A study on a class of $q$-Euler polynomials under the symmetric group of degree $n$

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Abstract

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1. Introduction
Throughout the paper, we make use of the following notations: $\mathbb{Z}_p$ denotes topological closure of $\mathbb{Z}$, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes topological closure of $\mathbb{Q}$, and $\mathbb{C}_p$ indicates the field of $p$-adic completion of an algebraic closure of $\mathbb{Q}_p$, in which $p$ be a fixed odd prime number. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* := \mathbb{N} \cup \{0\}$. For $d$ an odd positive number with $(p,d) = 1$, let

$$X := X_d = \lim_{n \to \infty} \frac{\mathbb{Z}}{dp^n \mathbb{Z}}$$

and $X_1 = \mathbb{Z}_p$.

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and
\[ a + dp^N \mathbb{Z}_p = \{ x \in X \mid x \equiv a \mod dp^N \}, \]
where \( a \in \mathbb{Z} \) lies in \( 0 \leq a < dp^N \) cf. \([1, 2, 5, 10] \).

The normalized absolute value according to the theory of \( p \)-adic analysis is given by \(|p|_p = p^{-1}\). The notation \( q \) can be considered as an indeterminate a complex number \( q \in \mathbb{C} \) with \(|q| < 1\), or a \( p \)-adic number \( q \in \mathbb{C}_p \) with \(|q - 1|_p < p^{-r} \) and \( q^x = \exp(x \log q) \) for \(|x|_p \leq 1\). It is always clear in the content of the paper. For any \( x \), the \( q \)-analogue of \( x \) is defined as \([x]_q = \frac{1 - q^x}{1 - q}\). Note that \( \lim_{q \to 1} [x]_q = x \) cf. \([1, 2, 5, 10]\).

Let \( C(\mathbb{Z}_p) \) be the space of continuous functions on \( \mathbb{Z}_p \). For \( f \in C(\mathbb{Z}_p) \), Kim \([6]\) originally constructed fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \), as follows:
\[
I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu_{-1}(x) = \lim_{n \to \infty} \sum_{x=0}^{p^n-1} f(x) (-1)^x, \tag{1.1}
\]
and also
\[
I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{1.2}
\]
where \( f_1(x) = f(x+1) \).

The \( \lambda \)-Euler polynomials \( E_n(\lambda, x) \) (usually called Apostol-Euler polynomials) for \( \lambda \in \mathbb{C} \) were defined by the power series expansion at \( t = 0 \):
\[
\sum_{n=0}^{\infty} E_n(\lambda, x) \frac{t^n}{n!} = \frac{2}{\lambda e^t + 1} e^{xt}, \tag{1.2}
\]
\((|t| < \pi \text{ when } \lambda = 1; |t| < \log (-\lambda) \text{ when } \lambda \neq 1; 1^0 := 1)\).

Taking \( x = 0 \) in the Eq. (1.2), we have \( E_n(\lambda, 0) := E_n(\lambda) \) that is known as \( n \)-th \( \lambda \)-Euler numbers (see for details, \([3, 4, 11]\)).

In \([9]\), Kim et al. defined \( q \)-extension of \( \lambda \)-Euler polynomials by the following fermionic \( p \)-adic integral on \( \mathbb{Z}_p \):
\[
E_{n,q}(\lambda, x) = \frac{[2]^q}{2} \int_{\mathbb{Z}_p} \lambda^y [x + y]^n d\mu_{-1}(y). \tag{1.3}
\]
Letting \( x = 0 \) into the Eq. (1.3) yields \( E_{n,0}(\lambda, 0) := E_{n,q}(\lambda) \) called \( n \)-th \( q \)-extension of \( \lambda \)-Euler numbers, cf. \([9]\).

By taking \( q \to 1^- \) in the Eq. (1.3), we have
\[
\lim_{q \to 1^-} E_{n,q}(\lambda, x) := E_n(\lambda, x) = \int_{\mathbb{Z}_p} \lambda^y (x+y)^n d\mu_{-1}(y). \tag{1.3}
\]

Motivated by the paper of Kim et al. \([10]\), we study \( q \)-extension of \( \lambda \)-Euler polynomials earlier given by Kim et al \([9]\). We derive some new symmetric identities for these polynomials, using fermionic \( p \)-adic invariant integral over the \( p \)-adic number field introduced by Kim \([6]\), under symmetric group of degree \( n \) denoted by \( S_n \).

We now give some interesting identities derived from the fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \) in the next section.

2. New symmetric identities for \( E_{n,q}(\lambda, x) \) under \( S_n \)

Let \( w_i \) be odd natural numbers where \( i = \{1, 2, \cdots, n\} \). From the Eqs. (1.1) and (1.3), we consider
\[
\int_{\mathbb{Z}_p} \lambda^y \prod_{j=1}^{n-1} w_i [x \prod_{j=1}^{n-1} \left( \prod_{j=1}^{w_i} y_j \right)^{x+w_n \sum_{j=1}^{n-1} \left( \prod_{j=1}^{w_i} w_i \right) k_j} t]^q d\mu_{-1}(y)
\]
Therefore, we derive that

\[
\begin{aligned}
I &= \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} (-1)^y \lambda^y \prod_{j=1}^{n-1} w_j \left( \prod_{j=1}^{n} \left( \Pi_{i=1}^{j-1} \Pi_{j \neq i}^{n} \right) y \right) \times e^{\lambda \left( \Pi_{j=1}^{n} \Pi_{j \neq i}^{n} \right) y} t_q d\mu_- (y) \\
&= \lim_{N \to \infty} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} \sum_{m=0}^{w_n-1} (-1)^m y (m + w_n y) \prod_{j=1}^{n-1} w_j \\
&\quad \times e^{\lambda \left( \Pi_{j=1}^{n} \Pi_{j \neq i}^{n} \right) y} t_q d\mu_- (y).
\end{aligned}
\]

(2.1)

Observe that the Eq. (2.1) is invariant under any permutation \( \sigma \in S_n \). Hence, we state the following theorem.

**Theorem 2.1.** Let \( w_i \) be odd natural numbers where \( i = \{1, 2, \cdots, n\} \). Then the following

\[
\prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} (-1)^y \lambda^y \prod_{j=1}^{n-1} w_j \left( \prod_{j=1}^{n} \left( \Pi_{i=1}^{j-1} \Pi_{j \neq i}^{n} \right) y \right) \times e^{\lambda \left( \Pi_{j=1}^{n} \Pi_{j \neq i}^{n} \right) y} t_q d\mu_- (y)
\]

holds true for any \( \sigma \in S_n \).

By using the definition of \([x]_q\), we have

\[
\begin{aligned}
&\left[ \left( \prod_{j=1}^{n-1} w_j \right) y + \left( \prod_{j=1}^{n} \Pi_{j=1}^{n} w_j \right) x + w_n \sum_{j=1}^{n-1} \left( \Pi_{i=1}^{j-1} w_i \right) k_j \right]_q \\
&= \left[ \prod_{j=1}^{n-1} w_j \right]_q \left[ y + w_n x + \sum_{j=1}^{n-1} \frac{w_n}{w_1} k_j \right]_q w_1 w_2 \cdots w_{n-1} \\
&= \left[ \prod_{j=1}^{n-1} w_j \right]_q \left[ y + w_n x + \sum_{j=1}^{n-1} \frac{w_n}{w_j} k_j \right]_q w_1 w_2 \cdots w_{n-1}.
\end{aligned}
\]

(2.2)
It follows from the Eq. (2.2) that

$$\int_{\mathbb{Z}} \lambda^n \prod_{j=1}^{n-1} w_j e^{\left(\prod_{j=1}^{n-1} w_j y + \left(\prod_{j=1}^{n} w_j x + w_n \sum_{j=1}^{n-1} \frac{w_n}{w_j} k_j \right) t \right)} \frac{d \mu_{-1}(y)}{m!} = \sum_{m=0}^{\infty} \sum_{\pi \subseteq \{1, 2, \ldots, n\} \backslash \{1\}} \left( \prod_{i \in \pi} \frac{(\prod_{j=1}^{n} w_j)^{m_i}}{m_i!} \lambda^{w_{\sigma(i)}(n)} \sum_{j=1}^{n-1} \frac{w_n}{w_j} k_j \right) \left( \prod_{i \in \pi} \frac{w_n}{w_j} k_j \right) \left( \prod_{i \in \pi} \frac{w_n}{w_j} k_j \right)$$

From Eq. (2.3), we have

$$\int_{\mathbb{Z}} \lambda^n \prod_{j=1}^{n-1} w_j \left( \prod_{j=1}^{n} w_j y + \left(\prod_{j=1}^{n} w_j x + w_n \sum_{j=1}^{n-1} \frac{w_n}{w_j} k_j \right) \right) \frac{d \mu_{-1}(y)}{m!} = \sum_{m=0}^{\infty} \sum_{\pi \subseteq \{1, 2, \ldots, n\} \backslash \{1\}} \left( \prod_{i \in \pi} \frac{(\prod_{j=1}^{n} w_j)^{m_i}}{m_i!} \lambda^{w_{\sigma(i)}(n)} \sum_{j=1}^{n-1} \frac{w_n}{w_j} k_j \right) \left( \prod_{i \in \pi} \frac{w_n}{w_j} k_j \right) \left( \prod_{i \in \pi} \frac{w_n}{w_j} k_j \right)$$

Thus, by Theorem 2.1 and Eq. (2.4), we derive the following theorem.

**Theorem 2.2.** Let $w_i$ be odd natural numbers where $i = \{1, 2, \ldots, n\}$. For $m \geq 0$, the following holds true for any $\sigma \in S_n$.

$$\sum_{m=0}^{\infty} \sum_{\pi \subseteq \{1, 2, \ldots, n\} \backslash \{1\}} \left( \prod_{i \in \pi} \frac{(\prod_{j=1}^{n} w_j)^{m_i}}{m_i!} \lambda^{w_{\sigma(i)}(n)} \sum_{j=1}^{n-1} \frac{w_n}{w_j} k_j \right) \left( \prod_{i \in \pi} \frac{w_n}{w_j} k_j \right) \left( \prod_{i \in \pi} \frac{w_n}{w_j} k_j \right)$$

By using the definitions of $[x]_q$ and binomial theorem, we can write:

$$\sum_{m=0}^{\infty} \sum_{\pi \subseteq \{1, 2, \ldots, n\} \backslash \{1\}} \left( \prod_{i \in \pi} \frac{(\prod_{j=1}^{n} w_j)^{m_i}}{m_i!} \lambda^{w_{\sigma(i)}(n)} \sum_{j=1}^{n-1} \frac{w_n}{w_j} k_j \right) \left( \prod_{i \in \pi} \frac{w_n}{w_j} k_j \right) \left( \prod_{i \in \pi} \frac{w_n}{w_j} k_j \right)$$

Taking $\int_{\mathbb{Z}} \lambda^n \prod_{j=1}^{n-1} w_j \frac{d \mu_{-1}(y)}{m!}$ on the both sides of the above equation gives

$$\int_{\mathbb{Z}} \lambda^n \prod_{j=1}^{n-1} w_j \left( \prod_{j=1}^{n} w_j y + \left(\prod_{j=1}^{n} w_j x + w_n \sum_{j=1}^{n-1} \frac{w_n}{w_j} k_j \right) \right) \frac{d \mu_{-1}(y)}{m!} = \sum_{m=0}^{\infty} \sum_{\pi \subseteq \{1, 2, \ldots, n\} \backslash \{1\}} \left( \prod_{i \in \pi} \frac{(\prod_{j=1}^{n} w_j)^{m_i}}{m_i!} \lambda^{w_{\sigma(i)}(n)} \sum_{j=1}^{n-1} \frac{w_n}{w_j} k_j \right) \left( \prod_{i \in \pi} \frac{w_n}{w_j} k_j \right) \left( \prod_{i \in \pi} \frac{w_n}{w_j} k_j \right)$$
\[ \times q \sum_{j=1}^{m-1} \left( \prod_{i \neq j}^{m-1} w_i \right) k_j \int_{Z_p} \lambda^j \prod_{j=1}^{m-1} w_j [y + w_n x]^l w_n^j w_{n-1}^j d\mu(y) \]

\[ = \frac{2}{[2]^q} \sum_{l=0}^{m} \left( \frac{l}{m} \right) \left[ \prod_{j=1}^{m-1} w_j \right]^l w_n^j w_{n-1}^j E_{l,q\prod_{i \neq j}^{m-1} w_i} \left( \lambda^j w_n^j w_{n-1}^j, w_n x \right) \]

As a result of the Eq. (2.5), we obtain

\[ \sum_{j=1}^{m-1} \left( \prod_{i \neq j}^{m-1} w_i \right) k_j \int_{Z_p} \lambda^j \prod_{j=1}^{m-1} w_j [y + w_n x + \sum_{j=1}^{n-1} w_n k_j] q^{w_n^j w_{n-1}^j} \]

\[ = \frac{2}{[2]^q} \sum_{l=0}^{m} \left( \frac{l}{m} \right) \left[ \prod_{j=1}^{m-1} w_j \right]^l w_n^j w_{n-1}^j E_{l,q\prod_{i \neq j}^{m-1} w_i} \left( \lambda^j w_n^j w_{n-1}^j, w_n x \right) U_{m,qw_n,\lambda w_n} (w_1, w_2, ..., w_{n-1} | l), \]

where

\[ U_{m,q,\lambda}(w_1, w_2, ..., w_{n-1} | l) = \prod_{s=1}^{n-1} \sum_{k_s=0}^{w_n^s-1} (-1)^{\sum_{i=1}^{s-1} k_i} \lambda^{\sum_{j=1}^{s-1} k_j} \prod_{j=1}^{s-1} w_j^{k_j} \prod_{j=1}^{s-1} \left( \sum_{i \neq j}^{m-1} w_i \right)^{k_j} q^{m-k_j}. \]

Last of all, by (2.5), we get the following theorem.

**Theorem 2.3.** Let \( w_i \) be odd natural numbers where \( i = \{1, 2, \ldots, n\} \) and let \( m \geq 0 \). Then the following expression

\[ \sum_{l=0}^{m} \left( \frac{n}{m} \right) \left[ \prod_{j=1}^{n-1} w_{\sigma(j)} \right]^l q^{w_{\sigma(n)}} E_{l,q\prod_{i \neq j}^{n-1} w_{\sigma(n)}^j} \left( \lambda^{w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(n-1)}}, w_{\sigma(n)}^j x \right) U_{m,q,\lambda w_{\sigma(n)}} (w_{\sigma(1)}, w_{\sigma(2)}, ..., w_{\sigma(n-1)} | l) \]

holds true for some \( \sigma \in S_n \).
References