Positive solutions to a class of $q$-fractional difference boundary value problems with $\phi$-Laplacian operator

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Abstract

By virtue of the upper and lower solutions method, as well as the Schauder fixed point theorem, the existence of positive solutions to a class of $q$-fractional difference boundary value problems with $\phi$-Laplacian operator is investigated. The conclusions here extend existing results. ©2016 All rights reserved.

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1. Introduction

In recent years, the fractional $q$-difference boundary value problems have received more attention as a new research direction by scholars both at home and abroad (see [1, 2, 4–6]). In [2], the author studied positive solutions to a class of $q$-fractional difference boundary value problems. In [6], the authors used $u_0$-concave operator fixed point theorem to study the following fractional difference boundary value problems

$$\begin{cases}
(D^\alpha_q y)(x) = -f(x, y(x)), & 0 < x < 1, \quad 2 < \alpha \leq 3, \\
y(0) = (D^\alpha_q y)(0) = 0, & (D^\alpha_q y)(1) = 0.
\end{cases}$$

An iterative sequence of positive solutions was established. In [4], the authors used a fixed point theorem on posets to study the existence and uniqueness of positive solutions to a class of $q$-fractional difference boundary value problems with $p$-Laplacian operator:

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\[
\begin{cases}
D_q^\gamma (\phi_{\mu} (D_q^\alpha u(t))) + f(t,u(t)) = 0, & 0 < t < 1, 2 < \alpha < 3, \\
u(0) = (D_q u)(0) = 0, & (D_q u)(1) = \beta (D_q u)(\eta).
\end{cases}
\]

Motivated by the aforementioned work, we investigate the existence of positive solutions to a class of q-fractional difference boundary value problems with \(\phi\)-Laplacian operator:

\[
\begin{cases}
D_q^\gamma (\phi_{\mu} (D_q^\alpha u(t))) = f(t,u(t)), & 0 < t < 1, \\
u(0) = u(1) = (D_q u)(0) = (D_q u)(1) = 0,
\end{cases}
\]

where \(1 < \alpha, \beta < 2, D_q^\gamma\) is the Riemann–Liouville fractional order derivative, the nonlinear term \(f(t,u(t))\) \(\in ([0,1] \times [0, +\infty), (0, +\infty))\) and \(\phi\)-Laplacian is defined by

\[
\phi_{\mu}(s) = |s|^{\mu - 2}s, \mu > 1, (\phi_{\mu})^{-1} = \phi_v, 1/\mu + 1/v = 1.
\]

2. Preliminaries

In the following section we give the definition of Riemann–Liouville fractional q-order derivative for \(q \in [0,1]\). One can refer to [3] for other related definitions and basic knowledge.

**Definition 2.1.** The q-derivative of a function \(f(x)\) is given by

\[
(D_q f)(x) = \frac{f(x) - f(qx)}{(1 - q)x}, (D_q f)(0) = \lim_{x \to 0} (D_q f)(x),
\]

and higher order q-derivatives are defined by

\[
(D_q^n f)(x) = f(x), \quad (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), \quad n \in \mathbb{N}.
\]

**Definition 2.2.** The q-integral of \(f(x)\) on the interval \([0, b]\) is given by

\[
(I_q f)(x) = \int_{0}^{x} f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n)q^n, \quad x \in [0, b].
\]

If the q-integral for the function \(f(x)\) on the interval \([a, b]\) exists, then

\[
\int_{a}^{b} f(t) d_q t = \sum_{n=0}^{\infty} \int_{0}^{a} f(t) d_q t - \int_{0}^{b} f(t) d_q t, \quad a \in [0, b].
\]

\[
(I_q^0 f)(x) = f(x), \quad (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.
\]

**Definition 2.3.** Let \(\alpha > 0\) and \(f(x)\) be a function defined on \([0,1]\). The fractional q-integral of the Riemann–Liouville type is

\[
(I_q^0 f)(x) = f(x),
\]

\[
(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_{0}^{x} (x - qt)^{(\alpha - 1)} f(t) d_q t, \quad \alpha > 0, \quad x \in [0,1],
\]

where the \(\Gamma_q(\alpha)\) function is defined by

\[
\Gamma_q(\alpha) = \frac{(1-q)^{\alpha - 1}}{(1-q)^{\alpha - 1}},
\]

and \((1-q)^{\alpha}\) is defined by

\[
(1-q)^{0} = 1, \quad (1-q)^{\alpha} = \prod_{k=0}^{\alpha-1} (1-q^k), \quad \alpha \in \mathbb{N} \setminus \{0, -1, -2, \ldots\}.
\]
Definition 2.4. The fractional $q$-derivative of the Riemann–Liouville type of order $\alpha > 0$ is defined by
\[
(D_q^\alpha f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \quad \alpha > 0, \quad x \in [0, 1],
\]
where $m$ is the smallest integer greater than or equal to $\alpha$. In the particular case,
\[
(t_0^\alpha f)(x) = f(x).
\]
Let
\[
(G_\alpha)(t, s) = \begin{cases} 
(t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}, & 0 < s \leq t \leq 1, \\
(t(1-s))^{\alpha-1}, & 0 < t \leq s \leq 1, \quad \alpha > 0.
\end{cases}
\]
$G_\alpha$ is a nonnegative continuous function on $[0, 1] \times [0, 1]$.

Lemma 2.5 [2]. Let $1 < \alpha \leq 2$ and suppose that $y(t) \in C[0, 1]$. Then
\[
\begin{cases}
(D_q^\alpha u)(t) + y(t) = 0, & 0 < t < 1, \\
u(0) = u(1) = 0,
\end{cases}
\]
is equivalent to
\[
u(t) = \int_0^1 G_\alpha(t, qs)y(s)d_qs.
\]
If $y(t) \geq 0, t \in [0, 1]$, then $\nu(t) \geq 0, t \in [0, 1]$.

Lemma 2.6 [5]. Let $y(t) \in C[0, 1], 1 < \alpha, \beta \leq 2$. Then the fractional $q$-difference
\[
\begin{cases}
(D_q^\alpha u)(t) + y(t) = 0, & 0 < t < 1, \\
u(0) = u(1) = 0, \quad (D_q^\alpha u)(0) = (D_q^\alpha u)(1) = 0
\end{cases}
\]
is equivalent to
\[
u(t) = \int_0^1 \left(G_\alpha(t, qs)\phi_\psi \left(\int_0^1 G_\beta(s, q\tau)y(\tau)d_q\tau\right)\right)d_qs.
\]
Suppose
\[
E = \{ u|u, \phi_\mu(D_q^\alpha u) \in C^2[0, 1] \}.
\]
The following definitions are about the upper and lower solutions to problem (1.1).

Definition 2.7. A function $\varphi(t) \in E$ is called a lower solution to (1.1), if it satisfies
\[
\begin{cases}
(D_q^\alpha \phi_\mu(D_q^\alpha \varphi(t))) \leq f(t, \varphi(t)), & 0 < t < 1, \\
\varphi(0) \leq 0, \quad \varphi(1) \leq 0, \quad D_q^\alpha \varphi(0) \geq 0, \quad D_q^\alpha \varphi(1) \geq 0.
\end{cases}
\]

Definition 2.8. A function $\psi(t) \in E$ is called an upper solution to (1.1), if it satisfies
\[
\begin{cases}
(D_q^\alpha \phi_\mu(D_q^\alpha \psi(t))) \geq f(t, \psi(t)), & 0 < t < 1, \\
\psi(0) \leq 0, \quad \psi(1) \leq 0, \quad D_q^\alpha \psi(0) \geq 0, \quad D_q^\alpha \psi(1) \geq 0.
\end{cases}
\]

3. Main results

According to Lemma 2.6, we can define an operator as follows:
\[
Tu(t) = \int_0^1 \left(G_\alpha(t, qs)\phi_\psi \left(\int_0^1 G_\beta(s, q\tau)f(\tau, u(\tau))d_q\tau\right)\right)d_qs, \quad u \in E.
\]
By the continuity of $G_\alpha, G_\beta, f$ and using the Arzela–Ascoli theorem, we can get that $T : E \to E$ is completely
continuous operator, and the existence of a solution to problem (1.1) is equivalent to the existence of a fixed point of $T$.

Suppose that the following assumptions are satisfied

(H1) $f(t,u) \in C([0,1] \times [0, +\infty), [0, +\infty))$, and $f$ is increasing with respect to the second variable.

(H2) there exists a $c < 1$ and a $k \in [0, 1]$, such that

$$f(t,ku) \geq k^{c(\mu-1)}f(t,u), \; \forall t \in [0,1],$$

where $\mu > 1$.

**Lemma 3.1.** If $u$ is a positive solution to (1.1), then there exist $m_1, m_2 > 0$, such that

$$m_1 \rho(t) \leq u(t) \leq m_2 \rho(t),$$

where

$$\rho(t) = \int_0^1 \left( G_\alpha(t,qs)\phi_v \left( \int_0^1 G_\beta(s,q\tau)y(\tau)dq\tau \right) \right) dqds.$$

**Proof.** It follows from $u \in C[0,1]$, so there exist an $M > 0$ such that $|u(t)| \leq M, t \in [0,1]$. By (H2) we can take

$$m_1 = \min_{t \in [0,1], u \in [0,M]} v^{-\frac{1}{2}}f(t,u(t)) > 0,$$

$$m_2 = \max_{t \in [0,1], u \in [0,M]} v^{-\frac{1}{2}}f(t,u(t)) > 0.$$

So

$$m_1 \rho(t) \leq u(t) = \int_0^1 \left( G_\alpha(t,qs)\phi_v \left( \int_0^1 G_\beta(s,q\tau)y(\tau)dq\tau \right) \right) dqds \leq m_2 \rho(t).$$

This completes the proof.

**Theorem 3.2.** Suppose that (H1) and (H2) are satisfied. Then (1.1) has a positive solution.

**Proof.** We prove the theorem in three steps as follows.

**Step 1.** The existence of upper and lower solutions for (1.1). Let

$$\eta(t) = \int_0^1 \left( G_\alpha(t,qs)\phi_v \left( \int_0^1 G_\beta(s,q\tau)y(\tau)dq\tau \right) \right) dqds.$$

Then by Lemma 2.6 we obtain a positive solution to the problem

$$\begin{cases} D^\beta_q(\phi_t(D^\alpha_qu(t))) = f(t,\rho(t)), \; 0 < t < 1, \\ u(0) = u(1) = 0, \; D^\alpha_qu(0) = D^\alpha_qu(1) = 0. \end{cases}$$

(3.1)

Furthermore,

$$\eta(0) = \eta(1) = 0, \; D^\alpha_q\eta(0) = D^\alpha_q\eta(1) = 0.$$  (3.2)

By Lemma 3.1 there exist $k_1, k_2 > 0$, such that

$$k_1 \rho(t) \leq \eta(t) \leq k_2 \rho(t), \; \forall t \in [0,1].$$

Let

$$\xi_1(t) = \delta_1 \eta(t), \; \xi_2(t) = \delta_2 \eta(t),$$

where

$$0 < \delta_1 < \min\left\{ \frac{1}{k_2}, \frac{1}{k_1} \right\}, \; \delta_2 > \max\left\{ \frac{1}{k_1}, \frac{1}{k_2} \right\}. $$
From (3.3), we have

\[ f(t, \xi_1(t)) = f(t, \delta_1(t)) = f(t, \frac{\eta(t)}{\rho(t)}) \]

\[ \geq (\delta_1 \frac{\eta(t)}{\rho(t)})^{c(\mu-1)} f(t, \rho(t)) \]

\[ \geq (\delta_1 k_1)^{c(\mu-1)} f(t, \rho(t)) \geq \delta_1^{-1} f(t, \rho(t)). \]  

(3.3)

and

\[ D_q^\beta (\phi_\mu (D_q^\alpha \xi_1(t))) = D_q^\beta (\phi_\mu (D_q^\alpha \delta_1 \eta(t))) = \delta_1^{-1} D_q^\beta (\phi_\mu (D_q^\alpha \eta(t))) \]

From (3.3), we have

\[ \xi_1(0) = \xi_1(1) = 0, \quad D_q^\alpha \xi_1(0) = D_q^\alpha \xi_1(1) = 0. \]

By Definition 2.7, \( \xi_1(t) \) is a lower solution to [1.1].

On the other hand, by the definition of \( \xi_2(t) \), we can obtain

\[ \delta_2^{-1} f(t, \rho(t)) = \delta_2^{-1} f(t, \frac{\rho(t)}{\rho_2(t)}) \xi_2(t) = \delta_2^{-1} f(t, \frac{\rho(t)}{\delta_2 \xi_2(t)}) \delta_2 \xi_2(t) \]

\[ \geq \delta_2^{-1} \left( \frac{\rho(t)}{\delta_2 \eta(t)} \right)^{c(\mu-1)} f(t, \xi_2(t)) \geq \delta_2^{-1} \left( \frac{\rho(t)}{\delta_2 k_2} \right)^{c(\mu-1)} f(t, \xi_2(t)) \]

\[ \geq \delta_2^{-1} \left( \frac{1}{\delta_2 \eta(t)} \right)^{c(\mu-1)} f(t, \xi_2(t)) \geq \delta_2^{-1} \left( \frac{1}{\delta_2} \right)^{c(\mu-1)} f(t, \xi_2(t)) \]

\[ = f(t, \xi_2(t)). \]

So

\[ D_q^\beta (\phi_\mu (D_q^\alpha \xi_2(t))) = D_q^\beta (\phi_\mu (D_q^\alpha \delta_2 \eta(t))) \]

\[ = \delta_2^{-1} D_q^\beta (\phi_\mu (D_q^\alpha \eta(t))) = \delta_2^{-1} f(t, \rho(t)) \]

\[ \geq f(t, \xi_2(t)). \]

Similarly

\[ \xi_2(0) = \xi_2(1) = 0, \quad D_q^\alpha \xi_2(0) = D_q^\alpha \xi_2(1) = 0. \]

By Definition 2.8, \( \xi_2(t) \) is an upper solution to [1.1].

**Step 2.** We prove that the following problem has a positive solution:

\[ \begin{cases} 
D_q^\beta (\phi_\mu (D_q^\alpha u(t))) = g(t, u(t)), & 0 < t < 1, \\
u(0) = u(1) = 0, & D_q^\alpha u(0) = D_q^\alpha u(1) = 0.
\end{cases} \]  

(3.4)

where

\[ g(t, u(t)) = \begin{cases} 
f(t, \xi_1(t)), & u(t) < \xi_1(t), \\
f(t, u(t)), & \xi_1(t) \leq u(t) \leq \xi_2(t), \\
f(t, \xi_2(t)), & u(t) > \xi_2(t).
\end{cases} \]

By Lemma 2.6, we need the following operator

\[ Au(t) = \int_0^1 \left( G_\alpha(t,qs) \phi_v \left( \int_0^1 G_\beta(s,\tau) g(\tau, u(\tau)) d_q \tau \right) \right) d_q s, \quad u \in C[0, 1]. \]

Now, we use the Schauder fixed point theorem to prove the existence of a fixed point of \( Au(t) \). In fact \( f(t, u) \) is increasing with respect to \( u \), so for any \( u \in C([0, 1], [0, +\infty)) \), there exist \( g(t, u(t)) \) such that

\[ f(t, \xi_1(t)) \leq g(t, u(t)) \leq f(t, \xi_2(t)). \]
Since $G_\alpha, G_\beta$ and $f$ are continuous, then by the Arzela–Ascoli theorem, $A$ is a compact operator. Thus, by using the Schauder fixed point theorem, $A$ has a fixed point, i.e., equation (3.4) has a positive solution, denoted by $u^*$.

**Step 3.**
To prove that $u^*$ is also a solution to (1.1), we only need to prove that
\[ \xi_1(t) \leq u^*(t) \leq \xi_2(t), \quad t \in [0, 1]. \] (3.5)

First we prove $u^*(t) \leq \xi_2(t), \ t \in [0, 1]$; one can prove another inequality in the same way.

Suppose $u^*(t) > \xi_2(t), \ t \in [0, 1]$; we have $g(t, u^*(t)) = f(t, \xi_2(t))$. We obtain
\[ D_q^\beta(\phi_\mu(D_q^\alpha u^*(t))) = f(t, \xi_2(t)). \]

On the other hand, $\xi_2(t)$ is an upper solution, so we have
\[ D_q^\beta(\phi_\mu(D_q^\alpha \xi_2(t))) \geq f(t, \xi_2(t)). \]

Let $z(t) = \phi_\mu(D_q^\alpha (D_q^\beta \xi_2(t))) - \phi_\mu(D_q^\alpha u^*(t)), \ t \in [0, 1]$. Therefore,
\[ D_q^\beta z(t) = D_q^\beta(\phi_\mu(D_q^\alpha \xi_2(t))) - D_q^\beta(\phi_\mu(D_q^\alpha u^*(t))) \geq f(t, \xi_2(t)) - f(t, \xi_2(t)) = 0. \]

Combined with the boundary conditions, $z(0) = z(1) = 0$ and by Lemma 2.5, we have $z(t) \leq 0, \ t \in [0, 1]$, which implies that
\[ \phi_\mu(D_q^\alpha \xi_2(t)) \leq \phi_\mu(D_q^\alpha u^*(t)), \ t \in [0, 1]. \]

Since $\phi_\mu$ is monotone increasing, we obtain $D_q^\alpha(\xi_2(t)) \leq D_q^\alpha(u^*(t)), \ t \in [0, 1], \ t \in [0, 1]$. Using Lemma 2.5, we get $\xi_2(t) - u^*(t) \geq 0, \ t \in [0, 1]$, a contradiction.

Inequality (3.5) shows that $u^*$ is also a positive solution to (1.1). Furthermore $f(t, 0) \neq 0$, that is to say, 0 is not a fixed point of the operator $T$, therefore, $u^*$ is a positive solution to (1.1). This completes the proof. \qed

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**References**


