Feng-Liu type fixed point results for multivalued mappings on JS-metric spaces

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Abstract

In this paper, we present a fixed point theorem for multivalued mappings on generalized metric space in the sense of Jleli and Samet [M. Jleli, B. Samet, Fixed Point Theory Appl., 2015 (2015), 61 pages]. In fact, we obtain as a special case both $b$-metric version and dislocated metric version of Feng-Liu’s fixed point result. ©2016 All rights reserved.

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1. Introduction and Preliminaries

Let $X$ be any nonempty set. An element $x \in X$ is said to be a fixed point of a multivalued mapping $T : X \rightarrow P(X)$ if $x \in Tx$, where $P(X)$ denotes the family of all nonempty subsets of $X$. Let $(X, d)$ be a
metric space. We denote the family of all nonempty closed and bounded subsets of $X$ by $CB(X)$ and the family of all nonempty closed subsets of $X$ by $C(X)$. For $A, B \in C(X)$, let
\[
H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\},
\]
where $d(x, B) = \inf \{d(x, y) : y \in B \}$. Then $H$ is called generalized Pompei-Hausdorff distance on $C(X)$. It is well known that $H$ is a metric on $CB(X)$, which is called Pompei-Hausdorff metric induced by $d$. We can find detailed information about the Pompeiu-Hausdorff metric in [3, 10].

Let $T : X \to CB(X)$. Then, $T$ is called multivalued contraction if there exists $L \in [0, 1)$ such that $H(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$ (see [16]). In 1969, Nadler [16] proved that every multivalued contraction on complete metric space has a fixed point. Then, the fixed point theory of multivalued contraction has been further developed in different directions by many authors, in particular, by Reich [17], Mizoguchi-Takahashi [15], Klim-Wardowski [14], Berinde-Berinde [2], Ćirić [4] and many others [5, 6, 12, 18]. Also, Feng and Liu [8] gave the following theorem without using generalized Pompei-Hausdorff distance. To state their result, we give the following notation for a multivalued mapping $T : X \to C(X)$: let $b \in (0, 1)$ and $x \in X$ define
\[
I_b^x(T) = \{ y \in Tx : bd(x, y) \leq d(x, Tx) \}.
\]

**Theorem 1.1** ([8]). Let $(X, d)$ be a complete metric space and $T : X \to C(X)$. If there exists a constant $c \in (0, 1)$ such that there is $y \in I_b^x(T)$ satisfying
\[
d(y, Ty) \leq cd(x, y)
\]
for all $x \in X$. Then $T$ has a fixed point in $X$ provided that $c < b$ and the function $x \to d(x, Tx)$ is lower semicontinuous.

As mentioned in Remark 1 of [8], we can see that Theorem [1.1] is a real generalization of Nadler’s.

The aim of this paper is to present Feng-Liu type fixed point results for multivalued mappings on some generalized metric space such as $b$-metric spaces and dislocated metric spaces. To do this, we will consider JS-metric on a nonempty set.

Let $X$ be a nonempty set and $D : X \times X \to [0, \infty]$ be a mapping. For every $x \in X$ define a set
\[
C(D, X, x) = \{ \{x_n\} \subset X : \lim_{n \to \infty} D(x_n, x) = 0 \}.
\]

In this case, we say that $D$ is a generalized metric in the sense of Jleli and Samet [11] (for short JS-metric) on $X$ if it satisfies the following conditions:

1. $(D_1)$ for every $(x, y) \in X \times X$, $D(x, y) = 0 \Rightarrow x = y$;
2. $(D_2)$ for every $(x, y) \in X \times X$, $D(x, y) = D(y, x)$;
3. $(D_3)$ there exists $c > 0$ such that for every $(x, y) \in X \times X$ and $\{x_n\} \in C(D, X, x)$,
\[
D(x, y) \leq c \limsup_{n \to \infty} D(x_n, y).
\]

In this case $(X, D)$ is said to be JS-metric space. Note that, if $C(D, X, x) = \emptyset$ for all $x \in X$, then $(D_3)$ is trivially held. The class of JS-metric space is larger than many known class of metric space. For example, every standard metric space, every $b$-metric space, every dislocated metric space (in the sense of Hitzler-Seda [9]), and every modular space with the Fatou property is a JS-metric space. For more details see [11].

Let $(X, D)$ be a JS-metric space, $x \in X$, and $\{x_n\}$ be a sequence in $X$. If $\{x_n\} \in C(D, X, x)$, then $\{x_n\}$ is said to be converges to $x$. If $\lim_{n,m \to \infty} D(x_n, x_{n+m}) = 0$, then $\{x_n\}$ is said to be Cauchy sequence. If every
Cauchy sequence in \((X, D)\) is convergent, then \((X, D)\) is said to be complete. By Proposition 2.4 of [1], we see that every convergent sequence in \((X, D)\) has a unique limit. That is, if \(\{x_n\} \in C(D, X) \cap C(D, X, y)\), then \(x = y\).

After the introducing the JS-metric space, Jleli and Samet [11] presented some fixed point results including Banach contraction and Ćirić type quasicontraction mappings.

2. Main result

Let \((X, D)\) be a JS-metric space and \(U \subseteq X\). We say that \(U\) is sequentially open if for each sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} D(x_n, x) = 0\) for some \(x \in U\) is eventually in \(U\), that is, there exists \(n_0 \in \mathbb{N}\) such that \(x_n \in U\) for all \(n \geq n_0\). Let \(\tau_{JS}\) be the family of all sequentially open subsets of \(X\), then it is easy to see that \((X, \tau_{JS})\) is a topological space. Further, a sequence \(\{x_n\}\) is convergent to \(x\) in \((X, D)\) if and only if it is convergent to \(x\) in \((X, \tau_{JS})\). Let \(C(X)\) be the family of all nonempty closed subsets of \((X, \tau_{JS})\) and let \(\Lambda\) be the family of all nonempty subsets \(A\) of \(X\) satisfying the following property: for all \(x \in X\),

\[
D(x, A) = 0 \Rightarrow x \in A,
\]

where \(D(x, A) = \inf\{D(x, y) : y \in A\}\). In this case, \(C(X) = \Lambda\). Indeed, let \(A \in C(X)\) and \(x \in X\). If \(D(x, A) = 0\), then there exists a sequence \(\{x_n\}\) in \(A\) such that \(\lim_{n \to \infty} D(x, x_n) = 0\). Therefore, by the definition of the topology \(\tau_{JS}\), for any \(U \in \tau_{JS}\) including the point \(x\), there exists \(n_U \in \mathbb{N}\) such that \(x_n \in U\) for all \(n \geq n_U\). In this case, we have \(U \cap A \neq \emptyset\), that is, \(x \in \overline{A} = A\). Hence \(C(X) \subseteq \Lambda\). Now, let \(A \in \Lambda\). We will show that \(A \in C(X)\). Let \(x \in X\) and \(\{x_n\}\) be a sequence in \(X\) such that \(\lim_{n \to \infty} D(x_n, x) = 0\). If there exists a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) such that \(\{x_{n_k}\} \subset A\), then we get \(D(x, A) = 0\). Since \(A \in \Lambda\), then \(x \in A\). This is a contradiction. Therefore, there exists \(n_0 \in \mathbb{N}\) such that \(x_n \in X\setminus A\) for all \(n \geq n_0\). This shows that \(X\setminus A \in \tau_{JS}\), and so \(A \in C(X)\). As a consequence we get \(C(X) = \Lambda\).

Now we will consider the following special cases for \(\tau_{JS}\):

Let \((X, D)\) be a metric space. Then it is clear that \(\tau_{JS}\) coincides with the metric topology \(\tau_D\).

Let \((X, D)\) be a \(b\)-metric space. In this case, there are three topologies on \(X\) as follows: First is sequential topology \(\tau_s\), which is defined as in Definition 3.1 (3) of [1]. Second is the \(\tau_D\) topology [13], which is the family of all open subsets of \(X\) in the usual sense, that is, a subset \(U\) of \(X\) is open if for any \(x \in U\), there exists \(\varepsilon > 0\) such that

\[
B(x, \varepsilon) := \{y \in X : D(x, y) < \varepsilon\} \subseteq U.
\]

Third is the \(\tau^D\) topology, which the family of all finite intersections of

\[
C = \{B(x, \varepsilon) : x \in X, r > 0\},
\]

satisfies conditions (B1)–(B2) of (7), Proposition 1.2.1) is a base of \(\tau^D\). By Proposition 3.3 of [1], we know that \(\tau_s = \tau_D \subset \tau^D\). Also by Definition 2.1 and Theorem 3.4 of [1], we can see that \(\tau_{JS} = \tau_s\).

Let \((X, D)\) be a dislocated metric space in the sense of Hitzler and Seda [9]. In this case, the set of balls does not in general yield a conventional topology. However, by defining a new membership relation, which is more general than the classical membership relation from set theory, Hitzler and Seda [9] constructed a suitable topology on dislocated metric space as follows: Let \(X\) be a set. A relation \(\triangleleft \subseteq X \times P(X)\) is called \(d\)-membership relation on \(X\) if it satisfies the following property: for all \(x \in X\) and \(A, B \in P(X)\),

\[
x \triangleleft A \text{ and } A \subseteq B \text{ implies } x \triangleleft A.
\]

Let \(U_x\) be a nonempty collection of subsets of \(X\) for each \(x \in X\). If the following conditions are satisfied, then the pair \((U_x, \triangleleft)\) is called \(d\)-neighbourhood system for \(x\):

(i) if \(U \in U_x\), then \(x \triangleleft U\);

(ii) if \(U, V \in U_x\), then \(U \cap V \in U_x\);
(iii) if \( U \subseteq U_x \), then there is \( V \subseteq U \) with \( V \subseteq U_x \) such that for all \( y \in V \) we have \( U \subseteq U_y \);
(iv) if \( U \subseteq U_x \) and \( U \subseteq V \), then \( V \subseteq U_x \).

The \( d \)-neighbourhood system \( (U_x, \prec) \) generates a topology on \( X \). This topological space is called \( d \)-topological space and indicated as \((X, U, \prec)\), where \( U = \{U_x : x \in X\} \).

Now, let \((X, D)\) be a dislocated metric space in the sense of Hitzler and Seda [9]. Define a membership relation \( \prec \) as the relation
\[
\{(x, A) : \text{there exists } \varepsilon > 0 \text{ for which } B(x, \varepsilon) \subseteq A\}. \tag{2.1}
\]

In this case, by Proposition 3.5 of [9], we know that \((U_x, \prec)\) is \( d \)-neighbourhood system for \( x \) for each \( x \in X \), where \( U_x \) be the collection of all subsets \( A \) of \( X \) such that \( x \prec A \). By taking into account the Definition 2.2, Definition 3.8 and Proposition 3.9 of [9] we can see that the \( d \)-topology generated by (2.1) on \((X, D)\) coincides with the topology \( \tau_{JS} \).

Let \((X, D)\) be a generalized metric space and \( T : X \to C(X) \) be a multivalued mapping. For a constant \( b \in (0, 1) \) and \( x \in X \), we will consider the following set in our main result:
\[
I^c_0(T) = \{y \in Tx : bD(x, y) \leq D(x, Tx)\}.
\]

**Theorem 2.1.** Let \((X, D)\) be a complete generalized metric space and \( T : X \to C(X) \) be multivalued mapping. Suppose there exists a constant \( c > 0 \) such that for any \( x \in X \) there is \( y \in I^c_0(T) \) satisfying
\[
D(y, Ty) \leq cD(x, y). \tag{2.2}
\]

If there exists \( x_0 \in X \) such that \( D(x_0, Tx_0) < \infty \), then it can be constructed a sequence \( \{x_n\} \) in \( X \) satisfying:
(i) \( x_{n+1} \in Tx_n \);
(ii) \( D(x_n, x_{n+1}) < \infty \);
(iii) \( bD(x_{n+1}, x_{n+2}) \leq cD(x_n, x_{n+1}) \) and \( bD(x_{n+1}, Tx_{n+1}) \leq cD(x_n, Tx_n) \).

If this constructed sequence is Cauchy and the function \( f(x) = D(x, Tx) \) is lower semicontinuous, then \( T \) has a fixed point.

Now consider the following important remarks, before giving the proof of Theorem 2.1.

**Remark 2.2.** If \((X, D)\) is a metric space (or dislocated metric space in the sense of Hitzler and Seda [9]) and \( c < b \), then the mentioned sequence in Theorem 2.1 is Cauchy. Indeed, since \( D \) has triangular inequality, for \( m, n \in \mathbb{N} \) with \( m > n \), we get from (iii),
\[
D(x_n, x_m) \leq D(x_n, x_{n+1}) + \cdots + D(x_{m-1}, x_m)
\leq \left(\frac{c}{b}\right)^n D(x_0, x_1) + \cdots + \left(\frac{c}{b}\right)^{m-1} D(x_0, x_1)
\leq \frac{(c/b)^n}{1 - (c/b)} D(x_0, x_1).
\]
Since \( c < b \), then \( \{x_n\} \) is Cauchy sequence.

**Remark 2.3.** If \((X, D)\) is a \( b \)-metric space with \( b \)-metric constant \( s \) and \( sc < b \), then the mentioned sequence in Theorem 2.1 is Cauchy. Indeed, in this case, we have
\[
D(x, y) \leq s[D(x, z) + D(z, y)]
\]
Therefore, for \( m, n \in \mathbb{N} \) with \( m > n \), we get from (iii),
Assume that $s c < b$, then the sequence $\{x_n\}$ is a Cauchy sequence.

Proof of Theorem 2.1. First observe that, since $Tx \in C(X)$ for all $x \in X$, $I_b^x(T)$ is nonempty. Let $x_0 \in X$ be such that $D(x_0, Tx_0) < \infty$. Then, from (2.2), there exists $x_1 \in I_b^{x_0}(T)$ such that

$$D(x_1, Tx_1) \leq cD(x_0, x_1).$$

Note that, since $x_1 \in I_b^{x_0}(T)$, then $x_1 \in Tx_0$ and

$$bD(x_0, x_1) \leq D(x_0, Tx_0) < \infty.$$

For $x_1 \in X$, there exists $x_2 \in I_b^{x_1}(T)$ such that

$$D(x_2, Tx_2) \leq cD(x_1, x_2).$$

By the way, we can construct a sequence $\{x_n\}$ in $X$ such that $x_{n+1} \in I_b^{x_n}(T)$ and

$$D(x_{n+1}, Tx_{n+1}) \leq cD(x_n, x_{n+1})$$

for all $n \in \mathbb{N}$. Note that, since $D(x_0, Tx_0) < \infty$, then $D(x_n, x_{n+1}) < \infty$ for all $n \in \mathbb{N}$.

Again, since $x_{n+1} \in I_b^{x_n}(T)$, we have $x_{n+1} \in Tx_n$ and

$$bD(x_n, x_{n+1}) \leq D(x_n, Tx_n)$$

for all $n \in \mathbb{N}$. Therefore from (2.3) and (2.4), we get

$$bD(x_{n+1}, x_{n+2}) \leq D(x_{n+1}, Tx_{n+1}) \leq cD(x_n, x_{n+1}),$$

and

$$D(x_{n+1}, Tx_{n+1}) \leq cD(x_n, x_{n+1}) \leq \frac{c}{b}D(x_n, Tx_n).$$

Hence (i), (ii), and (iii) hold. Furthermore, from (2.5) and (2.6), we get

$$\lim_{n \to \infty} D(x_n, x_{n+1}) = \lim_{n \to \infty} D(x_n, Tx_n) = 0.$$

Now, if $\{x_n\}$ is a Cauchy sequence then by the completeness of $(X, D)$, there exists $z \in X$ such that $x_n \in C(D, X, z)$, that is, $\lim_{n \to \infty} D(x_n, z) = 0$. Therefore, by the lower semicontinuity of the function $f(x) = D(x, Tx)$, we get

$$0 \leq D(z, Tz) = f(z) \leq \liminf_{n \to \infty} f(x_n) = \liminf_{n \to \infty} D(x_n, Tx_n) = 0.$$

Since $Tz \in C(X)$, we get $z \in Tz$.

By taking into account Remark 2.2 and Remark 2.3, we obtain the following results from Theorem 2.1.

**Corollary 2.4** (Feng-Liu’s fixed point theorem). Let $(X, d)$ be a complete metric space and $T : X \to C(X)$ be multivalued mapping. Suppose there exists a constant $c > 0$ such that for any $x \in X$ there is $y \in I_b^x(T)$ satisfying

$$d(y, Ty) \leq cd(x, y).$$

Then $T$ has a fixed point provided that $c < b$ and the function $f(x) = d(x, Tx)$ is lower semicontinuous.
Corollary 2.5 (Feng-Liu’s fixed point theorem on $b$-metric space). Let $(X,d)$ be a complete $b$-metric space with $b$-metric constant $s$ and $T : X \rightarrow C(X)$ be multivalued mapping. Suppose there exists a constant $c > 0$ such that for any $x \in X$ there is $y \in I^*_b(T)$ satisfying
\[ d(y,Ty) \leq cd(x,y). \]
Then $T$ has a fixed point provided that $sc < b$ and the function $f(x) = d(x,Tx)$ is lower semicontinuous.

Corollary 2.6 (Feng-Liu’s fixed point theorem on dislocated metric space). Let $(X,d)$ be a complete dislocated metric space and $T : X \rightarrow C(X)$ be multivalued mapping. Suppose there exists a constant $c > 0$ such that for any $x \in X$ there is $y \in I^*_b(T)$ satisfying
\[ d(y,Ty) \leq cd(x,y). \]
Then $T$ has a fixed point provided that $c < b$ and the function $f(x) = d(x,Tx)$ is lower semicontinuous.

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References