Dislocated quasi-b-metric spaces and fixed point theorems for cyclic weakly contractions

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Abstract

In this paper, we introduce the notions of type dqb-cyclic-weak Banach contraction, dqb-cyclic-\(\phi\)-contraction and derive the existence of fixed point theorems on dislocated quasi-b-metric spaces. Our main theorem extends and unifies existing results in the recent literature. ©2016 All rights reserved.

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1. Introduction and Preliminaries

Banach contraction principle was introduced in 1922 by Banach \cite{ref3}. In 2001, Rhoades \cite{ref7} introduced weakly contractive as follows:

(i) A mapping \(T : X \to X\) is said to be a weakly contractive if for all \(x, y \in X\),

\[
d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)),
\]

where \(\phi : [0, \infty) \to [0, \infty)\) is a continuous and nondecreasing function such that \(\phi(t) = 0\) if and only if \(t = 0\). If one takes \(\phi(t) = (1 - k)t\), where \(0 < k < 1\), a weak contraction reduces to a Banach contraction.

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Now, we recall the definition of cyclic map. Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and $T : A \cup B \to A \cup B$, then $T$ is called a cyclic map iff $T(A) \subseteq B$ and $T(B) \subseteq A$. In 2003, Kirk et al. \cite{Kirk2003} introduced cyclic contraction as follows:

(ii) A cyclic map $T : A \cup B \to A \cup B$ is said to be a cyclic contraction if there exists $a \in [0, 1)$ such that

$$d(Tx, Ty) \leq ad(x, y)$$

for all $x \in A$ and $y \in B$.

In 2013, K. Zoto \cite{Zoto2013} introduced $d$-cyclic-$\phi$-contraction follows:

(iii) A cyclic map $T : A \cup B \to A \cup B$ is said to be a $d$-cyclic-$\phi$-contraction if $\phi \in \Phi$ such that

$$d(Tx, Ty) \leq \phi(d(x, y))$$

for all $x \in A$, $y \in B$, where $\Phi$ the family of non-decreasing functions: $\phi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t > 0$, where $n$ is the $n$-th iterate of $\phi$.

**Lemma 1.1.** Suppose that the function $\phi : [0, \infty) \to [0, \infty)$ is non-decreasing, then for each $t > 0$, $\lim_{n \to \infty} \phi^n(t) = 0$ implies $\phi(t) < t$.

If $(X, d)$ is complete metric spaces, at least one of (i), (ii) and (iii) holds, then $T$ has a unique fixed point (see\cite{Kirk2003, Zoto2013}). Recently, Klin-eam and Suanoom \cite{Klin-eam2013} introduced dislocated quasi b-metric spaces, which is a new generalization of quasi b-metric space (see\cite{Bashirov2013}), b-metric-like space (see\cite{Bashirov2012}), b-metric space (see\cite{Bashirov2010}), metric space, etc. as follows:

**Definition 1.2** (\cite{Klin-eam2013}). Let $X$ be a nonempty set. Suppose that the mapping $d : X \times X \to [0, \infty)$ such that constant $s \geq 1$ satisfies the following conditions:

1. $d(x, y) = d(y, x) = 0$ implies $x = y$ for all $x, y \in X$;
2. $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

The pair $(X, d)$ is then called a dislocated quasi $b$-metric space (or simply dqb-metric). The number $s$ is called to be the coefficient of $(X, d)$.

**Remark 1.3.** When, in addition, the conditions $d(x, y) = d(y, x)$ and $d(x, x) = 0$ are true, then $d$ is a $b$-metric.

**Definition 1.4.** Let $\{x_n\}$ be a sequence in a dqb-metric space $(X, d)$.

1. A sequence $\{x_n\}$ dislocated quasi-$b$-converges (for short, dqb-converges) to $x \in X$ if

$$\lim_{n \to \infty} d(x_n, x) = 0 = \lim_{n \to \infty} d(x, x_n).$$

In this case $x$ is called a dqb-limit of $\{x_n\}$ and we write $(x_n \to x)$.

2. A sequence $\{x_n\}$ is called dislocated quasi-$b$-Cauchy (for short, dqb-Cauchy), if

$$\lim_{n,m \to \infty} d(x_n, x_m) = 0 = \lim_{n,m \to \infty} d(x_m, x_n).$$

3. A dqb-metric space $(X, d)$ is complete if every dqb-Cauchy sequence is dqb-convergent in $X$.

Moreover, they introduced the notion of dqb-cyclic-Banach and dqb-cyclic-Kannan mapping and derive the existence of fixed point theorems for such space.

In this paper, we study the properties of dislocated quasi-$b$-metric spaces and introduce dqb-cyclic-weak Banach contraction, dqb-cyclic-$\phi$-contraction and derive the existence of fixed point theorems in dislocated quasi-$b$-metric spaces. Our main theorem extends and unifies existing results in the recent literature.
2. Main results

Every dislocated quasi-b-metric space \((X, d)\) can be considered as a topological space on which the topology is introduced by taking, for any \(x \in X\), the collection \(\{B_r(x) | r > 0\}\) as a base of the neighborhood filter of the point \(x\). Here the ball \(B_r(x)\) is defined by the equality \(B_r(x) = \{y \in X | \max\{d(x, y), d(y, x)\} < r\}\).

**Definition 2.1** \([6]\). Let \(X\) be topological space. Then \(X\) is said to be Hausdorff topological space if for any distinct points \(x, y \in X\), there exists two open sets \(G\) and \(H\) such that \(x \in G\), \(y \in H\) and \(G \cap H = \emptyset\).

**Proposition 2.2.** Every dqb-metric space is Hausdorff topological space.

**Proof.** Let \(x\) and \(y\) be two distinct points in \(X\). Then \(d(x, y) > 0\) and \(d(y, x) > 0\). Choose \(\delta = \frac{d(x,y)}{2s}\). Then, there exists
\[
B_\delta(x) = \{z \in X | \max\{d(x, z), d(z, x)\} < \delta\}
\]
and
\[
B_\delta(y) = \{z \in X | \max\{d(y, z), d(z, y)\} < \delta\}
\]
such that \(x \in B_\delta(x)\) and \(y \in B_\delta(y)\).

To show that \(B_\delta(x) \cap B_\delta(y) = \emptyset\), suppose that \(B_\delta(x) \cap B_\delta(y) \neq \emptyset\). Then, there exists \(z \in B_\delta(x) \cap B_\delta(y)\). We have
\[
d(x, y) \leq s d(x, z) + s d(z, y)
\]
\[
\leq s \max\{d(x, z), d(z, x)\} + s \max\{d(y, z), d(z, y)\}
\]
\[
< s \delta + s \delta = d(x, y).
\]
So, \(d(x, y) < d(x, y)\) which is a contradiction. Therefore \(B_\delta(x) \cap B_\delta(y) = \emptyset\). \(\square\)

**Proposition 2.3.** Every dqb-convergent sequence in a dqb-metric space \((X, d)\) is dqb-Cauchy sequence.

**Proof.** Suppose that \(\{x_n\}\) is dqb-convergent. Then there exists \(x \in X\) such that \(x_n \to x\), that is
\[
\lim_{n \to \infty} d(x_n, x) = 0 = \lim_{n \to \infty} d(x, x_n).
\]
Consider,
\[
d(x_n, x_m) \leq s d(x_n, x) + s d(x, x_m).
\]
Taking limit as \(n, m \to \infty\) we obtain
\[
\lim_{n,m \to \infty} d(x_n, x_m) = 0.
\]
Similarly,
\[
\lim_{n,m \to \infty} d(x_m, x_n) = 0.
\]
Therefore \(\{x_n\}\) is dqb-Cauchy. \(\square\)

**Definition 2.4.** A subset \(S\) of a dqb-metric space \((X, d)\) is bounded if there exists \(\bar{x}\), \(M \in (0, \infty)\) such that \(d(x, \bar{x}) \leq M\) for all \(x \in S\).

**Proposition 2.5.** Every dqb-convergent sequence in a dqb-metric space \((X, d)\) is bounded sequence.

**Proof.** Suppose that \(\{x_n\}\) is dqb-convergent. Then there exists \(x \in X\) such that \(x_n \to x\), that is
\[
\lim_{n \to \infty} d(x_n, x) = 0 = \lim_{n \to \infty} d(x, x_n).
\]
Let \(\epsilon = 1\). Then there exists \(n_0 \in \mathbb{N}\) such that \(d(x_n, x) < \epsilon\) and \(d(x, x_n) < \epsilon\) for all \(n \geq n_0\). Choose
\[
K = \max\{d(x_1, x), d(x_2, x), ..., d(x_{n_0 - 1}, x), 1\}.
\]
Thus, \(d(x_n, x) \leq K\) for all \(n \in \mathbb{N}\) and so \(\{x_n\}\) is bounded sequence. \(\square\)
**Proposition 2.6.** Every dqb-Cauchy sequence in a dqb-metric space \((X, d)\) is bounded sequence.

**Proof.** Suppose that \(\{x_n\}\) is dqb-Cauchy. Then

\[
\lim_{n \to \infty} d(x_n, x_m) = 0 = \lim_{n \to \infty} d(x_m, x_n).
\]

Let \(\epsilon = 1\). Then there exists \(n_0 \in \mathbb{N}\) such that \(d(x_n, x_m) < 1\) and \(d(x_m, x_n) < 1\) for all \(n, m \geq n_0\). Let \(p\) be any point in the space and let

\[
k = \max_{i \leq m} d(x_i, p).
\]

The maximum exists, since \(\{x_i : i \leq m\}\) is a finite set. If \(n \leq m\), then \(d(x_n, p) \leq k\). If \(n > m\), then

\[
d(x_n, p) \leq d(x_n, x_m) + d(x_m, p) \leq 1 + k \quad \text{for all} \quad n \in \mathbb{N}.
\]

Therefore \(\{x_n\}\) is bounded sequence. \(\Box\)

The next two propositions for subsequence follow immediately from definitions of dqb-convergent sequence and dqb-Cauchy sequence respectively.

**Proposition 2.7.** Every subsequence of dqb-convergent sequence in a dqb-metric space \((X, d)\) is dqb-convergent sequence.

**Proposition 2.8.** Every subsequence of dqb-Cauchy sequence in a dqb-metric space \((X, d)\) is dqb-Cauchy sequence.

**Proposition 2.9.** Let \(\{x_n\}\) be sequence in a dqb-metric space \((X, d)\). Then \(x_n \to x\) if and only if \(d(x_n, x) \to 0\) and \(d(x, x_n) \to 0\).

**Proof.** Suppose that \(x_n \to x\). Then

\[
\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0.
\]

Thus \(d(x_n, x) \to 0\) and \(d(x, x_n) \to 0\).

Conversely, Suppose that \(d(x_n, x) \to 0\) and \(d(x, x_n) \to 0\). Then

\[
\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0.
\]

By definition of dqb-convergent sequence, we get \(x_n \to x\). \(\Box\)

**Proposition 2.10.** Let \(\{x_n\}\) be sequence in a dqb-metric space \((X, d)\). If \(x_n \to x\) and \(x_n \to y\), then \(x = y\).

**Proof.** Suppose that \(x_n \to x\) and \(x_n \to y\). Then

\[
\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = \lim_{n \to \infty} d(x_n, y) = \lim_{n \to \infty} d(y, x_n) = 0.
\]

Consider,

\[
0 \leq d(x, y) \leq sd(x, x_n) + sd(x_n, y)
\]

and

\[
0 \leq d(y, x) \leq sd(y, x_n) + sd(x_n, x).
\]

Taking limit as \(n, m \to \infty\), we obtain

\[
d(x, y) = d(y, x) = 0.
\]

Therefore \(x = y\). \(\Box\)

Now, we begin with introducing the property of a continuous function.

**Definition 2.11.** Suppose that \((X, d_X)\) and \((Y, d_Y)\) are dislocated quasi-b-metric spaces, \(E \subset X\), \(f : E \to Y\) and \(p \in E\). Then \(f\) is continuous at \(p\) iff for all \(\epsilon > 0\) there exists \(\delta > 0\) such that

\[
\max\{d_Y(f(x), f(p)), d_Y(f(p), f(x))\} < \epsilon
\]

for all \(x \in E\), when \(\max\{d_X(x, p), d_X(p, x)\} < \delta\).
Theorem 2.12. Let \((X,d_X)\) and \((X,d_Y)\) be dislocated quasi-b-metric spaces, \(E \subset X\), \(f: E \to Y\) and \(p \in E\). Then \(f\) is continuous at \(p\) if and only if for every dislocated quasi-b-converges sequence \(\{x_n\}\) in \(X\), \(\lim_{n \to \infty} f x_n = fx\).

Proof. Suppose that \(f\) is continuous at \(p\) and \(\{x_n\}\) converges to \(p\). Let \(\epsilon > 0\). Then there exists \(\delta > 0\) such that \(\max\{d_Y(f x, fp), d_Y(fp, fx)\} < \epsilon\), when \(\max\{d_X(x, p), d_X(p, x)\} < \delta\) for all \(x \in E\).

Since \(\{x_n\}\) converges to \(p\), there exists \(N \in \mathbb{N}\) such that \(\max\{d_X(x_n, p), d_Y(p, x_n)\} < \delta\) for all \(n \geq N\). Since \(f\) is continuous at \(p\), we have \(\max\{d_Y(f x_n, fp), d_Y(fp, fx_n)\} < \epsilon\), for all \(n \geq N\).

Hence \(\lim_{n} f x_n = fx\).

Conversely, let \(x \in X\) and assume in the contrary that

\[\exists \epsilon > 0 \quad \forall \delta > 0: \max\{d_X(x, p), d_X(p, x)\} < \delta, \max\{d_Y(f x, fp), d_Y(fp, fx)\} \geq \epsilon.\]

Applying these successively for all \(\delta = \frac{1}{k}\), we find a sequence \(\{x_k\}\) such that \(\max\{d_X(x_k, p), d_X(p, x_k)\} < \frac{1}{k}\) and \(\max\{d_Y(f x_k, fp), d_Y(fp, fx_k)\} \geq \epsilon\). Thus

\[\lim_{k \to \infty} x_k = p.\]

By assumption, we have

\[\lim_{k \to \infty} f x_k = fp.\]

Hence, there exists a \(k_0\) such that for all \(k > k_0\)

\[\max\{d_Y(f x_k, fp), d_Y(fp, fx_k)\} < \epsilon,\]

which is a contradiction. \(\square\)

Definition 2.14. Let \(A\) and \(B\) be nonempty closed subsets of a dislocated quasi-b-metric spaces \((X,d)\). A cyclic map \(T : A \cup B \to A \cup B\) is said to be a dqb-cyclic-weak contraction or dqb-cyclic-weakly contraction if for all \(x \in A\), \(y \in B\),

\[sd(Tx,Ty) \leq d(x,y) - \psi(d(x,y)),\] \hspace{1cm} (2.1)

where \(\psi: [0, \infty) \to [0, \infty)\) is a continuous and nondecreasing function such that \(\psi(t) = 0\) if and only if \(t = 0\).

Lemma 2.15. Let \((X,d_X)\) and \((Y,d_Y)\) be dislocated quasi-b-metric spaces and \(A\) and \(B\) be nonempty closed subsets of a dislocated quasi-b-metric spaces \((X,d)\). Consider a cyclic map \(T : A \cup B \to A \cup B\). If \(T\) is dqb-cyclic-weak contraction, then \(T\) is continuous.

Proof. Let \(\epsilon > 0\), all \(x \in A \cup B\) and fixed \(p \in A \cup B\). Suppose that \(\max\{d_X(x, p), d_C(p, x)\} < \delta\). Choose \(\epsilon = \frac{\delta}{s}\). Since \(T\) is dqb-cyclic-weak contraction, we have

\[sd(Tx,Tp) \leq d(x,p) - \psi(d(x,p)) \leq d(x,p) < \delta\]

and

\[sd(Tp,Tx) \leq d(p,x) - \psi(d(p,x)) \leq d(p,x) < \delta.\]

So, \(d(Tx,Tp) < \epsilon\) and \(d(Tp,Tx) < \epsilon\). Thus \(T\) is continuous at \(p\) and hence \(T\) is continuous on \(A \cup B\). \(\square\)
Theorem 2.16. Let $A$ and $B$ be nonempty subsets of a complete dislocated quasi-$b$-metric space $(X, d)$. Let $T$ be a cyclic mapping that satisfies the condition a dqb-cyclic-weak contraction. Then, $T$ has a unique fixed point in $A \cap B$.

Proof. Let $x \in A$ be fixed. Using contractive condition in assumptions, we have

\[
\begin{align*}
    d(T^2x, Tx) &\leq sd(T^2x, Tx) \\
              &= sd(T(Tx), Tx) \\
              &\leq d(Tx, x) - \psi(d(Tx, x)), \\
              &\leq d(Tx, x)
\end{align*}
\]

and

\[
\begin{align*}
    d(Tx, T^2x) &\leq sd(Tx, T^2x) \\
              &= sd(Tx, T(Tx)) \\
              &\leq d(x, Tx) - \psi((x, Tx)), \\
              &\leq d(x, Tx).
\end{align*}
\]

So

\[
    d(T^3x, T^2x) \leq d(T^2x, Tx) - \psi(d(T^2x, Tx))
\]

and

\[
    d(T^2x, T^3x) \leq d(Tx, T^2x) - \psi(d(Tx, T^2x)).
\]

For all $n \in \mathbb{N}$, we get

\[
    d(T^{n+2}x, T^{n+1}x) \leq d(T^{n+1}x, T^nx) - \psi(d(T^{n+1}x, T^nx))
\]

and

\[
    d(T^{n+1}x, T^{n+2}x) \leq d(T^nx, T^{n+1}x) - \psi(d(T^n x, T^{n+1}x)).
\]

Set $\varsigma_n = d(T^{n+1}x, T^nx)$ and $\tau_n = d(T^nx, T^{n+1}x)$. By inequalities (2.6) and (2.7), we get

\[
    \varsigma_{n+1} \leq \varsigma_n - \psi(\varsigma_n) \leq \varsigma_n
\]

and

\[
    \tau_{n+1} \leq \tau_n - \psi(\tau_n) \leq \tau_n.
\]

Thus $\{\varsigma_n\}$ and $\{\tau_n\}$ are decreasing sequences of non-negative real numbers, and hence possess a $\lim_{n \to \infty} \varsigma_n = \varsigma \geq 0$ and $\lim_{n \to \infty} \tau_n = \tau \geq 0$. Suppose that $\varsigma > 0$. Since $\psi$ is nondecreasing, $\psi(\varsigma_n) \geq \psi(\varsigma) > 0$. By inequality (2.5), we have $\varsigma_{n+1} \leq \varsigma_n - \psi(\varsigma)$. Thus $\varsigma_{N+m} \leq \varsigma_m - N\psi(\varsigma)$, a contradiction for $N$ large enough. Therefore $\varsigma = 0$.

Similarly, $\tau = 0$.

Next, we prove that $\{T^nx\}$ is a Cauchy sequence. Suppose that $\{T^nx\}$ is not Cauchy, then there exist $\epsilon > 0$ and subsequence $\{T^{m_k}x\}$ and $\{T^{n_k}x\}$ with $m_k > n_k \geq n$ such that $d(T^{m_k}x, T^{n_k}x) \geq \epsilon$ and $d(T^{m_k-1}x, T^{n_k}x) < \epsilon$. Now, we consider

\[
\begin{align*}
    sd(T^{m_k}x, T^{n_k}x) &\leq d(T^{m_k-1}x, T^{n_k-1}x) - \psi(d(T^{m_k-1}x, T^{n_k-1}x)) \\
                        &\leq d(T^{m_k-1}x, T^{n_k-1}x),
\end{align*}
\]
which implies that
\[ s\epsilon \leq d(T^{m_k-1}x, T^{n_k-1}x). \] (2.11)

Take limit inferior in (2.11) as \( k \to \infty \), we get
\[ \epsilon s \leq \lim \inf d(T^{m_k-1}x, T^{n_k-1}x). \] (2.12)

We have
\[ d(T^{m_k-1}x, T^{n_k-1}x) \leq sd(T^{m_k-1}x, T^{n_k}x) + sd(T^{n_k}x, T^{n_k-1}x) \]
\[ < s\epsilon + sd(T^{n_k}x, T^{n_k-1}x). \] (2.13)

Take limit superior in (2.13) as \( k \to \infty \), we get
\[ \lim \sup d(T^{m_k-1}x, T^{n_k-1}x) \leq s\epsilon. \] (2.14)

By (2.12) and (2.14), we get
\[ \lim d(T^{m_k-1}x, T^{n_k-1}x) = s\epsilon. \] (2.15)

Letting \( k \to \infty \) in (2.10), by property of \( \psi \) and (2.15), we get
\[ s\epsilon \leq s\epsilon - \psi(s\epsilon) < s\epsilon, \] (2.16)

which is a contradiction. Hence \( \{ T^n x \} \) is a dqb-Cauchy sequence. Since \( (X, d) \) is complete, we have \( \{ T^n x \} \) converges to some \( z \in X \). We note that, \( \{ T^{2n} x \} \) is a sequence in \( A \) and \( \{ T^{2n-1} x \} \) is a sequence in \( B \) in a way that both sequences tend to same limit \( z \). Since \( A \) and \( B \) are closed, we have \( z \in A \cap B \) and hence \( A \cap B \neq \emptyset \). The continuity of \( T \) implies that the limit is a fixed point. Finally, to prove the uniqueness of fixed point, let \( z^* \in X \) be another fixed point of \( T \) such that \( Tz^* = z^* \). Then, we have
\[ d(z, z^*) = d(Tz, Tz^*) \leq sd(Tz, Tz^*) \leq d(z, z^*) - \psi(d(z, z^*)) \leq d(z, z^*). \] (2.17)

On the other hand,
\[ d(z^*, z) = d(Tz^*, Tz) \leq sd(Tz^*, Tz) \leq d(z^*, z) - \psi(d(z, z^*)) \leq d(z^*, z). \] (2.18)

By forms (2.17) and (2.18), we obtain that \( d(z, z^*) = d(z^*, z) = 0 \), this implies that \( z^* = z \). Therefore \( z \) is a unique fixed point of \( T \). This completes the proof. \( \square \)

**Example 2.17.** Let \( X = [-1, 1] \) and \( T : A \cup B \to A \cup B \) be defined by \( Tx = -\frac{x}{3} \) and \( \psi(t) = \frac{t}{50} \). Suppose that \( A = [-1, 0] \) and \( B = [0, 1] \). Defined the function \( d : X^2 \to [0, \infty) \) by
\[ d(x, y) = |x - y|^2 + \frac{|x|}{10} + \frac{|y|}{11}. \]

We see that \( d \) is a dislocated quasi-b-metric on \( X \) (see[5]).

Let \( x \in A \). Then \(-1 \leq x \leq 0 \). So, \( 0 \leq \frac{x}{3} \leq \frac{1}{3} \). Thus, \( Tx \in B \). On the other hand, let \( x \in B \). Then \( 0 \leq x \leq 1 \). So, \( \frac{1}{3} \leq \frac{x}{3} \leq 0 \). Thus, \( Tx \in A \).

Hence, the map \( T \) is cyclic on \( X \), because \( T(A) \subset B \) and \( T(B) \subset A \).

Next, we consider
\[ 2d(Tx, Ty) = 2(|Tx - Ty|^2 + \frac{1}{10}|Tx| + \frac{1}{11}|Ty|) \]
\[ = 2(|-\frac{x}{3} - \frac{y}{3}|^2 + \frac{1}{10}|\frac{-x}{3}| + \frac{1}{11}|\frac{-y}{3}|) \]
Thus, T satisfies dqb-cyclic-weak contraction of Theorem 2.16 and 0 is the unique fixed point of T.

**Definition 2.18.** Let A and B be nonempty subsets of a dislocated quasi-b-metric spaces. (X,d). A cyclic mapping T: A ∪ B → A ∪ B is said to be a dqb-cyclic-φ-contraction and if there exists k ∈ [0, 1) and s ≥ 1 such that

\[
\phi(Tx, Tx) \leq \phi(x, y)
\]

(2.19)

for all x ∈ A, y ∈ B, where Φ the family of non-decreasing functions: Φ : [0, ∞) → [0, ∞) such that \(\sum_{n=1}^{\infty} \phi^n(t) < \infty\) for each t > 0, where n is the n-th iterate of \(\phi\).

**Theorem 2.19.** Let A and B be nonempty closed subsets of a complete dislocated quasi-b-metric space (X,d). Let T be a cyclic mapping that satisfies the condition a dqb-cyclic-φ-contraction. Then, T has a unique fixed point in A ∩ B.

**Proof.** Let x ∈ A be fixed, then using contractive condition of theorem, we have

\[
sd(Tx, Ty) \leq \phi(d(x, y))
\]

and

\[
sd(Tx, T^2x) = sd(T(Tx), Tx)
\]

\[
\leq \phi(d(Tx, x))
\]

Inductively, we have for all n ∈ N, we get

\[
s^n d(T^{n+1}x, T^nx) \leq \phi^n(d(Tx, x))
\]

and

\[
s^n d(T^nx, T^{n+1}x) \leq \phi^n(d(x, Tx)).
\]

Let \(\epsilon > 0\) be fixed and \(n(\epsilon) \in \mathbb{N}\), such that

\[
\sum_{n \geq n(\epsilon)} \phi^n(d(Tx, x)) < \epsilon
\]

and

\[
\sum_{n \geq n(\epsilon)} \phi^n(d(x, Tx)) < \epsilon.
\]

Let \(n, m \in \mathbb{N}\) with \(m > n > n(\epsilon)\), using the triangular inequality, we have:

\[
d(T^m x, T^n x) \leq s^{m-n} d(T^m x, T^{m-1} x) + s^{m-n-1} d(T^{m-1} x, T^{m-2} x) + ... + d(T^{n+1} x, T^n x)
\]

\[
\leq s^{m-1} d(T^m x, T^{m-1} x) + s^{m-2} d(T^{m-1} x, T^{m-2} x) + ... + s^n d(T^{n+1} x, T^n x)
\]

\[
\leq \phi^{m-1}(d(Tx, x)) + \phi^{m-2}(d(Tx, x)) + \phi^{m-3}(d(Tx, x)) + ... + \phi^n(d(Tx, x))
\]

\[
= \phi^{m-1}(k(d(x, Tx)))
\]

\[
\leq \sum_{n \geq n(\epsilon)} \phi^n(d(x, Tx)) < \epsilon.
\]
Similarly,
\[ d(T^n x, T^m x) < \epsilon. \]

Thus \( \{T^n x\} \) is a Cauchy sequence. Since \((X, d)\) is complete, we have \( \{T^n x\} \) converges to some \( z \in X \). We note that \( \{T^{2n} x\} \) is a sequence in \( A \) and \( \{T^{2n-1} x\} \) is a sequence in \( B \) in a way that both sequences tend to same limit \( z \). Since \( A \) and \( B \) are closed, we have \( z \in A \cap B \) and then \( A \cap B \neq \emptyset \). Now, we will show that \( Tz = z \). By using (2.19), consider
\[ d(z, Tz) \leq sd(z, T^{2n} x) + sd(T^{2n} x, Tz) \leq sd(z, T^{2n} x) + d(T^{2n-1} x, z). \]

Taking limit as \( n \to \infty \) in above inequality, we have
\[ d(z, Tz) = 0. \]

Similarly considering form (2.19), we get
\[ d(Tz, z) \leq sd(Tz, T^{2n} x) + sd(T^{2n} x, z) \leq d(Tz, T^{2n-1} x) + sd(T^{2n} x, z). \]

Taking limit as \( n \to \infty \) in above inequality, we have
\[ d(Tz, z) = 0. \]

Hence \( d(z, Tz) = d(Tz, z) = 0 \). This implies that \( Tz = z \) that is \( z \) is a fixed point of \( T \).

Finally, to prove the uniqueness of fixed point, let \( z^* \in X \) be another fixed point of \( T \) such that \( Tz^* = z^* \). Then, we have
\[ d(z^*, z) \leq sd(Tz^*, T^n x) + sd(T^n x, Tz) \leq \phi(d(Tz^*, T^n x)) + \phi(d(T^n x, Tz)) \]
and on the other hand,
\[ d(z^*, z) \leq sd(Tz, T^n x) + sd(T^n x, Tz^*) \leq \phi(d(Tz, T^n x)) + \phi(d(T^n x, Tz^*)). \]

Letting \( n \to \infty \) we obtain that \( d(z^*, z) = d(z^*, z) = 0 \), which implies that \( z^* = z \). Therefore \( z \) is a unique fixed point of \( T \). This completes the proof. \( \square \)

**Example 2.20.** Let \( X = [-1, 1] \) and \( T : A \cup B \to A \cup B \) be defined by \( Tx = -\frac{x}{5} \). Suppose that \( A = [-1, 0] \) and \( B = [0, 1] \). Defined the function \( d : X^2 \to [0, \infty) \) by
\[ d(x, y) = |x - y|^2 + \frac{|x|}{10} + \frac{|y|}{11}. \]

We see that \( d \) is a dislocated quasi-b-metric on \( X \), where \( s = 2 \). Let \( x \in A \). Then \(-1 \leq x \leq 0 \). So, \( 0 \leq \frac{x}{s} \leq \frac{1}{5} \).

Thus, \( Tx \in B \). On the other hand, let \( x \in B \). Then \( 0 \leq x \leq 1 \). So, \( \frac{1}{5} \leq \frac{x}{s} \leq 0 \). Thus, \( Tx \in A \).

Hence the map \( T \) is cyclic on \( X \), because \( T(A) \subset B \) and \( T(B) \subset A \).

Next, we consider
\[ sd(Tx, Ty) = 2d(Tx, Ty) \]
\[ = 2(|Tx - Ty|^2 + \frac{1}{10}|Tx| + \frac{1}{11}|Ty|) \]
\[ = 2(\frac{-x}{5} - \frac{-y}{5})^2 + \frac{1}{10}\frac{|x|}{5} + \frac{1}{11}\frac{|y|}{5} \]
$$= \frac{2}{3} \left( \frac{3}{25} |x-y|^2 + \frac{3}{50} |x| + \frac{3}{55} |y| \right)$$
$$\leq \frac{2}{3} \left( |x-y|^2 + \frac{5}{50} |x| + \frac{5}{55} |y| \right)$$
$$= \frac{2}{3} \left( |x-y|^2 + \frac{1}{10} |x| + \frac{1}{11} |y| \right)$$
$$= \phi(d(x, y)),$$

where the function $\phi \in \Phi$ is $\phi(t) = \frac{2t}{3}$. Clearly, 0 is the unique fixed point of $T$.

The following corollary can be taken as a particular case of Theorem 2.19 if we take $\phi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1)$. That is the dqb-cyclic-Banach contraction, in the setting of dislocated quasi-b-metric spaces.

**Corollary 2.21.** Let $A$ and $B$ be nonempty closed subsets of a complete dislocated quasi-b-metric space $(X, d)$. Let $T$ be a cyclic mapping that satisfies the condition a dqb-cyclic-Banach contraction; that is, if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \quad (2.22)$$

for all $x \in A$, $y \in B$ and $s \geq 1$ and $sk \leq 1$. Then, $T$ has a unique fixed point in $A \cap B$.

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**References**


