 = 1 \) and \( y_0 = \frac{9}{8} \).

It is clear that
\[ H(Tx_0, Ty_0) = H \left( \left[ 0, \frac{1}{2} \right], \left[ 0, \frac{1}{4} \right] \right) = \frac{1}{4} \leq \frac{1}{8} \epsilon = cd(x_0, y_0), \quad \forall \epsilon \in [0, 1), \]
\[ H(Tx_0, Ty_0) = \frac{1}{4} \leq \frac{1}{8} \varphi(d(x_0, y_0)) = \varphi(d(x_0, y_0))d(x_0, y_0) \]
for any mapping \( \varphi : (0, +\infty) \to [0, 1) \) with each of (1.3) and (1.5).

**Example 3.4.** Let \( X = [0, 1] \) be endowed with the Euclidean metric \( d = | \cdot | \). Let \( T : X \to CL(X), \)
\[ F : (0, +\infty) \to \mathbb{R}, \eta : (0, +\infty) \to (0, +\infty) \]
be defined by
\[ T(x) = \begin{cases} \left\{ \frac{x^2}{3} \right\}, & x \in [0, \frac{17}{36}] \cup \left( \frac{17}{36}, 1 \right], \\ \left\{ \frac{1}{8}, \frac{5}{48} \right\}, & x = \frac{17}{36}, \end{cases} \]
\[ F(t) = \ln t, \quad \forall t \in (0, +\infty), \]
\[ \eta(t) = \begin{cases} \ln 10, & t \in [0, \frac{1}{10}], \\ \ln \frac{1}{5}, & t \in (\frac{1}{10}, \frac{1}{5}), \\ \ln \frac{9}{10}, & t \in [\frac{1}{5}, +\infty). \end{cases} \]

It is easy to see that
\[ f(x) = d(x, Tx) = \begin{cases} x - \frac{x^2}{3}, & x \in [0, \frac{17}{36}] \cup \left( \frac{17}{36}, 1 \right], \\ \frac{25}{72}, & x = \frac{17}{36}. \end{cases} \]
is lower semi-continuous in \( X \) and
\[ \lim_{s \to 0^+} \eta(s) = \ln 10 > 0. \]

Put \( x \in X - Tx \). In order to verify (a6), we consider the following two possible cases:

**Case 1.** Let \( x \in (0, \frac{17}{36}] \cup \left( \frac{17}{36}, 1 \right] - \left\{ \frac{x^2}{3} \right\} \). Put \( y = \frac{x^2}{3} \in \left( \frac{x^2}{3} \right) - \left\{ \frac{x^2}{3} \right\} = Tx - Ty \). Note that \( x - \frac{x^2}{3} \in (0, \frac{2}{3}] \).

Assume that \( x - \frac{x^2}{3} \in (0, \frac{1}{10}) \). It follows that
\[ \frac{1}{3} \left( x + \frac{x^2}{3} \right) < x - \frac{x^2}{3} < \frac{1}{10}. \]
which yields that
\[ \ln \frac{1}{3} \left( x + \frac{x^2}{3} \right) + \ln 10 < 0. \]

Consequently, we have
\[
F(d(x, y)) = \ln \left( x - \frac{x^2}{3} \right) \leq \ln \left( x - \frac{x^2}{3} \right) + \frac{1}{2} \ln 10 = F(f(x)) + \frac{1}{2} \eta(d(x, y)),
\]
and
\[
F(f(y)) + \eta(d(x, y)) = \ln \left( \frac{x^2}{3} - \frac{x^4}{27} \right) + \ln 10
\]
\[
= \ln \frac{1}{3} \left( x + \frac{x^2}{3} \right) + \ln \left( x - \frac{x^2}{3} \right) + \ln 10
\]
\[
< \ln \left( x - \frac{x^2}{3} \right)
\]
\[
= F(d(x, y)).
\]

Assume that \( x - \frac{x^2}{3} \in \left[ \frac{1}{10}, \frac{1}{5} \right) \). It follows that
\[
F(d(x, y)) = \ln \left( x - \frac{x^2}{3} \right) \leq \ln \left( x - \frac{x^2}{3} \right) + \frac{1}{2} \ln \frac{1}{(x - \frac{x^2}{3})} = F(f(x)) + \frac{1}{2} \eta(d(x, y)),
\]
and
\[
F(f(y)) + \eta(d(x, y)) = \ln \left( \frac{x^2}{3} - \frac{x^4}{27} \right) + \ln \frac{1}{(x - \frac{x^2}{3})}
\]
\[
= \ln \frac{1}{3} \left( x + \frac{x^2}{3} \right) + \ln \left( x - \frac{x^2}{3} \right) + \ln \frac{1}{(x - \frac{x^2}{3})}
\]
\[
= \ln \frac{1}{3} \left( x + \frac{x^2}{3} \right) < \ln \left( x - \frac{x^2}{3} \right)
\]
\[
= F(d(x, y)).
\]

Assume that \( x - \frac{x^2}{3} \in \left[ \frac{1}{5}, +\infty \right) \). It follows that
\[
F(d(x, y)) = \ln \left( x - \frac{x^2}{3} \right) \leq \ln \left( x - \frac{x^2}{3} \right) + \frac{1}{2} \ln \frac{9}{4} = F(f(x)) + \frac{1}{2} \eta(d(x, y)),
\]
and
\[
F(f(y)) + \eta(d(x, y)) = \ln \left( \frac{x^2}{3} - \frac{x^4}{27} \right) + \ln \frac{9}{4}
\]
\[
= \ln \frac{1}{3} \left( x + \frac{x^2}{3} \right) + \ln \left( x - \frac{x^2}{3} \right) + \ln \frac{9}{4}
\]
\[
\leq \ln \frac{4}{9} + \ln \left( x - \frac{x^2}{3} \right) + \ln \frac{9}{4} = \ln \left( x - \frac{x^2}{3} \right)
\]
\[
= F(d(x, y)).
\]

Case 2. Let \( x = \frac{17}{36} \). Put \( y = \frac{1}{5} \in \left\{ \frac{1}{5}, \frac{5}{18} \right\} - \left\{ \frac{1}{192} \right\} = T x - Ty \). It follows that
\[
F(d(x, y)) = \ln \frac{25}{72} \leq \ln \frac{25}{72} + \frac{1}{2} \ln \frac{9}{4} = F(f(x)) + \frac{1}{2} \eta(d(x, y)),
\]
and
\[
F(f(y)) + \eta(d(x, y)) = \ln \frac{23}{192} + \ln \frac{9}{4} < -1.31 < -1.06 < \ln \frac{25}{72} = F(d(x, y)).
\]
That is, (a6) holds. It follows from Theorem 2.4 that \( T \) has a fixed point in \( X \). However, the mappings \( T \) does not satisfy (1.1), (1.2) and (1.4) in Theorems 1.1-1.3, respectively. In fact, put \( x_0 = \frac{1}{2} \) and \( y_0 = \frac{17}{36} \). It is clear that

\[
H(Tx_0, Ty_0) = H\left(\frac{1}{12}, \frac{1}{8}, \frac{5}{48}\right) = \frac{1}{24} \leq \frac{1}{36} c = cd(x_0, y_0), \quad \forall c \in [0, 1),
\]

for any mapping \( \varphi : (0, +\infty) \to [0, 1) \) with each of (1.3) and (1.5).

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References


