Some properties of the quasicompact-open topology on $C(X)$

Deniz Tokat*, İsmail Osmanoğlu

Department of Mathematics, Faculty of Arts and Sciences, Nevşehir Hacı Bektaş Veli University, 50300 Nevşehir, Turkey.

Communicated by R. Saadati

Abstract

This paper introduces quasicompact-open topology on $C(X)$ and compares this topology with the compact-open topology and the topology of uniform convergence. Then it examines submetrizability, metrizability, separability, and second countability of the quasicompact-open topology on $C(X)$. ©2016 All rights reserved.

Keywords: Function space, set-open topology, compact-open topology, quasicompactness, separability, submetrizability, second countability.

2010 MSC: 54C35, 54D65, 54E35.

1. Introduction and Preliminaries

There are several natural topologies that can be placed on $C(X)$ of all continuous real-valued functions on space $X$. The idea of defining a topology on $C(X)$ emerges from the studies of convergence of sequences of functions. The two major classes of topologies on $C(X)$ are the set-open topologies and the uniform topologies. The well-known set-open topologies are the point-open topology (or the topology of pointwise convergence) and the compact-open topology. The compact-open topology was introduced by Fox [6] in 1945 and soon after was developed by Arens in [2] and by Arens and Dugundji in [3]. It is shown in [12] that this topology is the proper setting to study sequences of functions converging uniformly on compact subsets. Thus, the compact-open topology is sometimes called the topology of uniform convergence on compact sets. Therefore, there have been many topologies that lie between the compact-open topology and...
the topology of uniform convergence, such as the $\sigma$-compact-open topology [9], the bounded-open topology [16], the pseudocompact-open topology [15], and the $C$-compact-open topology [20].

In the present paper, we introduce quasicompact-open topology on $C(X)$ and compare this topology with the compact-open topology and the topology of uniform convergence. We investigate the properties of the quasicompact-open topology on $C(X)$ such as submetrizability, metrizability, separability, and second countability.

A topological space $X$ is called functionally Hausdorff (or completely Hausdorff) if for any distinct points $x, y \in X$ there exists a continuous real function $f$ on $X$ such that $f(x) = 0$ and $f(y) = 1$, equivalently $f(x) \neq f(y)$. This property lies strictly between the Hausdorffness and the complete regularity.

Unless otherwise stated clearly, throughout this paper, all spaces are assumed to be functionally Hausdorff.

If $X$ and $Y$ are any two topological spaces with the same underlying set, then we use the notation $X = Y$, $X \leq Y$, and $X < Y$ to indicate, respectively, that $X$ and $Y$ have the same topology, that the topology on $Y$ is finer than or equal to the topology on $X$, and that the topology on $Y$ is strictly finer than the topology on $X$.

We denote $\bar{A}$ and $A^\circ$ the closure and the interior of a set $A$, respectively. If $A \subseteq X$ and $f \in C(X)$, then we use the notation $f|_A$ for the restriction of the function $f$ to the set $A$. As usual, $f(A)$ and $f^{-1}(A)$ are the image and the preimage of the set $A$ under the mapping $f$, respectively. We denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{R}$ the real line with the natural topology. Finally, the constant zero function in $C(X)$ is denoted by $f_0$.

2. The quasicompact-open topology and its comparison with other topologies

In this section, we define the quasicompact-open topology on $C(X)$ and also give some equivalent definitions. Then we compare the quasicompact-open topology with the compact-open topology and the topology of uniform convergence.

A subset $A$ of $X$ is called a zero-set if there is a continuous real-valued function $f$ defined on $X$ such that $A = \{x \in X : f(x) = 0\}$. The complement of a zero-set is called a cozero-set. A space $X$ is said to be quasicompact [7] if every covering of $X$ by cozero-sets admits a finite subcollection which covers $X$, also known as z-compact space. For more information see [7].

We recall that any compact space is quasicompact and the continuous image of a quasicompact space is quasicompact [4]. We also note that the closure of a quasicompact subset is quasicompact and any quasicompact space is pseudocompact [4].

Let $\alpha$ be a nonempty collection of subsets of a space $X$. Then various topologies on $C(X)$ has a subbase consisting of the sets $S(A, V) = \{f \in C(X) : f(A) \subseteq V\}$, where $A \in \alpha$ and $V$ is an open subset of real line $\mathbb{R}$, and the function space $C(X)$ endowed with these topologies is denoted by $C_{\alpha}(X)$. The topology defined in this way is called the set-open topology.

Now let $QC(X)$ denote the collection of all quasicompact subsets of $X$. For the quasicompact-open topology on $C(X)$, we take as subbase, the collection $\{S(A, V) : A \in QC(X), V \text{ is open in } \mathbb{R}\}$ and we denote the corresponding space by $C_q(X)$. Let $K(X)$ denote the collection of all compact subsets of $X$. The compact-open topology on $C(X)$ is defined similarly and is denoted by $C_k(X)$.

Let $\alpha = QC(X)$ and $\bar{\alpha} = \{\overline{A} : A \in \alpha\}$. Then note that the quasicompact-open topology is obtained if $\alpha$ is replaced by $\bar{\alpha}$. This is because for each $f \in C(X)$ we have $f(\overline{A}) \subseteq \overline{f(A)} = f(A)$.

The topology of uniform convergence on members of $\alpha$ has as base at each point $f \in C(X)$ the family of all sets of the form $B_A(f, \epsilon) = \{g \in C(X) : \sup |f(x) - g(x)| < \epsilon \text{ for all } x \in A\}$, where $A \in \alpha$ and $\epsilon > 0$. The space $C(X)$ having the topology of uniform convergence on $\alpha$ is denoted by $C_{\alpha,u}(X)$. For $\alpha = QC(X)$, we denote the corresponding space by $C_{q,u}(X)$. In the case that $\alpha = \{X\}$, the topology on $C(X)$ is called the topology of uniform convergence or uniform topology and denoted by $C_u(X)$.

There is another way to consider the quasicompact-open topology on $C(X)$. For each $A \in QC(X)$ and $\epsilon > 0$, we define the seminorm $p_A$ on $C(X)$ and $V_{A,\epsilon}$, as follow: $p_A(f) = \sup \{|f(x)| : x \in A\}$ and...
Thus, there exists an \( W \) and \( C \) such that \( (f(x) - \epsilon, f(x) + \epsilon) \subseteq V \) forms a neighborhood base at \( f \). This topology is locally convex since it is generated by a collections of seminorms and it is the same as the quasicompact-open topology on \( C(X) \).

It is also easy to see that this topology is Hausdorff. \( C_q(X) \), being a locally convex Hausdorff space, is a Tychoff space.

Now, we can compare the topologies. We have \( C_k(X) \subseteq C_q(X) \) since \( K(X) \subseteq QC(X) \). But to compare the quasicompact-open topology and the topology of uniform convergence, we need the following theorem.

**Theorem 2.1.** For any space \( X \), the quasicompact-open topology on \( C(X) \) is the same as the topology of uniform convergence on the quasicompact subsets of \( X \), that is, \( C_q(X) = C_{q,u}(X) \).

**Proof.** Assume that \( S(A,V) \) is a subsbasic open set in \( C_q(X) \) and \( f \in S(A,V) \). Recall that compact and quasicompact subsets of \( \mathbb{R} \) are equivalent. Since \( f(A) \) is compact and \( f(A) \subseteq V \), there exists \( \epsilon > 0 \) such that \( (f(A) - \epsilon, f(A) + \epsilon) \subseteq V \) (see [Corollary 4.1.14]). If \( g \in B_A(f,\epsilon) \) and \( x \in A \), then we obtain \( g(x) \in (f(x) - \epsilon, f(x) + \epsilon) \). Hence, we find \( g(A) \subseteq V \), i.e. \( g \in S(A,V) \). It follows that \( B_A(f,\epsilon) \subseteq S(A,V) \). Consequently, \( C_q(X) \subseteq C_{q,u}(X) \).

Now, let \( B_A(f,\epsilon) \) be a basic neighborhood of \( f \) in \( C_{q,u}(X) \). Then, there exist \( f(x_1), f(x_2), \ldots, f(x_n) \) in \( f(A) \) such that \( f(A) \subseteq \bigcup_{i=1}^n (f(x_i) - \frac{\epsilon}{2}, f(x_i) + \frac{\epsilon}{2}) \) since \( f(A) \) is compact. If we take \( V_i = (f(x_i) - \frac{\epsilon}{2}, f(x_i) + \frac{\epsilon}{2}) \) and \( W_i = (f(x_i) - \frac{\epsilon}{2}, f(x_i) + \frac{\epsilon}{2}) \), we find \( V_i \subseteq W_i \). Also \( f(A) \subseteq \bigcup_{i=1}^n V_i \subseteq \bigcup_{i=1}^n W_i \). Let \( A_i = A \cap f^{-1}(V_i) \), where clearly each \( A_i \) is quasicompact and \( A = \bigcup_{i=1}^n A_i \). We have \( f(A_i) \subseteq V_i \subseteq W_i \) and so \( f \in \bigcap_{i=1}^n S(A_i,W_i) \). Now we need to show that \( \bigcap_{i=1}^n S(A_i,W_i) \subseteq B_A(f,\epsilon) \). Suppose that \( g \in \bigcap_{i=1}^n S(A_i,W_i) \) and \( x \in A \). Thus, there exists an \( i \) such that \( x \in A_i \) and consequently, \( f(x) \in V_i \) and \( g(x) \in W_i \). Since \( |f(x) - g(x)| \leq |f(x) - f(x_i)| + |f(x_i) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \), then \( g \in B_A(f,\epsilon) \). Hence, \( C_{q,u}(X) \subseteq C_q(X) \).

**Corollary 2.2.** For any space \( X \), \( C_k(X) = C_{q,u}(X) \leq C_u(X) \).

From this result, we obtain the following.

**Corollary 2.3.** For any space \( X \), \( C_k(X) \leq C_q(X) \leq C_u(X) \).

Note that in a perfectly normal space, every open set is a cozero-set and consequently, a quasicompact space is compact. Thus, for a perfectly normal space \( X \), \( C_k(X) = C_q(X) \).

**Theorem 2.4.** For any space \( X \), \( C_q(X) = C_u(X) \) if and only if \( X \) is quasicompact.

**Proof.** Let \( C_q(X) = C_u(X) \). We know that \( C_q(X) = C_{q,u}(X) \) by Theorem 2.1. So, \( C_u(X) = C_{q,u}(X) \).

Thus, \( B_X(f,\epsilon) \) in \( C_u(X) \) is also basic neighborhood of \( f \) in \( C_{q,u}(X) \) and so \( X \) is quasicompact.

Conversely, suppose that \( X \) is quasicompact. It follows that for each \( f \in C(X) \) and each \( \epsilon > 0 \), \( B_X(f,\epsilon) \) is a basic open set in \( C_q(X) \). Consequently, \( C_q(X) = C_u(X) \).

We know that for a compact space \( X \), \( C_k(X) = C_u(X) \). Then we can give the following example.

**Example 2.5.** For any compact space \( X \), \( C_k(X) = C_q(X) = C_u(X) \).

If \( X \) is both realcompact and pseudocompact, then it is compact [8, Problem 5H]. Also every Lindelöf space is realcompact [8, Theorem 8.2]. Thus, we get the following result.

**Theorem 2.6.** For any Lindelöf space \( X \), \( C_k(X) = C_q(X) \).

**Proof.** We know that every quasicompact space is pseudocompact. Considering the above description, Lindelöf quasicompact space is compact and consequently, \( C_k(X) = C_q(X) \) by Example 2.5.

Since every countable or second countable space is Lindelöf, we obtain the following result.

**Corollary 2.7.** For any countable or second countable space \( X \), \( C_k(X) = C_q(X) \).
Example 2.8. Let $X$ denote the set of positive integers endowed with the particular point topology \cite[Example 9]{22}. The space $X$ is a quasicompact, but not compact. Thus, we obtain $C_k(X) \leq C_q(X) = C_u(X)$.

Example 2.9. Let $X$ be the prime integer topology \cite[Example 61]{22}. The space $X$ is a quasicompact, but not compact \cite{1}. This yields $C_k(X) \leq C_q(X) = C_u(X)$.

Example 2.10. Let $X = \mathbb{R}$ and define a topology on $X$ by requiring that a neighborhood of a point $x$ is any set containing $x$ which contains all the rationals in an open interval around $x$ \cite{21}. The space $X$ is quasicompact, but not compact \cite{4}. It follows that $C_k(X) \leq C_q(X) = C_u(X)$.

Example 2.11. Hewitt’s example \cite{11} of a regular space $X$ on which every continuous real-valued function is constant is a quasicompact space which is not compact \cite{13}. For this space $X$, we have $C_k(X) \leq C_q(X) = C_u(X)$.

Example 2.12. Let $X$ be the skyline space \cite{10}. The space $X$ is a quasicompact, but not compact \cite{14}. Hence, we obtain $C_k(X) \leq C_q(X) = C_u(X)$.

Example 2.13. Let $X = \mathbb{N}$ and define a topology on $X$ by taking every odd integer to be open and a set $U$ is open if for every even integer $p \in U$, the predecessor and the successor of $p$ are also in $U$ \cite{14}. From this it follows that $C_k(X) \leq C_q(X) = C_u(X)$.

3. Main Results on $C_q(X)$

In this section, we study the submetrizability, metrizability, separability, and second countability of $C_q(X)$. First, we provide some natural functions which play a useful role in studying the topological properties of function spaces.

If $f : X \to Y$ is a continuous function, then the induced function of $f$, denoted by $f^* : C(Y) \to C(X)$ is defined by $f^*(g) = g \circ f$ for all $g \in C(Y)$.

Given a nonempty set $X$ a topological space $Y$, a function $f : X \to Y$ is called almost onto if $f(X)$ is dense in $Y$.

Theorem 3.1. Let $f : X \to Y$ be a continuous function between two spaces $X$ and $Y$. Then we have the following.

1. $f^* : C_q(Y) \to C_q(X)$ is continuous;
2. for normal space $Y$, if $f$ is one-to-one, then $f^* : C_q(Y) \to C_q(X)$ is almost onto;
3. $f^* : C(Y) \to C(X)$ is one-to-one if and only if $f$ is almost onto \cite{19}.

Proof. (1) Let $g \in C_q(Y)$ and $S(A,V)$ be a basic neighborhood of $f^*(g)$ in $C_q(X)$. It is easily seen that $f^*(g) = g \circ f \in S(A,V)$ if and only if $g \in S(f(A), V)$. Then $f^*(S(f(A), V)) = S(A,V)$ and consequently, $f^*$ is continuous.

The proof of (2) is similar to 2(a) in \cite{18}.\hfill $\square$

Another kind of useful function on function spaces is the sum function. Let $\{X_i : i \in I\}$ be a family of topological spaces. If $\oplus X_i$ denotes their topological sum, then the sum function $s$ is defined by $s : C(\oplus X_i) \to \prod\{C(X_i) : i \in I\}$ where $s(f) = f|_{X_i}$ for each $f \in C(\oplus X_i)$.

Theorem 3.2. Let $\{X_i : i \in I\}$ be a family of spaces. Then the sum function $s : C(\oplus X_i) \to \prod\{C(X_i) : i \in I\}$ is a homeomorphism.

Proof. The proof is similar to Theorem 4.10 in \cite{15}.\hfill $\square$

A space $X$ is said to be submetrizable if it has a weaker metrizable topology, equivalently if there exists a metrizable space $Y$ and a continuous bijection $f : X \to Y$ from the space $X$ onto $Y$.

In a topological space a $G_\delta$-set is a set which can be written as the intersection of a countable collection of open sets.
Remark 3.3.

1. For any space $X$, if the set $\{(x,x) : x \in X\}$ is a $G_\delta$-set (resp. zero-set) in the product space $X \times X$, then $X$ is said to have a $G_\delta$-diagonal (resp. zero-set diagonal). Every submetrizable space $X$ has a $G_\delta$-diagonal. Consequently, every submetrizable space $X$ has a zero-set diagonal since a zero-set is a $G_\delta$-set.

2. A space $X$ is called an $E_0$-space if every point in the space is a $G_\delta$-set. The submetrizable spaces are $E_0$-spaces.

Proposition 3.4. If $X$ is a submetrizable space then all quasicompact subsets of $X$ are $G_\delta$-sets.

Proof. Let $X$ be submetrizable. Then there exists a continuous bijection $f : X \to Y$ from the space $X$ onto a metrizable space $Y$. Let $A$ be a quasicompact subset of $X$. Then $f(A)$ is compact in the metric space $Y$. Since a closed set in a metric space is a $G_\delta$ set, $f(A)$ is a $G_\delta$-set in $Y$. In other words, $f(A) = \cap_{n=1}^\infty G_n$, where $G_n$ is an open subset of $Y$ for each $n$. It follows that $A = \cap_{n=1}^\infty f^{-1}(G_n)$ and so $A$ is a $G_\delta$-set. 

A space $X$ is called $\sigma$-quasicompact if there exists a sequence $\{A_n\}$ of quasicompact sets in $X$ such that $X = \cup_{n=1}^\infty A_n$. By using this fact we obtain the following result.

Theorem 3.5. For any space $X$, the following are equivalent.

1. $C_q(X)$ is submetrizable.
2. Every quasicompact subset of $C_q(X)$ is a $G_\delta$-set in $C_q(X)$.
3. Every compact subset of $C_q(X)$ is a $G_\delta$-set in $C_q(X)$.
4. $C_q(X)$ is an $E_0$-space.
5. $X$ is $\sigma$-quasicompact.
6. $C_q(X)$ has a zero-set diagonal.
7. $C_q(X)$ has a $G_\delta$-diagonal.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follow from Proposition 3.4

(4) $\Rightarrow$ (5) If $C_q(X)$ is an $E_0$-space, then the constant zero function $f_0$ defined on $X$ is a $G_\delta$-set. Suppose that $\cap_{n=1}^\infty B_{A_n}(f_0,\epsilon_n) = \{f_0\}$ where each $A_n$ is quasicompact subset in $X$ and $\epsilon_n > 0$. We need to show that $X = \cup_{n=1}^\infty A_n$. Assume that $x_0 \in X \setminus \cup_{n=1}^\infty A_n$. Hence there exists a continuous function $f_1 : X \to [0,1]$ such that $f_1(x) = 0$ for all $x \in \cup_{n=1}^\infty A_n$ and $f_1(x_0) = 1$. Since $f_1(x) = 0$ for all $x \in A_n$, $f_1 \in B_{A_n}(f_0,\epsilon_n)$ for all $n$ and thus, $f_1 \in \cap_{n=1}^\infty B_{A_n}(f_0,\epsilon_n) = \{f_0\}$, that is, $f_1$ is the zero function on $X$. But $f_1(x_0) = 1$. This contradicts the hypothesis, hence $X$ is $\sigma$-quasicompact.

(5) $\Rightarrow$ (4) Assume that $X$ is $\sigma$-quasicompact and $f \in C_q(X)$. Now we need to prove that $\{f\} = \cap_{n=1}^\infty B_{A_n}(f,\frac{1}{n})$. Let $g \in \cap_{n=1}^\infty B_{A_n}(f,\frac{1}{n})$ and $x \in X$. Then there exists $m \in \mathbb{N}$ such that $x \in A_m$ for all $n \geq m$. Then we find $|g(x) - f(x)| \leq \frac{1}{n}$ for all $n \geq m$. Thus $g(x) = f(x)$ and consequently $C_q(X)$ is an $E_0$-space.

(5) $\Rightarrow$ (1) Suppose that $X = \cup_{n=1}^\infty A_n$, where each $A_n$ is quasicompact. Let $S = \oplus\{A_n : n \in \mathbb{N}\}$ be the topological sum of the $A_n$ and let $\phi : S \to X$ be the natural function. Thus, the induced function $\phi^* : C_q(X) \to C_q(S)$ defined by $\phi^*(f) = f \circ \phi$ is continuous. We need to show that $\phi^*$ is one-to-one. Let $\phi^*(g_1) = \phi^*(g_2)$. So, $g_1$ and $g_2$ are equal on $\cup_{n=1}^\infty A_n$. So $g_1 - g_2 \in \cap_{n=1}^\infty B_{A_n}(f_0,\epsilon_n) = \{f_0\}$. Hence, $g_1 = g_2$ and consequently, $\phi^*$ is one-to-one. By Theorem 3.2, $C_q(\oplus\{A_n : n \in \mathbb{N}\})$ is homeomorphic to $\prod\{C_q(A_n) : n \in \mathbb{N}\}$. But each $C_q(A_n)$ is metrizable by Theorem 2.4. Since $C_q(S)$ is metrizable and $\phi^*$ is a continuous injection, $C_q(X)$ is submetrizable.

The implications (1) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (4) are immediate from Remark 3.3.

Lemma 3.6. In a completely regular submetrizable space, the notions of compactness and quasicompactness coincide.
Proof. Since pseudocompact completely regular submetrizable space is metrizable \cite{17} Corollary 2.7] and every quasicompact space is pseudocompact, then the notions of compactness and quasicompactness coincide.

Corollary 3.7. Let \( X \) be \( \sigma \)-quasicompact. Then compact and quasicompact subsets of \( C_q(X) \) are equivalent.

Proof. If \( X \) is \( \sigma \)-quasicompact, then \( C_q(X) \) is submetrizable by Theorem 3.5. Also we know that \( C_q(X) \) is Tychonoff (completely regular Hausdorff). Hence, compact and quasicompact subsets of \( C_q(X) \) are equivalent by Lemma 3.6.

A space \( X \) is called a \( q \)-space if for each point \( x \in X \), there exists a sequence \( \{U_n : n \in \mathbb{N}\} \) of neighborhoods of \( x \) such that if \( x_n \in U_n \) for each \( n \), then \( \{x_n : n \in \mathbb{N}\} \) has a cluster point. This fact yields the following theorem.

Theorem 3.8. For any space \( X \), the following are equivalent.
1. \( C_q(X) \) is metrizable.
2. \( C_q(X) \) is first countable.
3. \( C_q(X) \) is a \( q \)-space.
4. \( X \) is hemiquasicompact; that is, there exists a sequence of quasicompact sets \( \{A_n\} \) in \( X \) such that for any quasicompact subset \( A \) of \( X \), \( A \subseteq A_n \) holds for some \( n \).

Proof. (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) are all immediate.

(3) \( \Rightarrow \) (4) Suppose that \( C_q(X) \) is a \( q \)-space. Hence, there exists a sequence \( \{U_n : n \in \mathbb{N}\} \) of neighborhoods of the zero function \( f_0 \) in \( C_q(X) \) such that if \( g_n \in U_n \) for each \( n \), then \( \{g_n : n \in \mathbb{N}\} \) has a cluster point in \( C_q(X) \). Now for each \( n \), there exists a quasicompact subset \( A_n \) of \( X \) and \( \epsilon_n > 0 \) such that \( f_0 \in B_{A_n}(f_0, \epsilon_n) \subseteq U_n \). Let \( A \) be a quasicompact subset of \( X \). If possible, suppose that \( A \) is not a subset of \( A_n \) for any \( n \in \mathbb{N} \). Then for each \( n \in \mathbb{N} \), there exists \( a_n \in A \setminus A_n \). So for each \( n \in \mathbb{N} \), there exists a continuous function \( g_n : X \to \mathbb{R} \) such that \( g_n(a_n) = n \) and \( g_n(x) = 0 \) for all \( x \in A_n \). It is clear that \( g_n \in B_{A_n}(f_0, \epsilon_n) \). Suppose that this sequence has a cluster point \( g \) in \( C_q(X) \). Then for each \( k \in \mathbb{N} \), there exists a positive integer \( n_k > k \) such that \( g_{n_k} \in B_A(g, 1) \). Thus, \( g(a_{n_k}) > g_{n_k}(a_{n_k}) - 1 = n_k - 1 \geq k \) for all \( k \in \mathbb{N} \). But this means that \( g \) is unbounded on the quasicompact set \( A \). Hence, the sequence \( \{g_n\}_{n \in \mathbb{N}} \) cannot have a cluster point in \( C_q(X) \) and consequently, \( C_q(X) \) fails to be a \( q \)-space. Thus, \( X \) must be hemiquasicompact.

(4) \( \Rightarrow \) (1) Here we need the well-known result which says that if the topology of a locally convex Hausdorff space is generated by a countable family of seminorms, then it is metrizable(see page 119 in \cite{23}). Now the locally convex topology on \( C(X) \) generated by the countable family of seminorms \( \{p_{A_n} : n \in \mathbb{N}\} \) is metrizable and weaker than the quasicompact-open topology. But since for each quasicompact set \( A \) in \( X \), there exists \( A_n \) such that \( A \subseteq A_n \), the locally convex topology generated by the family of seminorms \( \{p_A : A \in QC(X)\} \), that is, the quasicompact-open topology is weaker than the topology generated by the family of seminorms \( \{p_{A_n} : n \in \mathbb{N}\} \). Hence, \( C_q(X) \) is metrizable.

Proposition 3.9. Let \( X \) be locally compact and second countable. Then \( C_q(X) \) is second countable.

Proof. Since regular second countable space \( X \) is metrizable by Urysohn’s Metrization Theorem, then \( C_k(X) = C_q(X) \). We know that \( C_k(X) \) is second countable by \cite{18} it follows that \( C_q(X) \) is second countable.

Theorem 3.10. For any space \( X \), the following are equivalent.
1. \( C_q(X) \) is separable.
2. \( C_k(X) \) is separable.
3. \( X \) has a weaker separable metrizable topology.
Proof. (1) ⇒ (2) is straightforward and proof of (2) ⇒ (3) was given in [18].

(3) ⇒ (1). If X has a weaker separable metrizable topology, then X is embeddable into Hilbert cube $I^\omega$ (see [5] Theorem 4.2.10). Let $f : X \to I^\omega$ be a continuous injection. Then the induced function $f^* : C(I^\omega) \to C_q(X)$ is almost onto by Theorem 3.1. Since $C(I^\omega)$ is second countable by Proposition 3.9, then $C_q(X)$ must be separable. 

\[ \text{Corollary 3.11. Let } X \text{ be completely regular space. If } C_q(X) \text{ is separable, then } C_k(X) = C_q(X). \]

Proof. If $C_q(X)$ is separable, X is submetrizable. Since X is completely regular and submetrizable, compact and quasicompact subsets of X are equivalent by Lemma 3.6. Consequently, $C_k(X) = C_q(X)$.

\[ \text{Example 3.12. Since } \mathbb{R} \text{ is a separable metric space, } C_q(\mathbb{R}) \text{ is separable. Thus, we have } C_k(\mathbb{R}) = C_q(\mathbb{R}). \]

\[ \text{Example 3.13. Let } X \text{ be a countable discrete space. Then } C_q(X) \text{ is separable and so } C_k(X) = C_q(X). \]

\[ \text{Corollary 3.14. Let } X \text{ be quasicompact space. If } X \text{ is metrizable, then } C_q(X) \text{ is separable.} \]

Proof. If X is metrizable and quasicompact, then X is compact. Since X is compact and metrizable, then X is separable and consequently, $C_q(X)$ is separable.

Note that converse of Corollary 3.14 is not always true. If $C_q(X)$ is separable, then X is submetrizable. But a quasicompact submetrizable space need not be metrizable. An example of this, the space $E \cap [0,1]$ of [8] Problem 3J is quasicompact and submetrizable, but not metrizable. If X is completely regular, then it is metrizable by Corollary 2.7 in [17]. Then we can give the following theorem.

\[ \text{Theorem 3.15. Let } X \text{ be quasicompact and completely regular space. } C_q(X) \text{ is separable if and only if } X \text{ is compact and metrizable.} \]

Proof. If $C_q(X)$ is separable, then X is submetrizable by Theorem 3.10. Since quasicompact completely regular submetrizable space is metrizable, X is metrizable and by Lemma 3.6, X is compact.

The sufficiency part follows from Corollary 3.14.

A topological space is said to be hemicompact if it has a sequence of compact subsets such that every compact subset of the space lies inside some compact set in the sequence.

\[ \text{Theorem 3.16. For a locally compact space } X, \text{ the following are equivalent.} \]

1. $C_q(X)$ is second countable.
2. $C_k(X)$ is second countable.
3. X is hemicompact and submetrizable.

Proof. (1) ⇔ (2) If either $C_q(X)$ or $C_k(X)$ is second countable, then it is separable and submetrizable by Theorem 3.10. We know that regular separable space is normal. Consequently, $C_k(X) = C_q(X)$.

(2) ⇒ (3) If $C_k(X)$ is second countable, then it is submetrizable as well as it is separable. Hence, X is hemicompact and submetrizable.

(3) ⇒ (2) If X is hemicompact, then $C_k(X)$ is metrizable. Note that X, being hemicompact, is Lindelöf. Since X is also submetrizable, X has a separable metrizable compression and consequently, $C_k(X)$ is separable. Thus, $C_k(X)$ is second countable.

Considering Corollary 3.11, we obtain the following result.

\[ \text{Corollary 3.17. Let } X \text{ be a completely regular space. If } C_q(X) \text{ is second countable, then } C_k(X) = C_q(X). \]

Note that if X is locally compact, then X is hemicompact if and only if X is either Lindelöf or $\sigma$-compact in [5] Exercises 3.8.C]. Hence, by using Theorem 3.16 and Proposition 3.9, we have the following result.
Theorem 3.18. For a locally compact space $X$, the following statements are equivalent.

1. $C_q(X)$ is second countable.
2. $C_k(X)$ is second countable.
3. $X$ is hemicompact and submetrizable.
4. $X$ is $\sigma$-compact and submetrizable.
5. $X$ is Lindelöf and submetrizable.
6. $X$ is second countable.

Proof. From Theorem 3.16, we obtain $(1) \iff (2) \iff (3)$. Also by [5, Exercises 3.8.C], we get $(3) \iff (4) \iff (5)$. It is easy to see that $(6) \implies (1)$ from Proposition 3.9.

Now, it is sufficient to show that $(5) \implies (6)$. Since $X$ is locally compact, for each $x \in X$, there exists an open set $V_x$ in $X$ such that $x \in V_x$ and $V_x$ is compact. Note that $\{V_x : x \in X\}$ is an open cover of $X$. But $X$ is Lindelöf and consequently, there exists a countable subset $\{x_n : n \in \mathbb{N}\}$ of $X$ such that $X = \bigcup_{n=1}^{\infty} V_{x_n}$. Since $X$ is separable submetrizable by Theorem 3.10 and each $V_{x_n}$ is compact, each $V_{x_n}$ is metrizable and so each $V_{x_n}$ is second countable. Consequently, each $V_{x_n}$ is also second countable and $X$ becomes the union of a countable family of second countable open subsets of $X$. Hence, $X$ is second countable.

Acknowledgement

The authors would like to thank the referee for his (or her) valuable suggestions which greatly improved the paper.

References