The stability of sextic functional equation in fuzzy modular spaces

Kittipong Wongkum\textsuperscript{a,b}, Poom Kumam\textsuperscript{a,b,c,}\textsuperscript{*}

\textsuperscript{a}Department of Mathematics, Faculty of Science, King Mongkut’s University of Technology Thonburi (KMUTT), 126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand.

\textsuperscript{b}Theoretical and Computational Science (TaCS) Center, Science Laboratory Building, Faculty of Science, King Mongkut’s University of Technology Thonburi (KMUTT), 126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand.

\textsuperscript{c}Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan.

Communicated by Y. J. Cho

Abstract

By using the fixed point technique, we prove the stability of sextic functional equations. Our results are studied and proved in the framework of fuzzy modular spaces (briefly, $\mathcal{FM}$-spaces). The lower semi continuous (briefly, l.s.c.) and $\beta$-homogeneous are necessary conditions for this work. ©2016 All rights reserved.

Keywords: Stability, sextic mapping, fuzzy modular space.

2010 MSC: 46A80, 39B82.

1. Introduction

In 1940 during a conference at Wisconsin University, S. M. Ulam [16] presented the following question concerning stability of group homomorphisms:

Let $G_1$ be a group and let $G_2$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $g : G_1 \to G_2$ with $d(f(x), g(x)) < \epsilon$ for all $x \in G_1$?

When the homomorphisms are stable? So, we are interested in this question, that is, if a mapping is almost a homomorphism, then there exists an exact homomorphism that must be close. In following year, Hyers [7] was the first to give an affirmative answer to Ulam’s question for the case where $G_1$ and $G_2$ are

\textsuperscript{*}Corresponding authors.

Email addresses: kittipong.wong@mail.kmutt.ac.th (Kittipong Wongkum), poom.kum@kmutt.ac.th (Poom Kumam)

Received 2015-07-15
Banach spaces. After that, a generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [14]. Later, the stability problems of various functional equation have been extensively investigated by many authors [3, 4].

One of the interesting functional equations studied is the system of additive-quadratic-cubic functional equations [6]:

\[
\begin{align*}
 f(ax_1 + bx_2, y, z) + f(ax_1 - bx_2, y, z) &= 2af(x_1, y, z), \\
 f(x, ay_1 + by_2, z) + f(x, ay_1 - by_2, z) &= 2a^2f(x, y_1, z) + 2b^2f(x, y_2, z), \\
 f(x, y, az_1 + bz_2) + f(x, y, az_1 - bz_2) &= ab^2(f(x, y, z_1 + z_2) + f(x, y, z_1 - z_2)) + 2a(a^2 - b^2)f(x, y, z_1),
\end{align*}
\]

where \(a, b \in \mathbb{Z} \setminus \{0\}\) with \(a \neq \pm 1, \pm b\).

The function \(f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) defined by \(f(x, y, z) = cxy^2z^3\) is a solution of the system (1.1). In particular, letting \(y = z = x\), we get a sextic function \(h : \mathbb{R} \to \mathbb{R}\) in one variable given by \(h(x) := f(x, x, x) = cx^6\).

The concept of modular spaces was introduced by Nakano [12]. Soon after, the notation of modular spaces was redefined and generalized by Musielak and Orlicz [11]. In 2007, Nourouzi [13] presented probabilistic modular spaces related to the theory of modular spaces.

After that, Shen and Chen [15] following the idea of probabilistic modular spaces and the definition of fuzzy metric spaces based on George and Veeramani’s sense [5], applied fuzzy concept to the classical notions of modular and modular spaces, and in 2013, Shen and Chen [15] presented the concept of a fuzzy modular space. After that, Kumam [16, 17], Wongkum and et al [18] studied fixed points and some properties in modular or fuzzy modular spaces.

In this paper, we investigate the generalized Ulam-Hyers-Rassias (briefly, UHR) stability of a sextic functional equations from linear spaces into \(\mathcal{F}\mathcal{M}\)-spaces, by using some ideas of [2, 18].

2. Preliminaries

In this section, conventionally, we write throughout the paper \(\mathbb{R}\), \(\mathbb{C}\), and \(\mathbb{N}\) to denote respectively the set of all reals, complexes, and nonnegative integers.

Moreover, we recall some basic definitions and properties of a fuzzy modular space.

**Definition 2.1** ([17]). A fuzzy set \(A\) in \(X\) is a function with domain \(X\) and value in \([0, 1]\).

**Definition 2.2** ([1]). A triangular norm (briefly, t-norm) is a function \(* : [0, 1] \times [0, 1] \to [0, 1]\) satisfying, for each \(a, b, c, d \in [0, 1]\), the following conditions:

1. \(a * 1 = a\);
2. \(a * b \leq c * d\) whenever \(a \leq c, b \leq d\);
3. \(a * b = b * a\); and \((a * b) * c = a * (b * c)\).

**Definition 2.3.** Let \(X\) be a vector space over a field \(\mathbb{K}\) (\(\mathbb{R}\) or \(\mathbb{C}\)). A generalized functional \(\rho : X \to [0, \infty]\) is called a modular if for arbitrary \(x, y \in X\),

\begin{align*}
&(m1) \quad \rho(x) = 0 \text{ if and only if } x = 0, \\
&(m2) \quad \rho(\alpha x) = \rho(x) \text{ for every scalar } \alpha \text{ with } |\alpha| = 1, \\
&(m3) \quad \rho(z) \leq \rho(x) + \rho(y), \text{whenever } z \text{ is a convex combination of } x \text{ and } y.
\end{align*}

The corresponding modular space, denoted by \(X_\rho\), is then defined by

\[X_\rho := \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.
\]
Remark 2.4. Note that for a fixed $x \in X_{\rho}$, the valuation $\gamma \in K \mapsto \rho(\gamma x)$ is increasing.

Unlike a norm, a modular needs not be continuous or convex in general. However, it often occurs that some weaker form of them are assumed.

Remark 2.5. In case a modular $\rho$ is convex, one has $\rho(x) \leq \delta \rho(\frac{1}{\delta} x)$ for all $x \in X_{\rho}$, provided that $0 < \delta \leq 1$.

Definition 2.6. Let $X_{\rho}$ be a modular space and $\{x_n\}$ be a sequence in $X_{\rho}$. Then,

(i) $\{x_n\}$ is $\rho$-convergent to a point $x \in X_{\rho}$ and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(ii) $\{x_n\}$ is called $\rho$-Cauchy if for all $\epsilon > 0$, we have $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in \mathbb{N}$.

(iii) A subset $K \subset X_{\rho}$ is called $\rho$-complete if any $\rho$-Cauchy sequence is $\rho$-convergent.

Another unnatural behavior one usually encounter is that the convergence of a sequence $\{x_n\}$ to $x$ does not imply that $\{cx_n\}$ converges to $cx$, where $c$ is chosen from the corresponding scalar field. Thus, many mathematicians imposed some additional conditions for a modular to meet in order to make the multiples of $\{x_n\}$ converge naturally. Such preferences are referred to mostly under the term related to the $\Delta_2$-conditions.

A modular $\rho$ is said to satisfy the $\Delta_2$-condition if there exists $\kappa \geq 2$ such that $\rho(2x) \leq \kappa \rho(x)$ for all $x \in X_{\rho}$. Some authors varied the notion so that only $\kappa > 0$ is required and called it the $\Delta_2$-type condition. In fact, one may see that these two notions coincide. There are still a number of equivalent notions related to the $\Delta_2$-conditions.

Remark 2.7. We have to be very careful about the convergence behaviors on multiples and sums of $\rho$-convergent sequences. In general, we suppose that $\{x_n^1\}, \{x_n^2\}, \ldots, \{x_n^{2k}\}$, for some $k \in \mathbb{N}$, are sequences in $X_{\rho}$ in which they $\rho$-converge to the points $x^1, x^2, \ldots, x^{2k} \in X_{\rho}$, respectively. Then, the averaged sequence $\left\{ \frac{1}{2^k} \sum_{i=1}^{2k} x_n^i \right\}$ $\rho$-converges to $\frac{1}{2^k} \sum_{i=1}^{2k} x^i$.

In [8], Khamsi proved a series of fixed point theorems in modular spaces where the modulars do not satisfy $\Delta_2$-conditions. His results exploit one unifying hypothesis in which the boundedness of an orbit is assumed.

Definition 2.8. Given a modular space $X_{\rho}$, a nonempty subset $C \subset X_{\rho}$, and a mapping $T : C \to C$. The orbit of $T$ around a point $x \in X_{\rho}$ is the set

$$O(x) := \{x, Tx, T^2x, \ldots\}.$$  

The quantity $\delta_{\rho}(x) := \sup\{\rho(u - v) : u, v \in O(x)\}$ is then associated and is called the orbital diameter of $T$ at $x$. In particular, if $\delta_{\rho}(x) < \infty$, we say that $T$ has a bounded orbit at $x$.

Lemma 2.9 ([8]). Let $X_{\rho}$ be a modular space whose the induced modular is l.s.c. and $C \subset X_{\rho}$ be a $\rho$-complete subset. If $T : C \to C$ is a $\rho$-contraction, i.e., there is a constant $k \in [0, 1)$ such that

$$\rho(Tx - Ty) \leq kp(x - y), \quad \forall x, y \in X_{\rho},$$

and $T$ has a bounded orbit at a point $x_0 \in X_{\rho}$, then the sequence $\{T^n x_0\}$ is $\rho$-convergent to a point $w \in C$.

Definition 2.10 ([15]). Let $V$ be a real or complex vector space with a zero $\theta$, $*$ a continuous triangular norm, and $\mu$ a fuzzy set on the product $V \times \mathbb{R}^+$. Suppose that the following properties hold for $x, y \in V$ and $s, t > 0$:

(FM1) $\mu(x, t) > 0$;

(FM2) $\mu(x, t) = 1$ for all $t > 0$ if and only if $x = \theta$;

(FM3) $\mu(x, t) = \mu(-x, t)$;
(FM4) \( \mu(z, s + t) \geq \mu(x, s) \cdot \mu(y, t) \) whenever \( z \) is the convex combination between \( x \) and \( y \);

(FM5) the mapping \( t \mapsto \mu(x, t) \) is continuous at each fixed \( x \in V \).

Then, we write \((V, \mu, *)\) to represent the space with the pre-defined properties. In particular, we call \( \mu \) a fuzzy modular and the triple \((V, \mu, *)\) a fuzzy modular space (briefly, \( \mathcal{F} \mathcal{M} \)-space).

It is worth noting that every fuzzy modular is non-decreasing with respect to \( t > 0 \).

**Example 2.11.** Let \( X \) be a real or complex vector space and \( \rho \) be a modular on \( X \). Take the \( t \)-norm \( a \cdot b = \min \{a, b\} \). For every \( t \in (0, \infty) \), define \( \mu(x, t) = \frac{t}{1 + \rho(x)} \) for all \( x \in X \). Then \((X, \mu, *)\) is a \( \mathcal{F} \mathcal{M} \)-space.

**Remark 2.12.** Note that the above conclusion still holds even if the \( t \)-norm is replaced by \( a \cdot b = \min\{a + b - 1, 0\} \) and \( a \cdot b = \max\{a + b - 1, 0\} \), respectively.

**Definition 2.13.** Let \((X, \mu)\) be a \( \mathcal{F} \mathcal{M} \)-space, \( \{x_n\} \) be a sequence in \( X \) and \( x \in X \).

1. The sequence \( \{x_n\} \) with \( x_n \in (X, \mu) \) is said to be \( \mu \)-convergent to \( x \) (write \( x_n \xrightarrow{\mu} x \)) if, for any \( t > 0 \) and \( \lambda (0, 1) \), there exists a positive integer \( n_0 \) such that
   \[
   \mu(x_n - x, t) > 1 - \lambda
   \]
   for all \( n \geq n_0 \)

2. The sequence \( \{x_n\} \) with \( x_n \in (X, \mu) \) is called a \( \mu \)-Cauchy sequence if, for any \( t > 0 \) and \( \lambda (0, 1) \), there exists a positive integer \( n_0 \) such that
   \[
   \mu(x_n - x_m, t) > 1 - \lambda
   \]
   for all \( n, m \geq n_0 \).

3. Every \( \mu \)-convergent sequence in \( \mathcal{F} \mathcal{M} \)-space is \( \mu \)-Cauchy sequence. If each \( \mu \)-Cauchy sequence is \( \mu \)-convergent sequence in a \( \mathcal{F} \mathcal{M} \)-space \((X, \mu)\), then \((X, \mu)\) is called a \( \mu \)-complete \( \mathcal{F} \mathcal{M} \)-space.

Shen and Chen [15] also studied the topological properties of a fuzzy modular space with a special property that for every \( x \in V \) and a non-zero real \( \lambda \), the equality

\[
\mu(\lambda x, t) = \mu\left(x, \frac{t}{|\lambda|^{\beta}}\right)
\]

holds for some fixed \( \beta \in (0, 1] \). If the fuzzy modular \( \mu \) has this property, we shall say that it is \( \beta \)-homogeneous.

The \( \mu \)-ball in \((V, \mu, *)\) is the set of the form

\[
B(x, r, t) := \{y \in V | \mu(x - y, t) > 1 - r\},
\]

where \( r \in (0, 1) \) and \( t > 0 \).

Now, suppose that \( \mu \) is \( \beta \)-homogeneous for some \( \beta \in (0, 1] \). According to Shen and Chen [15], the family \( \mathcal{B} \) of all \( \mu \)-balls forms a base for a first-countable Hausdorff topology, written as \( \mathcal{T}_\mu \). With the notion of the \( \mu \)-balls, it is easy to see that a sequence \((x_n)\) in \( V \) \( \mu \)-converges (i.e. it converges in the topology \( \mathcal{T}_\mu \)) to its \( \mu \)-limit \( x \in V \) if and only if \( \mu(x - x_n, t) \to 1 \) as \( n \to \infty \) for all \( t > 0 \). Note here that the \( \mu \)-limit is unique if it does exists after all. It is then natural to say that \((x_n)\) is \( \mu \)-Cauchy if for any given \( \varepsilon \in (0, 1) \) and \( t > 0 \), there exists \( N \in \mathbb{N} \) with \( \mu(x_m - x_n, t) > 1 - \varepsilon \) whenever \( m, n > N \). We say that \( \mu \)-complete if every \( \mu \)-Cauchy sequence converge.

From here, let us turn to a typical example of a triangular norm which is defined by \((a \ast b) = \min\{a, b\}\). This triangular norm has a very special property that if \( \ast' \) be an arbitrary triangular norm, then \((a \ast' b) \leq (a \ast b)\) for all \( a, b \in [0, 1] \). With this property, it is suitable to call this \( \ast \) a strongest triangular norm. As is
claimed by Shen and Chen [15], if \( V \) is a real vector space equipped with a \( \beta \)-homogeneous fuzzy modular \( \mu \) and a strongest triangular norm \( * \), then a \( \mu \)-convergent sequence is \( \mu \)-Cauchy. The authors also mentioned that if \( * \) is not the strongest one, such implementation is not always true.

We say that \( \mathcal{FM} \)-space \((X, \mu, *)\) satisfies the lower semi continuous if, for any sequence \( x_n \) of \( X \) and \( \mu \)-converging to a point \( x \in X \),
\[
\mu(x, t) \leq \liminf_{n \to \infty} \mu(x_n, t)
\]
for all \( t > 0 \).

**Theorem 2.14** ([8]). Let \( X_p \) be a modular space satisfying l.s.c. property. Let \( C \) be a \( \rho \)-complete nonempty subset of \( X_p \) and \( T: C \to C \) be a quasi-contraction, that is, there exists \( K < 1 \) such that
\[
\rho(T(x) - T(y)) \leq K \max\{\rho(x - y), \rho(x - T(x)), \rho(y - T(y)), \rho(x - y), \rho(y - T(x))\}.
\]
Let \( X \in C \) such that
\[
\delta_{\mu}(x) := \sup\{\rho(T^n(x) - T^m(x)) : m, n \in \mathbb{N}\} < \infty.
\]
Then \( \{T^n(x)\} \) \( \rho \)-converges to a point \( w \in C \). Moreover, if \( \rho(w - T(w)) < \infty \) and \( \rho(x - T(w)) < \infty \), then the \( \rho \)-limit of \( T^n(x) \) is a fixed point of \( T \). Furthermore, if \( w^* \) is any fixed point of \( T \) in \( C \) such that \( \rho(w - w^*) < \infty \), then one has \( w = w^* \).

In this section, we assume that \( \mu \) is a fuzzy modular on \( V \) with the l.s.c. (in the fuzzy modular sense) and \((V, \mu, *)\) is a \( \mu \)-complete \( \beta \)-homogeneous \( \mathcal{FM} \)-space with \( \beta \in (0, 1] \) and \( * \) is defined by minimum t-norm. Also, we establish the conditional UHR stability of sextic functional equations in a \( \mathcal{FM} \)-space.

**Theorem 2.15.** Let \( E \) be a linear space and \((V, \mu, *)\) be a \( \mu \)-complete \( \beta \)-homogeneous \( \mathcal{FM} \)-space and \( p \in \{-1, 1\} \) be fixed. Suppose that \( f: E \times E \times E \to (V, \mu, *) \) satisfies the condition \( f(x, 0, z) = 0 \) and the inequalities of the form:
\[
\begin{align*}
\mu(f(ax_1 + bx_2, y, z) + f(ax_1 - bx_2, y, z) - 2af(x_1, y, z), t) & \geq \tau(x_1, x_2, y, z, t), \\
\mu(f(x, ay_1 + by_2, z) + f(x, ay_1 - by_2, z) - 2b^2f(x, y_1, z) - 2b^2f(x, y_2, z), t) & \geq \varsigma(x, y_1, y_2, z, t), \\
\mu(f(x, y, az_1 + bz_2) + f(x, z, -az_1 - bz_2) - ab^2f(x, y, z_1 + z_2) + f(x, y, z_1 - z_2) - 2(a^2 - b^2)f(x, y, z_1), t) & \geq \upsilon(x, y, z_1, z_2, t),
\end{align*}
\]
where \( \tau, \varsigma, \upsilon : E^4 \to \triangle \), and \( \triangle \) is the set of all non-decreasing functions, are given functions such that
\[
\begin{align*}
&\lim_{n \to \infty} \tau(a^n x_1, a^n x_2, a^n y, a^n z, a^{6 \beta n} t) = 1, \\
&\lim_{n \to \infty} \varsigma(a^n x, a^n y_1, a^n y_2, a^n z, a^{6 \beta n} t) = 1, \\
&\lim_{n \to \infty} \upsilon(a^n x, a^n y, a^n z_1, a^n z_2, a^{6 \beta n} t) = 1
\end{align*}
\]
for all \( x, x_i, y, y_i, z, z_i \in E, i = 1, 2 \). Assume that
\[
\Phi(x, y, z, t) := \varsigma(\frac{p+1}{2} x, \frac{p+1}{2} y, \frac{p-1}{2} z, 0, a^{(9 - 3p) \beta t/2^{3+2}}) \\
\times \varsigma(\frac{p+1}{2} y, 0, a^{(6 - 3p) \beta t/2^{3+2}}) \\
\times \tau(\frac{p+1}{2} x, 0, a^{(4 - 3p) \beta t/2})
\]
for all \( x, y, z \in E \).
has the property:
\[ \Phi(a^nx, a^ny, a^nz, a^{6\beta}Lt) \geq \Phi(x, y, z, t) \] (2.5)
for all \( x, y, z \in E \) with a constant \( 0 < L < \frac{1}{\beta} \). Then there exists a unique sextic function \( s : E \times E \times E \to (V, \mu, *) \) satisfying the system (1.1) such that
\[ \mu(s(x, y, z) - f(x, y, z), \frac{2\beta}{1 - 2\beta L}t) \geq \Phi(x, y, z, t). \] (2.6)

**Proof.** Let \( x_1 = 2x \) and \( x_2 = 0 \) and replacing \( y, z \) by \( 2y, 2z \) in (2.1), respectively, we get
\[ \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), t) \geq \tau(2x, 0, 2y, 2z, t) \] (2.7)
for all \( x, y, z \in E \).

Let \( y_1 = 2y \) and \( y_2 = 0 \) and replacing \( x, z \) by \( 2ax, 2az \) in (2.2), respectively, we have
\[ \mu(2f(2ax, 2ay, 2z) - 2a^2f(2ax, 2ay, 2z), t) \geq \varsigma(2ax, 2y, 0, 2z, t) \] (2.8)
for all \( x, y, z \in E \).

Let \( z_1 = 2z \) and \( z_2 = 0 \) and replacing \( x, y \) by \( 2ax, 2ay \) in (2.3), respectively, we obtain
\[ \mu(2f(2ax, 2ay, 2az) - 2a^3f(2ax, 2ay, 2z), t) \geq \upsilon(2ax, 2ay, 2z, 0, t) \] (2.9)
for all \( x, y, z \in E \). Since \( \mu \) is \( \beta \)-homogeneous. We note that, since
\[
\mu(2f(2ax, 2ay, 2az) - 2a^3f(2ax, 2ay, 2z), t) \\
\geq \mu(\frac{1}{a^3}(2f(2ax, 2ay, 2az) - 2a^3f(2ax, 2ay, 2z)), t).
\]
Hence, since \( \mu \) is \( \beta \)-homogeneous, it follows from (2.8) and (2.9) that
\[
\mu(2f(2ax, 2ay, 2z) - 2a^2f(2ax, 2y, 2z) \\
+ 2f(2ax, 2ay, 2az) - 2a^3f(2ax, 2ay, 2z), t) \\
\geq \mu(2f(2ax, 2ay, 2z) - 2a^2f(2ax, 2y, 2z) \\
+ 2a^{-3}f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z), t) \\
= \mu(2a^{-3}f(2ax, 2ay, 2az) - 2a^2f(2ax, 2y, 2z), t) \\
\geq \mu(a^{-3}f(2ax, 2ay, 2az) - a^2f(2ax, 2y, 2z), t) \\
= \mu(2a^{-3}f(2ax, 2ay, 2az) - \frac{2}{2}a^2f(2ax, 2y, 2z), t) \\
= \mu(2a^{-3}f(2ax, 2ay, 2az) - 2a^2f(2ax, 2y, 2z), 2^3t) \\
\geq \mu(2a^{-3}f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z) + 2f(2ax, 2ay, 2z) \\
- 2a^2f(2ax, 2y, 2z), 2^3t) \\
= \mu(2a^{-3}f(2ax, 2ay, 2az) - f(2ax, 2ay, 2z) + f(2ax, 2ay, 2z) \\
- a^2f(2ax, 2y, 2z), 2^3t) \\
\geq \mu(2a^{-3}f(2ax, 2ay, 2az) - f(2ax, 2ay, 2z) + f(2ax, 2ay, 2z) \\
- a^2f(2ax, 2y, 2z), t) \\
= \mu(\frac{1}{2}(2a^{-3}f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z)) + \frac{1}{2}(2f(2ax, 2ay, 2z) \\
- 2a^2f(2ax, 2y, 2z)), t/2 + t/2) \\
\geq \mu(2a^{-3}f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z), t/2)
\[ * \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z), t/2) \]
\[ = \mu(2f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2ay, 2z), t/2) \]
\[ * \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z), t/2) \]
\[ \geq v(2ax, 2ay, 2z, 0, a^{3\beta} t/2) * \varsigma(2ax, 2y, 0, 2z, t/2) \]

and hence

\[ \mu(2a^{-3} f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z), t) \]
\[ \geq \mu\left(\frac{1}{a^2} (2a^{-3} f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z)), t\right) \]
\[ = \mu(2a^{-5} f(2ax, 2ay, 2az) - 2f(2ax, 2y, 2z), t) \]
\[ = \mu\left((2a^{-5}) \frac{a^2}{a^2} f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z), t\right) \]
\[ = \mu\left(\frac{1}{a^2} (2a^{-3} f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z)), t\right) \]
\[ = \mu(2a^{-3} f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z), a^{2\beta} t) \]
\[ \geq \mu(2a^{-3} f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z) + 2f(2ax, 2ay, 2z) \]
\[ - 2a^2 f(2ax, 2y, 2z), a^{2\beta} t) \]
\[ = \mu(2a^{-3} f(2ax, 2ay, 2az) - f(2ax, 2ay, 2z) + f(2ax, 2ay, 2z) \]
\[ - a^2 f(2ax, 2y, 2z), a^{2\beta} t) \]
\[ = \mu(a^{-3} f(2ax, 2ay, 2az) - f(2ax, 2ay, 2z) + f(2ax, 2ay, 2z) \]
\[ - a^2 f(2ax, 2y, 2z), a^{2\beta} t/2^\beta) \]
\[ = \mu\left(\frac{1}{2} (2a^{-3} f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z)) + \frac{1}{2} (2f(2ax, 2ay, 2z) \]
\[ - 2a^2 f(2ax, 2y, 2z), a^{2\beta} t/2^\beta+1 + a^{2\beta} t/2^\beta+1) \]
\[ \geq \mu(2a^{-3} f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z), a^{2\beta} t/2^\beta+1) \]
\[ * \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z), a^{2\beta} t/2^\beta+1) \]
\[ = \mu\left((2a^{-3}) \frac{a^3}{a^3} f(2ax, 2ay, 2az) - 2a^3 f(2ax, 2ay, 2z), a^{2\beta} t/2^\beta+1) \right) \]
\[ * \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z), a^{2\beta} t/2^\beta+1) \]
\[ = \mu\left(2f(2ax, 2ay, 2z) - 2a^3 f(2ax, 2ay, 2z), a^{2\beta} t/2^\beta+1) \right) \]
\[ * \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z), a^{2\beta} t/2^\beta+1) \]
\[ \geq v(2ax, 2ay, 2z, 0, a^{3\beta} t/2^\beta+1) * \varsigma(2ax, 2y, 0, 2z, a^{2\beta} t/2^\beta+1) \]

for all \(x, y, z \in E\). By [27] and the last inequality, we get

\[ \mu(a^{-5} f(2ax, 2ay, 2az) - af(2x, 2y, 2z), t) \]
\[ = \mu(a^{-5} f(2ax, 2ay, 2az) - f(2ax, 2y, 2z) \]
\[ + f(2ax, 2y, 2z) - af(2x, 2y, 2z), t) \]
\[ = \mu\left(\frac{1}{2} (2a^{-5} f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z)) \]
\[ + \frac{1}{2} (2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z)), t/2 + t/2) \]
\[ \begin{align*}
&\geq \mu(2a^{-5}f(2ax, 2ay, 2az) - 2f(2ax, 2y, 2z), t/2) \\
&\quad \ast \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), t/2) \\
&= \mu(a^{-5}f(2ax, 2ay, 2az) - f(2ax, 2y, 2z), t/2^{\beta+1}) \\
&\quad \ast \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), t/2) \\
&= \mu\left(\frac{1}{2}(2a^{-5}f(2ax, 2ay, 2az) - 2a^{-2}f(2ax, 2ay, 2z)\right) \\
&\quad + \frac{1}{2}(2a^{-2}f(2ax, 2ay, 2z) - 2f(2ax, 2y, 2z)), t/2^{\beta+2} + t/2^{\beta+2}\right) \\
&\quad \ast \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), t/2) \\
&\geq \mu(2a^{-5}f(2ax, 2ay, 2az) - 2a^{-2}f(2ax, 2ay, 2z), t/2^{\beta+2}) \\
&\quad \ast \mu(2a^{-2}f(2ax, 2ay, 2az) - 2f(2ax, 2y, 2z), t/2^{\beta+2}) \\
&\quad \ast \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), t/2) \\
&= \mu(2f(2ax, 2ay, 2az) - 2a^{5}f(2ax, 2ay, 2z), a^{5}t/2^{\beta+2}) \\
&\quad \ast \mu(2f(2ax, 2ay, 2az) - 2a^{2}f(2ax, 2y, 2z), a^{2}t/2^{\beta+2}) \\
&\quad \ast \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), t/2) \\
&\geq \mu^{5}(2ax, 2ay, 2z, 0, a^{5}t/2^{\beta+2}) \ast \mu(2ax, 2y, 0, 2z, a^{2}t/2^{\beta+2}) \\
&\quad \ast \tau(2x, 0, 2y, 2z, t/2) \\
&\text{for all } x, y, z \in E. \text{ Therefore, we get} \\
\mu(a^{-6}f(2ax, 2ay, 2az) - f(2x, 2y, 2z), t) \\
= \mu((a^{-6})\frac{a}{a}f(2ax, 2ay, 2az) - \frac{a}{a}f(2x, 2y, 2z), t) \\
= \mu\left(\frac{1}{a}(a^{-5}f(2ax, 2ay, 2az) - af(2x, 2y, 2z)), t\right) \\
= \mu(a^{-5}f(2ax, 2ay, 2az) - af(2x, 2y, 2z), a^{2}t) \\
= \mu(a^{-5}f(2ax, 2ay, 2az) - f(2ax, 2y, 2z) \\
\quad + f(2ax, 2y, 2z) - af(2x, 2y, 2z), a^{4}t) \\
= \mu\left(\frac{1}{2}(2a^{-5}f(2ax, 2ay, 2az) - 2f(2ax, 2y, 2z)\right) \\
\quad + \frac{1}{2}(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z)) + a^{4}t/2 + a^{4}t/2) \\
&\geq \mu(2a^{-5}f(2ax, 2ay, 2az) - 2f(2ax, 2y, 2z), a^{4}t/2) \\
&\quad \ast \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), a^{4}t/2) \\
&= \mu(a^{-5}f(2ax, 2ay, 2az) - f(2ax, 2y, 2z), a^{4}t/2^{\beta+1}) \\
&\quad \ast \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), a^{4}t/2^{\beta+2}) \\
&\quad \ast \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), a^{4}t/2) \\
&= \mu(2f(2ax, 2ay, 2az) - 2a^{3}f(2ax, 2ay, 2z), a^{3}t/2^{\beta+2}) \\
\end{align*}\]
* μ(2f(2ax, 2ay, 2z) − 2a^2 f(ax, 2y, 2z), a^{3β} t/2^{β+2})
* μ(2f(2ax, 2y, 2z) − 2af(2x, 2y, 2z), a^{β} t/2)
≥ μ(2ax, 2ay, 2z, 0, a^{6β} t/2^{β+2}) * μ(2ax, 2y, 0, 2z, a^{3β} t/2^{β+2})
* τ(2x, 0, 2y, 2z, a^{β} t/2).

Replacing x, y and z by \( \frac{x}{2}, \frac{y}{2} \) and \( \frac{z}{3} \) in the last inequality, respectively, we get

\[
\begin{align*}
\mu\left( \frac{f(ax, ay, az)}{a^6} - f(x, y, z), t \right) &= \mu\left( \frac{f(ax, ay, az)}{a^6} - \frac{f(ax, y, z)}{a} + \frac{f(ax, y, z)}{a} - f(x, y, z), t \right) \\
&= \mu\left( \frac{1}{2} \left( \frac{2f(ax, ay, az)}{a^6} - \frac{2f(ax, y, z)}{a} \right) + \frac{1}{2} \left( \frac{2f(ax, y, z)}{a} - 2f(x, y, z) \right), t/2 + t/2 \right) \\
&≥ \mu\left( \frac{2f(ax, ay, az)}{a^6} - \frac{2f(ax, y, z)}{a}, t/2 \right) * \mu\left( \frac{1}{2} \left( \frac{2f(ax, y, z)}{a} - 2f(x, y, z) \right), t/2 \right) \\
&= \mu\left( \frac{1}{2} \left( \frac{2 \cdot 2f(ax, ay, az)}{a^6} - \frac{2 \cdot 2f(ax, ay, z)}{a^3} \right) + \frac{1}{2} \left( \frac{2 \cdot 2f(ax, ay, z)}{a^3} - \frac{2 \cdot 2f(ax, y, z)}{a} \right), t/2 \cdot 2 + t/2 \cdot 2 \right) \\
&≥ \mu\left( \frac{2 \cdot 2f(ax, ay, az)}{a^6} - \frac{2 \cdot 2f(ax, ay, z)}{a^3}, t/2 \cdot 2 \right) * \mu\left( \frac{2 \cdot 2f(ax, ay, z)}{a^3} - \frac{2 \cdot 2f(ax, y, z)}{a}, t/2 \cdot 2 \right) \\
&= \mu\left( \frac{2 \cdot 2f(ax, ay, az)}{a^6} - \frac{2 \cdot 2f(ax, ay, z)}{a^3}, t/2 \cdot 2 \right) * \mu\left( \frac{2 \cdot 2f(ax, ay, z)}{a^3} - \frac{2 \cdot 2f(ax, y, z)}{a}, t/2 \cdot 2 \right) \\
&≥ \mu(2f(ax, ay, az) - 2a^3 f(ax, ay, z), a^{6β} t/2^{β+2}) * μ(2f(ax, ay, z) - 2a^2 f(ax, y, z), a^{3β} t/2^{β+2})
&≥ \mu(2f(ax, ay, z) - 2a^2 f(ax, y, z), a^{3β} t/2^{β+2})
&≥ \mu(2f(ax, ay, z) - 2af(x, y, z), a^{β} t/2)
\end{align*}
\]

for all x, y, z ∈ E. Replacing x, y, z by \( a^{-1}x, a^{-1}y, a^{-1}z \) in (2.10), we get

\[
\begin{align*}
\mu\left( \frac{f(x, y, z)}{a^6} - f(a^{-1}x, a^{-1}y, a^{-1}z), t \right) &= \mu\left( \frac{1}{a^6} \left( \frac{f(x, y, z)}{a^6} - f(a^{-1}x, a^{-1}y, a^{-1}z) \right), t \right) \\
&= \mu\left( \frac{f(x, y, z)}{a^6} - f(a^{-1}x, a^{-1}y, a^{-1}z), a^{6β} t \right) \\
&= \mu\left( \frac{1}{2} \left( \frac{2f(x, y, z)}{a^6} - \frac{2f(x, y, z)}{a} \right), \frac{a^{6β} t}{2} + \frac{a^{6β} t}{2} \right)
\end{align*}
\]
\[
\geq \mu\left(\frac{2f(x,y,z) - 2f}{a^6}(x,a^{-1}y, a^{-1}z), a^{6\beta}t/2\right)
\]
\[
* \mu\left(\frac{2f}{a^6}(x,a^{-1}y, a^{-1}z) - 2f(a^{-1}x, a^{-1}y, a^{-1}z), a^{6\beta}t/2\right)
\]
\[
= \mu\left(\frac{f}{a^6}(x,y,z) - \frac{f}{a}(x,a^{-1}y, a^{-1}z), a^{6\beta}t/2^{\beta+1}\right)
\]
\[
* \mu\left(2f(x,a^{-1}y, a^{-1}z) - 2af(a^{-1}x,a^{-1}y,a^{-1}z), a^{7\beta}t/2\right)
\]
\[
= \mu\left(\frac{1}{2} \left(\frac{2f}{a^6}(x,y,z) - \frac{2}{a^3}f(x,y,a^{-1}z)\right)
\right)
\]
\[
+ \frac{1}{2} \left(\frac{2f}{a^6}(x,y,a^{-1}z) - \frac{2}{a}f(x,a^{-1}y,a^{-1}z), a^{6\beta}t/2^{\beta+2} + a^{6\beta}t/2^{\beta+2}\right)
\]
\[
* \mu\left(2f(x,a^{-1}y,a^{-1}z) - 2af(a^{-1}x,a^{-1}y,a^{-1}z), a^{7\beta}t/2\right)
\]
\[
\geq \mu\left(\frac{1}{a^6}(f(x,y,z) - 2a^3f(x,y,a^{-1}z)), a^{6\beta}t/2^{\beta+2}\right)
\]
\[
* \mu\left(\frac{1}{a^6}(2f(x,y,a^{-1}z) - 2a^2f(x,a^{-1}y,a^{-1}z)), a^{6\beta}t/2^{\beta+2}\right)
\]
\[
* \mu\left(2f(x,a^{-1}y,a^{-1}z) - 2af(a^{-1}x,a^{-1}y,a^{-1}z), a^{7\beta}t/2\right)
\]
\[
\geq \mu\left(2f(x,y,z) - 2a^3f(x,y,a^{-1}z)\right), a^{12\beta}t/2^{\beta+2}\right)
\]
\[
* \mu\left(2f(x,y,a^{-1}z) - 2a^2f(x,a^{-1}y,a^{-1}z)\right), a^{9\beta}t/2^{\beta+2}\right)
\]
\[
* \mu\left(2f(x,a^{-1}y,a^{-1}z) - 2af(a^{-1}x,a^{-1}y,a^{-1}z), a^{7\beta}t/2\right)
\]
\[
\geq v(a^{-1}x,y,a^{-1}z,0,a^{12\beta}t/2^{\beta+2}) * \zeta(x,a^{-1}y,0,a^{-1}z,a^{9\beta}t/2^{\beta+2})
\]
\[
\geq (a^{-1}x,0,a^{-1}y,a^{-1}z,a^{7\beta}t/2)
\]

but, we know that
\[
\mu\left(\frac{f(a^{-1}x,a^{-1}y,a^{-1}z)}{a^{-6}} - f(x,y,z), t\right) \geq \mu\left(\frac{f(x,y,z)}{a^6} - f(a^{-1}x,a^{-1}y,a^{-1}z), t\right)
\]

therefore
\[
\mu\left(\frac{f(a^{-1}x,a^{-1}y,a^{-1}z)}{a^{-6}} - f(x,y,z), t\right)
\]
\[
\geq v(a^{-1}x,y,a^{-1}z,0,a^{12\beta}t/2^{\beta+2}) * \zeta(x,a^{-1}y,0,a^{-1}z,a^{9\beta}t/2^{\beta+2})
\]
\[
* \tau(a^{-1}x,0,a^{-1}y,a^{-1}z,a^{7\beta}t/2)
\]

and so
\[
\mu\left(\frac{f(a^p x, a^p y, a^p z)}{a^{6p}} - f(x,y,z), t\right) \geq \Phi(x,y,z,t).
\]

(2.11)

Now, we consider the set
\[D = \{ h : E \times E \times E \to V : h(x,0,z) = 0 \text{ for all } x, z \in E \}\]

and introduce the modular \( \rho \) on \( D \) as follows:
\[\rho(h) = \inf\{ c > 0 : \mu(h(x,y,z), ct) \geq \Phi(x,y,z,t) \}\]

We know that \( \rho \) is even from \( \rho(-h) = \rho(h) \) and \( \rho(0) = 0 \). If \( \rho(h) = 0 \), then, for each \( c > 0 \),
\[\mu(h(x,y,z), ct) \geq \Phi(x,y,z,t)\]
for all $t > 1$ and $x, y \in E$. Now, if $\epsilon = ct$ is fixed and $t \to +\infty$, then $\mu(h(x, y, z), \epsilon) = 1$, which implies that $h = 0$. It is sufficient to show that $\rho$ satisfies the following condition:

$$\rho(\alpha g + \beta h) \leq \rho(g) + \rho(h)$$

if $\alpha + \beta = 1$ for all $\alpha, \beta \geq 0$. Let $\epsilon > 0$ be given. Then there exist $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \leq \rho(g) + \epsilon, \mu(g(x, y, z), c_1 t) \geq \Phi(x, y, z, t)$$

and

$$c_2 \leq \rho(h) + \epsilon, \mu(h(x, y, z), c_2 t) \geq \Phi(x, y, z, t).$$

If $\alpha + \beta = 1$ for all $\alpha, \beta \geq 0$, then we get

$$\mu(\alpha g(x, y, z) + \beta h(x, y, z), c_1 t + c_2 t) \geq \mu(g(x, y, z), c_1 t) * \mu(h(x, y, z), c_2 t)$$

and

$$\rho(\alpha g + \beta h) \leq c_1 + c_2 \leq \rho(g) + \rho(h) + 2\epsilon$$

thus

$$\rho(\alpha g + \beta h) \leq \rho(g) + \rho(h).$$

Now, we show that $\rho$ has the $\Delta_2$-condition, where $\kappa = 2^{\beta}$. For all $\epsilon > 0$, there exists $c > 0$ such that

$$c \leq \rho(h) + \epsilon, \mu(h(x, y, z), ct) \geq \Phi(x, y, z, t).$$

Since $(V, \mu, *)$ is a $\beta$-homogeneous $\mathcal{F}, \mathcal{M}$-space, we get

$$\mu(2h(x, y, z), 2^{\beta} ct) = \mu(h(x, y, z), ct) \geq \Phi(x, y, z, t),$$

where $\rho(2h) \leq 2^{\beta} c \leq 2^{\beta} \rho(h) + 2^{\beta} \epsilon$ and so $\rho(2h) \leq 2^{\beta} \rho(h)$. Thus $\rho$ satisfies the $\Delta_2$-condition with $\kappa = 2^{\beta}$.

Moreover, $\rho$ satisfies the l.s.c. (in the modular sense). Indeed, if the sequence $\{h_n\}$ in $\mathcal{D}$ is $\rho$-convergent to $h$, then we can easily see that $h_n(x, y, z)$ is $\mu$-convergent to $h(x, y, z)$ for all $x, y, z \in E$.

Let $\rho := \lim \inf_{n \to \infty} \rho(h_n) < \infty$ and $\rho(h) > \rho$. Then, we have

$$\mu(h(x, y, z), \rho t) < \Phi(x, y, z, t)$$

for all $t > 0$. Since $\mu$ satisfies the l.s.c. (in the fuzzy modular sense), we have

$$\limsup_{n \to \infty} \mu(h_n(x, y, z), \rho t) \leq \mu(h(x, y, z), \rho t) < \Phi(x, y, z, t).$$

From the last inequality, we know that there exists a positive integer $n_0 \in \mathbb{N}$ such that

$$\mu(h_n(x, y, z), \rho t) < \Phi(x, y, z, t)$$

and so $\rho(h_n) > \rho$ for all $n \geq n_0$. Thus $\lim \inf_{n \to \infty} \rho(h_n) > \rho$ where $n \to \infty$, which is a contradiction. Therefore, $\rho$ satisfies the l.s.c..

If $\delta > 0$ and $\lambda \in (0, 1)$ are given, it follows from $\Phi(x, y, z) \in \triangle$ that there exists $t_0 > 0$ such that $\Phi(x, y, z, t_0) > 1 - \lambda$. Let $\{h_n\}$ be a $\rho$-Cauchy sequence in $\mathcal{D}$ and let $\epsilon < \frac{\delta}{t_0}$ be given. Then there exists a positive integer $n_0 \in \mathbb{N}$ such that $\rho(h_{n} - h_{m}) \leq \epsilon$ for all $n, m \geq n_0$.

Now, by considering the definition of the modular $\rho$, we see that

$$\mu(h_n(x, y, z) - h_m(x, y, z), \delta) \geq \mu(h_n(x, y, z) - h_m(x, y, z), \epsilon t_0)$$

$$\geq \Phi(x, y, z, t_0)$$

$$> 1 - \lambda$$

(2.12)
for all $x, y, z \in E$ and $n, m \geq n_0$.

If $x, y$ and $z$ are arbitrary given points of $E$, then (2.12) implies that $\{h_n(x, y, z)\}$ is a $\mu$-Cauchy sequence in $(V, \mu, *)$. Since it is $\mu$-complete, it follows that $\{h_n(x, y, z)\}$ is $\mu$-convergent in $(V, \mu, *)$ for all $x, y, z \in E$. Thus, we can define
\[
h(x, y, z) = \lim_{n \to \infty} h_n(x, y, z),
\]
where a function $h : E \times E \times E \to (V, \mu, *)$ for all $x, y, z \in E$. Moreover, $\mu$ has the l.s.c.. Then, we have
\[
\rho(h_n - h) \leq \epsilon
\]
for all $n \geq n_0$. Thus $\{h_n\}$ is a $\rho$-convergent sequence in $D_\rho$. Therefore, $D_\rho$ is $\rho$-complete. Now, we consider the function $T : D_\rho \to D_\rho$ defined by
\[
T h(x, y, z) := a^{-6p}h(a^p x, a^p y, a^p z)
\]
for all $h \in D_\rho$. Let $g, h \in D_\rho$ and $c \in [0, \infty]$ be an arbitrary constant with $\rho(g - h) \leq c$. From the definition of $\rho$, we have
\[
\mu(g(x, y, z) - h(x, y, z), ct) \geq \Phi(x, y, z, t)
\]
for all $x, y, z \in E$. By the assumption and the last inequality, we get
\[
\mu(T g(x, y, z) - T h(x, y, z), Lct)
\]
\[
= \mu(a^{-6p}g(a^p x, a^p y, a^p z) - a^{-6p}h(a^p x, a^p y, a^p z), Lct)
\]
\[
= \mu(g(a^p x, a^p y, a^p z) - h(a^p x, a^p y, a^p z), a^6\beta Lct)
\]
\[
\geq \Phi(a^p x, a^p y, a^p z, a^6\beta L t)
\]
\[
\geq \Phi(x, y, z, t)
\]
for all $x, y, z \in E$ and so $\rho(T g - T h) \leq L \rho(g - h)$ for all $g, h \in D_\rho$, that is, $T$ is a $\rho$-contraction.

Now, we show that the $\rho$-strict mapping $T$ satisfies the conditions of Theorem (2.14). Observe that
\[
\mu(a^{-6p}f(a^{2p} x, a^{2p} y, a^{2p} z) - f(a^p x, a^p y, a^p z), t) \geq \Phi(a^p x, a^p y, a^p z, t)
\]
and so
\[
\mu(a^{-2(6)p}f(a^{2p} x, a^{2p} y, a^{2p} z) - a^{-6p}f(a^p x, a^p y, a^p z), L t)
\]
\[
= \mu(a^{-6p}f(a^{2p} x, a^{2p} y, a^{2p} z) - f(a^p x, a^p y, a^p z), a^6\beta L t)
\]
\[
\geq \Phi(a^p x, a^p y, a^p z, a^6\beta L t)
\]
\[
\geq \Phi(x, y, z, t).
\]
Thus, we get
\[
\mu\left(\frac{\Phi(a^{2p} x, a^{2p} y, a^{2p} z)}{a^{2(6)p}} - \frac{f(x, y, z)}{2^3 (L t + t)}\right) \geq \mu\left(\frac{\Phi(a^{2p} x, a^{2p} y, a^{2p} z)}{a^{2(6)p}} - \frac{f(a^p x, a^p y, a^p z)}{a^6 p}, L t\right) \geq \Phi(x, y, z)(t)
\]
for all $x, y, z \in E$. By replacing $x, y$ and $z$ by $a^p x, a^p y$ and $a^p z$ in (2.13), respectively, we get
\[
\mu(a^{-2(6)p}f(a^{3p} x, a^{3p} y, a^{3p} z) - f(a^p x, a^p y, a^p z), a^6\beta p 2^3 (L^2 t + Lt))
\]
\[
\begin{align*}
&\geq \Phi(a^p x, a^p y, a^p z), a^{6p} L t) \\
&\geq \Phi(x, y, z, t)
\end{align*}
\]
and so
\[
\mu(a^{-3(6)p} f(a^{3p} x, a^{3p} y, a^{3p} z) - a^{-6p} f(a^p x, a^p y, a^p z), 2^\beta (L^2 t + L t)) \geq \Phi(x, y, z, t).
\]
Therefore, we get
\[
\begin{align*}
&\mu\left( \frac{f(a^{3p} x, a^{3p} y, a^{3p} z)}{a^{6(3)p}} - f(x, y, z), 2^\beta \{2^\beta (L^2 t + L t) + t\} \right) \\
&\geq \mu\left( \frac{f(a^{3p} x, a^{3p} y, a^{3p} z)}{a^{6p}} - f(a^p x, a^p y, a^p z), 2^\beta (L^2 t + L t) \right) \\
&\quad \ast \mu\left( \frac{f(a^{3p} x, a^{3p} y, a^{3p} z)}{a^{6p}} - f(x, y, z), t \right) \\
&\geq \Phi(x, y, z, t)
\end{align*}
\]
for all \(x, y, z \in E\). By induction, we can easily see that
\[
\mu\left( \frac{f(a^{np} x, a^{np} y, a^{np} z)}{a^{6np}} - f(x, y, z), \left\{ (2^\beta L)^{n-1} + 2^\beta \sum_{i=1}^{n} (2^\beta L)^{i-1} \right\} t \right) \geq \Phi(x, y, z, t)
\]
for all \(x, y, z \in E\) and so
\[
\rho(T^n f - f) \leq (2^\beta L)^{n-1} + 2^\beta \sum_{i=1}^{n-1} (2^\beta L)^{i-1} \leq 2^\beta \sum_{i=1}^{n} (2^\beta L)^{i-1} \leq \frac{2^\beta}{1 - 2^\beta L}.
\] (2.14)

Next, we confirm that \(\delta_\rho(f) = \sup\{\rho(T^n f - T^m(f)) : n, m \in \mathbb{N}\} < \infty\). Since \(\rho\) satisfies the \(\Delta_2\)-condition with \(\kappa = 2^\beta\), it follows from the inequality (2.14) that
\[
\rho(T^n f - T^m f) \leq \frac{1}{2} \rho(2T^n f - 2f) + \frac{1}{2} \rho(2T^m f - 2f) \\
\leq \frac{\kappa}{2} \rho(T^n f - f) + \frac{\kappa}{2} \rho(T^m f - f) \\
\leq \frac{2^\beta}{1 - 2^\beta L}
\] (2.15)
for all \(n, m \in \mathbb{N}\). By the definition of \(\delta_\rho(f)\), we have \(\delta_\rho(f) < \infty\). Thus Theorem (2.14) shows that \(\{T^n f\}\) is \(\rho\)-convergent to a point \(s \in D_\rho\). Since \(\rho\) has the l.s.c., the inequality (2.14) gives \(\rho(T(s) - f) < \infty\).

If we replace \(m\) by \(m + 1\) in the inequality (2.15), then we obtain
\[
\rho(T^{n+1} f - T^n f) \leq \frac{2^{2\beta}}{1 - 2^\beta L}.
\]
Therefore, we get \(\rho(T(s) - s) \leq \frac{2^{2\beta}}{1 - 2^\beta L} < \infty\). Therefore, it follows from Theorem (2.14) that \(\rho\)-limit of \(\{T^n f\}, s \in D_\rho,\) is a fixed point of the mapping \(T\).

If we replace \(x_1, x_2, y\) and \(z\) by \(a^{np} x_1, a^{np} x_2, a^{np} y\) and \(a^{np} z\) in the inequality (2.1), respectively, then we obtain
\[
\begin{align*}
&\mu\left( \frac{f(a^{np}(ax_1 + bx_2), a^{np} y, a^{np} z)}{a^{6np}} + f(a^{np}(ax_1 - bx_2), a^{np} y, a^{np} z) \\
&\quad - 2a f(a^{np} x_1, a^{np} y, a^{np} z), t \right) \\
&= \mu(f(a^{np}(ax_1 + bx_2), a^{np} y, a^{np} z) + f(a^{np}(ax_1 - bx_2), a^{np} y, a^{np} z) \\
&\quad - 2a f(a^{np} x_1, a^{np} y, a^{np} z), a^{6np} t) \\
&\geq \tau(a^{np} x_1, a^{np} x_2, a^{np} y, a^{np} z), a^{6np} t),
\end{align*}
\] (2.16)
Similarly, by replacing \(x, y, y_1\) and \(z\) by \(a^{np}x, a^{np}y_1, a^{np}y_2\) and \(a^{np}z\) in the inequality \((2.2)\), respectively, we get
\[
\mu \left( f\left( f(a^{np}x, a^{np}(ay_1 + by_2), a^{np}z) + f(a^{np}x, a^{np}(ay_1 - by_2), a^{np}z) \right) - 2a^2 f\left( a^{np}x, a^{np}y_1, a^{np}z \right) \right) \geq \varepsilon \left( a^{np}x, a^{np}y_1, a^{np}y_2, a^{np}z \right), \quad (2.17)
\]
and, also by replacing \(x, y, z_1\) and \(z_2\) by \(a^{np}x, a^{np}y, a^{np}z_1\) and \(a^{np}z_2\) in the inequality \((2.3)\), respectively, we get
\[
\mu \left( f\left( f(a^{np}x, a^{np}y, a^{np}(az_1 + b_2z)) + f(a^{np}x, a^{np}y, a^{np}(az_1 - b_2z)) \right) - ab^2 f(a^{np}x, a^{np}y, a^{np}(z_1 + z_2)) + f(a^{np}x, a^{np}y, a^{np}(z_1 - z_2)) \right) \geq \nu \left( a^{np}x, a^{np}y, a^{np}z_1, a^{np}z_2 \right), \quad (2.18)
\]
for all \(x, x_i, y, y_1, z, z_i \in E, i = 1, 2\). Taking \(n \to \infty\) in the inequalities \((2.16), (2.17)\) and \((2.18)\), we deduce that \(s\) satisfies the system \((1.1)\), that is, \(s\) is sextic. It follows from the inequality \((2.14)\) that
\[
\rho(s - f) \leq \frac{2^\beta}{1 - 2^\beta L}.
\]
Hence \((2.5)\) holds. If \(s^*\) is another fixed point of \(T\), then we get
\[
\rho(s - s^*) \leq \frac{1}{2} \rho(2T(s) - 2f) + \frac{1}{2} \rho(2T(s^*) - 2f) \\
\leq \frac{\kappa}{2} \rho(T(s) - f) + \frac{\kappa}{2} \rho(T(s^*) - f) \\
\leq \frac{2^\beta}{1 - 2^\beta L} < \infty.
\]
Since \(T\) is \(\rho\)-contraction, we get
\[
\rho(s - s^*) = \rho(T(s) - T(s^*)) \\
\leq L\rho(s - s^*),
\]
which implies that \(\rho(s - s^*) = 0\) or \(s = s^*\). Since \(\rho(s - s^*) < \infty\), which proves the uniqueness of \(s\). This completes the proof.

**Concluding remarks**

Our results guarantee the generalized UHR stability of sextic mappings, whose codomain is equipped with a \(\beta\)-homogeneous and l.s.c. modular.

**Acknowledgements**

The first author was supported by Sisaket Rajabhat University through the Ph.D. program at KMUTT. The authors would like to thank the Higher Education Research Promotion and National Research University Project of Thailand’s Office of the Higher Education Commission for financial support (Under NRU59 Project No.59000399).
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