Some integral inequalities of the Hermite–Hadamard type for log-convex functions on co-ordinates

Yu-Mei Bai\textsuperscript{a}, Feng Qi\textsuperscript{b,c,}\textsuperscript{*}

\textsuperscript{a}College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, China.
\textsuperscript{b}Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300160, China.
\textsuperscript{c}Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China.

Communicated by Sh. Wu

Abstract

In the paper, the authors establish some new integral inequalities for log-convex functions on co-ordinates. These newly-established inequalities are connected with integral inequalities of the Hermite–Hadamard type for log-convex functions on co-ordinates. ©2016 All rights reserved.

Keywords: Log-convex functions, co-ordinates, integral inequality, Hermite–Hadamard type.

2010 MSC: 26A51, 26D15, 26D20, 26E60, 41A55.

1. Introduction

Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex function on an interval \( I \) and let \( a, b \in I \) such that \( a < b \). Then the double inequality

\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

holds. This double inequality is known in the literature as the Hermite–Hadamard integral inequality.

Definition 1.1. If a positive function \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} = (0, \infty) \) satisfies

\[
f(\lambda x + (1-\lambda)y) \leq [f(x)]^\lambda [f(y)]^{1-\lambda}
\]

for all \( x, y \in I \) and \( \lambda \in [0, 1] \), then we say that \( f \) is a logarithmically convex (or simply, log-convex) function on \( I \). If the above inequality is reversed, then we say that \( f \) is a log-concave function.

*Corresponding author

Email addresses: baiym2008@sohu.com (Yu-Mei Bai), qifeng618@gmail.com, qifeng618@hotmail.com (Feng Qi)

Received 2016-08-04
Equivalently, a function $f$ is log-convex on $I$ if and only if $f$ is positive and its logarithm $\ln f$ is convex on $I$. Moreover, if the second derivative $f''$ exists on $I$, then $f$ is log-convex if and only if $f f'' - (f')^2 \geq 0$.

A corresponding version of the Hermite–Hadamard integral inequality for log-convex functions was given in [5] as follows.

**Theorem 1.2 ([5]).** Suppose that $f : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}_+$ is a log-convex function on $[a, b]$. Then

$$f\left(\frac{a + b}{2}\right) \geq \frac{1}{b - a} \int_a^b f(x) \, dx \leq L(f(a), f(b)),$$

where $L(x, y)$ is the logarithmic mean

$$L(x, y) = \begin{cases} \frac{y - x}{\ln y - \ln x}, & x \neq y, \\ x, & x = y. \end{cases}$$

In [3, 4], the so-called convex functions on co-ordinates were introduced as follows.

**Definition 1.3 ([3, 4]).** A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be convex on co-ordinates on $\Delta$ with $a < b$ and $c < d$ if the partial mappings

$$f_y : [a, b] \rightarrow \mathbb{R}, \quad f_y(u) = f_y(u, y) \quad \text{and} \quad f_x : [c, d] \rightarrow \mathbb{R}, \quad f_x(v) = f_x(x, v)$$

are convex for all $x \in (a, b)$ and $y \in (c, d)$.

**Definition 1.4 ([3, 4]).** A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be convex on co-ordinates on $\Delta$ with $a < b$ and $c < d$ if the inequality

$$f(tx + (1 - t)z, \lambda y + (1 - \lambda)w) \leq t\lambda f(x, y) + t(1 - \lambda)f(x, w) + (1 - t)\lambda f(z, y) + (1 - t)(1 - \lambda)f(z, w)$$

holds for all $t, \lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$.

An inequality of the Hermite–Hadamard type for convex function on co-ordinates on a rectangle from the plane $\mathbb{R}^2$ was established in [3, 4] as follows.

**Theorem 1.5 ([3, 4] Theorem 2.2).** Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be convex on co-ordinates on $\Delta$ with $a < b$ and $c < d$. Then one has

$$f\left(\frac{a + b}{2}, \frac{c + d}{2}\right) \leq \frac{1}{2} \left[ \frac{1}{b - a} \int_a^b f\left(x, \frac{c + d}{2}\right) \, dx + \frac{1}{d - c} \int_c^d f\left(\frac{a + b}{2}, y\right) \, dy \right]$$

$$\leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx$$

$$\leq \frac{1}{4} \left[ \frac{1}{b - a} \int_a^b \left[ f(x, c) + f(x, d) \right] \, dx + \frac{1}{d - c} \int_c^d \left[ f(a, y) + f(b, y) \right] \, dy \right]$$

$$\leq \frac{1}{4} \left[ f(a, c) + f(b, c) + f(a, d) + f(b, d) \right].$$

In [1], Alomari and Darus introduced a class of log-convex functions on co-ordinates as follows.

**Definition 1.6 ([1]).** A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}_+$ is called log-convex on co-ordinates on $\Delta$ with $a < b$ and $c < d$ if

$$f(tx + (1 - t)z, \lambda y + (1 - \lambda)w) \leq [f(x, y)]^{t\lambda}[f(x, w)]^{t(1 - \lambda)}[f(z, y)]^{(1 - t)\lambda}[f(z, w)]^{(1 - t)(1 - \lambda)}$$

holds for all $t, \lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$. 
Remark 1.7. If \( f \) and \( g \) are both log-convex on co-ordinates on \( \Delta \), then their composite \( f \circ g \) is also log-convex on co-ordinates on \( \Delta \).

An inequality of the Hermite–Hadamard type for log-convex functions on co-ordinates on a rectangle from the plane \( \mathbb{R}^2 \) was established by Alomari and Darus in [1] as follows.

**Theorem 1.8** ([1, Theorem 3.3]). Suppose that \( f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^+ \) is log-convex on co-ordinates on \( \Delta \) for \( a < b \) and \( c < d \). Let

\[
A = \frac{f(a, c)f(b, d)}{f(b, c)f(a, d)}, \quad B = \frac{f(a, d)}{f(b, d)}, \quad \text{and} \quad C = \frac{f(b, c)}{f(b, d)}.
\]

Then the inequality

\[
I = \frac{1}{(b-a)(d-c)} \int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy \leq f(b, d) \times \begin{cases} 1, & A = B = C = 1, \\ \frac{B - 1}{\ln B} - \frac{C - 1}{\ln C}, & A = 1, \\ H(C), & B = 1, \\ H(B), & C = 1, \\ \frac{1}{2} \left( \frac{B - 1}{\ln B} + \frac{AB - 1}{\ln(AB)} \right), & A = B = 1, \\ \gamma + \ln(-\ln A) + Ei(1, -\ln A), & B = C = 1, \\ \int_{0}^{1} C^\beta AB - 1 \, d\beta, & A, B, C > 0, \\ 2 \ln \ln A, & -1 < \frac{\ln x}{\ln A} < 0, \\ 0, & \text{otherwise} \end{cases}
\]

holds, where \( \gamma \) is the Euler constant,

\[
H(x) = \frac{Ei(1, -\ln A) + \ln \ln x - Ei(1, -\ln(Ax)) - \ln(Ax)}{\ln A} + \begin{cases} 2 \ln \ln A - \ln(-\ln A), & -1 < \frac{\ln x}{\ln A} < 0, \\ 0, & \text{otherwise}; \end{cases}
\]

and

\[
Ei(x) = V.P. \int_{-x}^{\infty} \frac{e^{-t}}{t} \, dt
\]

is the exponential integral function.

For more and detailed information on this topic, please refer to the newly published papers [2, 6–25] and plenty of references therein.

2. Some new integral inequalities of the Hermite–Hadamard type

In this section, we prove some new inequalities of the Hermite–Hadamard type for log-convex functions on co-ordinates.
Theorem 2.1. Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}^+$ for $a < b$ and $c < d$ be log-convex on co-ordinates on $\Delta$. Then one has
\[
\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) \, dx \, dy \\
\leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b L(f(x, c), f(x, d)) \, dx + \frac{1}{d-c} \int_c^d L(f(a, y), f(b, y)) \, dy \right] \\
\leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b [f(x, c) + f(x, d)] \, dx + \frac{1}{d-c} \int_c^d [f(a, y) + f(b, y)] \, dy \right] \\
\leq \frac{1}{4} \left[ L(f(a, c), f(b, c)) + L(f(a, d), f(b, d)) + L(f(a, c), f(a, d)) + L(f(b, c), f(b, d)) \right] \\
\leq \frac{1}{4} [f(a, c) + f(b, c) + f(a, d) + f(b, d)],
\]
where $L(u, v)$ is the logarithmic mean.

Proof. For all $x, y > 0$, it is known that $L(x, y) \leq \frac{x + y}{2}$. Setting $y = \lambda c + (1 - \lambda)d$ for all $0 \leq \lambda \leq 1$, using the log-convexity of $f$, and by the arithmetic-geometric mean inequality, we obtain
\[
\int_c^d \int_a^b f(x, y) \, dx \, dy = \int_c^d \int_a^b f(x, \lambda c + (1 - \lambda)d) \, dx \, d\lambda \\
\leq \int_c^d \int_a^b \left[ f(x, c) \right]^{1 - \lambda} \left[ f(x, d) \right]^{\lambda} \, dx \, d\lambda \\
= \int_c^d \int_a^b L(f(x, c), f(x, d)) \, dx.
\]
Since $f(x, c) \leq [f(a, c)]^t [f(b, c)]^{1-t}$ and $f(x, d) \leq [f(a, d)]^t [f(b, d)]^{1-t}$ for each $t \in [0, 1]$, we have
\[
\frac{1}{2(b-a)} \int_a^b [f(x, c) + f(x, d)] \, dx \leq \frac{1}{2} \int_0^1 \left\{ [f(a, c)]^t [f(b, c)]^{1-t} + [f(a, d)]^t [f(b, d)]^{1-t} \right\} \, dt \\
= \frac{1}{2} \left[ L(f(a, c), f(b, c)) + L(f(a, d), f(b, d)) \right] \\
\leq \frac{1}{4} [f(a, c) + f(b, c) + f(a, d) + f(b, d)].
\]
By a similar argument, we can obtain
\[
\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) \, dx \, dy \\
\leq \frac{1}{d-c} \int_c^d L(f(a, y), f(b, y)) \, dy \\
\leq \frac{1}{2(d-c)} \int_c^d [f(a, y) + f(b, y)] \, dy \\
\leq \frac{1}{2} \left[ L(f(a, c), f(a, d)) + L(f(b, c), f(b, d)) \right] \\
\leq \frac{1}{4} [f(a, c) + f(b, c) + f(a, d) + f(b, d)].
\]
The proof of Theorem 2.1 is thus complete.

Example 2.2. The function $f(x, y) = x^2 y^2 + 1$ is log-convex on co-ordinates on $\Delta = [-1, 1]^2$. In Theorem 1.8 since $A = B = C = 1$, we have
\[
\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) \, dx \, dy = \frac{10}{9} < 2 = f(b, d).
By Theorem 2.1, we obtain

\[
\frac{1}{(b-a)(d-c)} \int_a^b f(x, y) \, dx \, dy = \frac{10}{9} < \frac{4}{3}
\]

\[
= \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b L(f(x, c), f(x, d)) \, dx + \frac{1}{d-c} \int_c^d L(f(a, y), f(b, y)) \, dy \right] < 2.
\]

**Theorem 2.3.** Let \( f: \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}_+ \) with \( a < b \) and \( c < d \) be log-convex on co-ordinates on \( \Delta \). Then

\[
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2} \left\{ \frac{1}{b-a} \int_a^b \left[ f\left(x, \frac{c+d}{2}\right) f\left(a+b-x, \frac{c+d}{2}\right) \right]^{1/2} \, dx \\
+ \frac{1}{d-c} \int_c^d \left[ f\left(\frac{a+b}{2}, y\right) f\left(\frac{a+b}{2}, c+d-y\right) \right]^{1/2} \, dy \right\} \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b \left( f\left(x, \frac{c+d}{2}\right) \right) \, dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) \, dy \right]
\]

\[(2.1)\]

\[
\leq \frac{1}{2(b-a)(d-c)} \int_a^b \int_a^b \left\{ [f(x, y) f(x, c+d-y)]^{1/2} + [f(x, y) f(a+b-x, y)]^{1/2} \right\} \, dx \, dy \\
\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_a^b f(x, y) \, dx \, dy.
\]

**Proof.** Utilizing the log-convexity of \( f \) leads to

\[
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = f\left(\frac{1}{2} [ta + (1-t)b + (1-t)a + tb], \frac{1}{2} \left( \frac{c+d}{2} + \frac{c+d}{2} \right) \right) \\
\leq \left[ f\left(\frac{1}{2} [ta + (1-t)b, \frac{c+d}{2} + \frac{c+d}{2}] \right) \right]^{1/2}
\]

\[(2.2)\]

for all \( 0 \leq t \leq 1 \). On using the change of the variable \( x = ta + (1-t)b \) for \( 0 \leq t \leq 1 \), integrating the inequality \[(2.2)\] over \( t \) on \([0, 1]\), and by the arithmetic-geometric mean inequality, we procure

\[
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2} \int_0^1 \left[ f\left(\frac{1}{2} [ta + (1-t)b, \frac{c+d}{2} + \frac{c+d}{2}] \right) \right]^{1/2} \, dt
\]

\[(2.3)\]

\[
= \frac{1}{b-a} \int_a^b \left[ f\left(x, \frac{c+d}{2}\right) f\left(a+b-x, \frac{c+d}{2}\right) \right]^{1/2} \, dx
\]

\[(2.4)\]

Using the log-convexity of \( f \), we find

\[
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq [f(x, \lambda c + (1-\lambda)d) f(x, (1-\lambda)c + \lambda d)]^{1/2}
\]

\[(2.5)\]
Similarly, we obtain
\[
f\left(\frac{a + b}{2} , \frac{c + d}{2} \right) \leq \frac{1}{d - c} \int_c^d \left[ f\left(\frac{a + b}{2} , y \right) f\left(\frac{a + b}{2} , c + d - y \right) \right]^{1/2} dy
\]
\[
\leq \frac{1}{d - c} \int_c^d f\left(\frac{a + b}{2} , y \right) dy
\]
\[
\leq \frac{1}{(b - a)(d - c)} \int_c^d \int_a^b \left[ f(x, y) f(a + b - x, y) \right]^{1/2} dx dy
\]
\[
\leq \frac{1}{(b - a)(d - c)} \int_c^d f(x, y) dx dy.
\]
A combination of (2.3), (2.5), and the last inequality gives the desired inequality (2.1). Theorem 2.3 is thus proved.

Making use of Theorem 2.3, we derive the following corollary.

**Corollary 2.4.** Let \( f, g : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}_+ \) with \( a < b \) and \( c < d \) be log-convex on co-ordinates on \( \Delta \). Then
\[
f\left(\frac{a + b}{2} , \frac{c + d}{2} \right) g\left(\frac{a + b}{2} , \frac{c + d}{2} \right)
\]
\[
\leq \frac{1}{2} \left\{ \frac{1}{b - a} \int_a^b \left[ f\left(x, \frac{c + d}{2} \right) g\left(x, \frac{c + d}{2} \right) \right]^{1/2} dx
\]
\[
+ \frac{1}{d - c} \int_c^d \left[ f\left(\frac{a + b}{2} , y \right) \right]^{1/2} dy
\]
\[
\leq \left\{ \frac{1}{2(b - a)(d - c)} \right\} \int_c^d \int_a^b \left[ f(x, y) g(x, c + d - y) \right]^{1/2} dx dy
\]
\[
\leq \frac{1}{2(b - a)(d - c)} \int_c^d \int_a^b \left[ f(x, y) g(x, c + d - y) \right]^{1/2} dx dy
\]

**Theorem 2.5.** Let \( f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}_+ \) with \( a < b \) and \( c < d \) be log-convex on co-ordinates on \( \Delta \). Then
\[
f\left(\frac{a + b}{2} , \frac{c + d}{2} \right)
\]
\[
\leq \frac{1}{2} \left[ \frac{1}{b - a} \int_a^b \left[ f\left(x, \frac{c + d}{2} \right) \right]^{1/2} dx
\]
\[
+ \frac{1}{d - c} \int_c^d \left[ f\left(\frac{a + b}{2} , y \right) \right]^{1/2} dy
\]
\[
\leq \frac{1}{(b - a)(d - c)} \int_c^d \int_a^b \left[ f(x, y) f(x, c + d - y) \right]^{1/4} dx dy
\]
\[
\leq \frac{1}{2(b - a)(d - c)} \int_c^d \int_a^b \left[ f(x, y) f(x, c + d - y) \right]^{1/2} + \left[ f(x, y) f(a + b - x, y) \right]^{1/2} dx dy
\]
\[
\leq \frac{1}{(b - a)(d - c)} \int_c^d \int_a^b f(x, y) dx dy.
\]
Proof. Since $f$ is log-convex on co-ordinates on $\Delta$, using the inequalities (2.3), (2.5), and the arithmetic-geometric inequality figures out

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{b-a} \int_{a}^{b} \left[ f\left(x, \frac{c+d}{2}\right) f\left(a+b-x, \frac{c+d}{2}\right) \right]^{1/2} dx$$

$$\leq \frac{1}{b-a} \int_{0}^{1} \int_{a}^{b} \left[ f(x, \lambda c + (1-\lambda)d) f(x, (1-\lambda)c + \lambda d) \right]^{1/4} dx \, d\lambda$$

$$\times f(a+b-x, \lambda c + (1-\lambda)d) f(a+b-x, (1-\lambda)c + \lambda d) \right]^{1/4} dx \, d\lambda$$

$$= \frac{1}{(b-a)(d-c)} \int_{c}^{d} \int_{a}^{b} \left[ f(x,y) f(x, c+d-y) f(a+b-x, y) \right]^{1/4} dx \, dy$$

$$\times f(a+b-x, c+d-y) \right]^{1/4} dx \, dy$$

$$\leq \frac{1}{2(b-a)(d-c)} \int_{c}^{d} \int_{a}^{b} \left\{ [f(x,y) f(x, c+d-y)]^{1/2} + [f(x,y) f(a+b-x, y)]^{1/2} \right\} dx \, dy$$

$$\leq \frac{1}{(b-a)(d-c)} \int_{c}^{d} \int_{a}^{b} \left[ f(x,y) \right]^{1/2} dx \, dy.$$

Similarly, we obtain

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{d-c} \int_{c}^{d} \left[ f\left(\frac{a+b}{2}, y\right) f\left(\frac{a+b}{2}, c+d-y\right) \right]^{1/2} dy$$

$$\leq \frac{1}{(b-a)(d-c)} \int_{c}^{d} \int_{a}^{b} \left[ f(x,y) f(x, c+d-y) f(a+b-x, y) \right]^{1/4} dx \, dy$$

$$\times f(a+b-x, c+d-y) \right]^{1/4} dx \, dy$$

$$\leq \frac{1}{2(b-a)(d-c)} \int_{c}^{d} \int_{a}^{b} \left\{ [f(x,y) f(x, c+d-y)]^{1/2} + [f(x,y) f(a+b-x, y)]^{1/2} \right\} dx \, dy$$

$$\leq \frac{1}{(b-a)(d-c)} \int_{c}^{d} \int_{a}^{b} \left[ f(x,y) \right]^{1/2} dx \, dy.$$

Hence, the proof of Theorem 2.5 is complete. 

Corollary 2.6. Let $f, g : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}_+$ with $a < b$ and $c < d$ be log-convex on co-ordinates on $\Delta$. Then

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

$$\leq \frac{1}{2} \left[ \frac{1}{b-a} \int_{a}^{b} \left[ f\left(x, \frac{c+d}{2}\right) g\left(x, \frac{c+d}{2}\right) f\left(\frac{a+b-x}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b-x}{2}, \frac{c+d}{2}\right) \right]^{1/2} dx \right]$$

$$+ \frac{1}{d-c} \int_{c}^{d} \left[ f\left(\frac{a+b}{2}, y\right) g\left(\frac{a+b}{2}, y\right) f\left(\frac{a+b}{2}, c+d-y\right) g\left(\frac{a+b}{2}, c+d-y\right) \right]^{1/2} dy \right]$$

$$\leq \frac{1}{(b-a)(d-c)} \int_{c}^{d} \int_{a}^{b} \left[ f(x,y) g(x,y) f(x, c+d-y) g(x, c+d-y) \right]^{1/4} dx \, dy$$

$$\times f(a+b-x, y) g(a+b-x, y) f(a+b-x, c+d-y) g(a+b-x, c+d-y) \right]^{1/4} dx \, dy$$

$$\leq \frac{1}{2(b-a)(d-c)} \int_{c}^{d} \int_{a}^{b} \left\{ [f(x,y) g(x,y) f(x, c+d-y) g(x, c+d-y)]^{1/2} \right.$$
Theorems 2.1 and 2.3 can be improved as follows.

**Corollary 2.7.** Under the conditions of Theorems 2.1 and 2.3, if \( f(x, y) = f_1(x)g_1(y) \) for \((x, y) \in \Delta\), then

\[
\left( a + b \right) g_1 \left( \frac{c + d}{2} \right) \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b \left[ f_1(x) f_1(a + b - x) \right]^{1/2} \, dx \right] g_1 \left( \frac{c + d}{2} \right) \\
+ \left( \frac{1}{d-c} \int_c^d \left[ g_1(x) g_1(c + d - y) \right]^{1/2} \, dy \right) f_1 \left( \frac{c + d}{2} \right)
\]

\[
\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left[ f_1(x)g_1(y)f_1(a + b - x)g_1(c + d - y) \right]^{1/2} \, dx \, dy
\]

\[
\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f_1(x)g_1(y) \, dx \, dy
\]

\[
\leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b L(f_1(x)g_1(c), f_1(x)g_1(d)) \, dx \right]
+ \left( \frac{1}{d-c} \int_c^d L(f_1(a)g_1(y), f_1(b)g_1(y)) \, dy \right)
\]

\[
\leq \frac{1}{4} \left[ \frac{g_1(c) + g_1(d)}{b-a} \int_a^b f_1(x) \, dx + \frac{f_1(a) + f_1(b)}{d-c} \int_c^d g_1(y) \, dy \right]
\]

\[
\leq \frac{1}{4} \left[ L(f_1(a), f_1(b))[g_1(c) + g_1(d)] + [f_1(a) + f_1(b)]L(g_1(c), g_1(d))] \right]
\]

\[
\leq \frac{1}{4} \left[ [f_1(a) + f_1(b)][g_1(c) + g_1(d)] \right].
\]

### 3. Conclusions

By the arithmetic-geometric inequality and other techniques, we establish some new integral inequalities for log-convex functions on co-ordinates. These newly-established inequalities are connected with integral inequalities of the Hermite–Hadamard type for log-convex functions on co-ordinates.

### Acknowledgment

This work was partially supported by the National Natural Science Foundation of China under Grant No. 11361038 and by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region under Grant No. NJZY16180, China.

The authors appreciate the communicating editor and anonymous referees for their kind help, careful corrections, and valuable comments on the original version of this paper.

**References**


