Fixed points of Bregman relatively nonexpansive mappings and solutions of variational inequality problems

Mohammed Ali Alghamdi$^a$, Naseer Shahzad$^{a, *}$, Habtu Zegeye$^b$

$^a$Operator Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia.

$^b$Department of Mathematics, University of Botswana, Pvt. Bag 00704 Gaborone, Botswana.

Communicated by Y. J. Cho

Abstract

In this paper, we propose an iterative scheme for finding a common point of the fixed point set of a Bregman relatively nonexpansive mapping and the solution set of a variational inequality problem for a continuous monotone mapping. We prove a strong convergence theorem for the sequences produced by the method. Our results improve and generalize various recent results. ©2016 All rights reserved.

Keywords: Bregman distance function, Bregman relatively nonexpansive mapping, fixed points of mappings, strong convergence, monotone mapping.

2010 MSC: 47H05, 47H09, 47J25, 49J40.

1. Introduction.

Let $E$ denote a real reflexive Banach space with norm $||.||$ and $E^*$ stands for the (topological) dual of $E$ endowed with the induced norm $||.||_*$. Let $C$ be a nonempty subset of $E$. A mapping $A : C \to E^*$ is said to be monotone if for any $x, y \in C$, we have

$$\langle Ax - Ay, x - y \rangle \geq 0.$$
We note that the class of monotone mappings includes the class of \( \gamma \)-inverse strongly monotone mappings, where a mapping \( A : C \to E^* \) is called \( \gamma \)-inverse strongly monotone if there exists a positive real number \( \gamma \) such that,
\[
\langle Ax - Ay, x - y \rangle \geq \gamma \|Ax - Ay\|^2, \quad \text{for all } x, y \in C.
\] (1.1)

The monotone mapping \( A \) is called maximal, if its graph \( G(A) = \{(x, y) : y \in Ax\} \) is not properly contained in the graph of any other monotone mapping.

The variational inequality problem for a monotone mapping \( A \) is the problem of finding a point \( x^* \in C \) satisfying
\[
\forall x \in C, \quad \langle Ax^*, x - x^* \rangle \geq 0.
\] (1.2)

We denote the solution set of this problem by \( VI(C, A) \). We note that if \( A \) is a continuous monotone mapping then the solution set \( VI(C, A) \) is always closed and convex.

The monotone variational inequalities were initially investigated by Kinderlehrer and Stampacchia in [9] and are related with the convex minimization problems, the zeros of monotone mappings and the complementarity problems. Consequently, many researchers have studied variational inequality problems for monotone mappings (see, e.g., [26, 27, 28, 31, 32]).

In this paper, \( f : E \to (-\infty, +\infty] \) is always a proper, lower semi-continuous and convex function with \( \text{dom} f = \{x \in E : f(x) < \infty\} \). For any \( x \in \text{int}(\text{dom} f) \) and any \( y \in E \), let \( f^0(x, y) \) be the right-hand derivative of \( f \) at \( x \) in the direction of \( y \), that is,
\[
f^0(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}.
\] (1.3)

The function \( f \) is said to be Gâteaux differentiable at \( x \), if \( \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t} \) exists for any \( y \). In this case, \( f^0(x, y) \) coincides with \( \nabla f(x) \), the value of the gradient \( \nabla f \) of \( f \) at \( x \). The function \( f \) is said to be Gâteaux differentiable if it is Gâteaux differentiable everywhere. The function \( f \) is said to be Fréchet differentiable at \( x \in E \) (see, for example, [1]), if for all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( \|x - y\| \leq \delta \) implies that
\[
|f(x) - f(y) - \langle x - y, \nabla f(y) \rangle| \leq \epsilon \|x - y\|.
\] (1.4)

The function \( f \) is said to be Fréchet differentiable, if it is Fréchet differentiable everywhere. The function \( f \) is said to be strongly coercive if
\[
\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = \infty.
\] (1.5)

Let \( f : E \to (-\infty, +\infty] \) be a Gâteaux differentiable function. The function \( Df : \text{dom} f \times \text{int}(\text{dom} f) \to [0, +\infty) \) defined by
\[
Df(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y \rangle,
\]
is called the Bregman distance with respect to \( f \) [3]. A Bregman projection [3] of \( x \in \text{int}(\text{dom} f) \) onto the nonempty closed and convex set \( C \subset \text{dom} f \) is the unique vector \( P_C^f(x) \in C \) satisfying
\[
D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.
\]

If \( E \) is a smooth Banach space, setting \( f(x) = \|x\|^2 \) for all \( x \in E \), we have \( \nabla f(x) = 2Jx \), where \( J \) is the normalized duality mapping from \( E \) into \( 2E^* \) defined by \( Jx := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \) and hence \( D_f(x, y) \) reduces to \( \phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \) for all \( x, y \in E \), which is the Lyapunov function introduced by Alber [1]. In this case, the Bregman projection is called the generalized projection, \( \Pi_C \) (see [1]). If, in addition, \( E = H \), a Hilbert space, then \( D_f(x, y) \) becomes \( \phi(x, y) = \|x - y\|^2 \) for \( x, y \in H \) and the Bregman projection reduces to the metric projection \( P_C \) from \( E \) onto \( C \).

A point \( x \in C \) is a fixed point of \( T : C \to C \) if \( Tx = x \) and we denote by \( F(T) \) the set of fixed points of \( T \); that is, \( F(T) = \{x \in C : Tx = x\} \). A point \( p \) in \( C \) is said to be an asymptotic fixed point of \( T \) (see...
follows. A mapping $T : C \to \text{int}(\text{dom } f)$ with $F(T) := \{ x \in D(T) : Tx = x \} \neq \emptyset$ is called:

(i) **quasi-Bregman nonexpansive** \[21\] if,

$$D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T);$$

(ii) **Bregman relatively nonexpansive** \[21\] if,

$$D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T), \text{ and } \hat{F}(T) = F(T).$$

When $E$ is a smooth Banach space and $f(x) = ||x||^2$ for all $x \in E$, the above definitions reduce to the following definitions using Lyapunov function.

A mapping $T : C \to \text{int}(\text{dom } f)$ with $F(T) \neq \emptyset$ is called:

(i) **quasi-nonexpansive** \[21\] if,

$$\phi(p, Tx) \leq \phi(p, x), \forall x \in C, p \in F(T);$$

(ii) **relatively nonexpansive** \[21\] if,

$$\phi(p, Tx) \leq \phi(p, x), \forall x \in C, p \in F(T), \text{ and } \hat{F}(T) = F(T).$$

Various methods have been introduced for approximating fixed points of relatively nonexpansive and quasi-nonexpansive mappings (see, e.g., \[8, 10, 13, 15, 21, 24, 30\]). In 2011, Zhang et al. \[39\] introduced an iteration method for finding fixed point of relatively nonexpansive mappings in a Banach space setting as follows.

**Theorem 1.1** (\[39\]). **Let $C$ be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space $E$ and let $T : C \to C$ be a relatively nonexpansive mapping. Let $\{x_n\}$ be a sequence in $C$ defined by $x_1 \in C$ and

\begin{align*}
x_{n+1} &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), n \geq 1, \quad (1.6)
\end{align*}

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n \to \infty} \alpha_n = 0$. If the interior of $F(T)$ is nonempty, then they proved that the sequence $\{x_n\}$ converges strongly to a fixed point of $T$.**

In 2005, Matsushita and Takahashi \[14\] proposed the following hybrid iteration method for a relatively nonexpansive mapping $T$ in a Banach space $E$. Let $C$ be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space $E$. Define the sequences $\{x_n\}$ by

\begin{align*}
x_0 &\in C = C_1, \text{ chosen arbitrary}, \\
y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), \\
C_n &= \{ z \in C : \phi(z, y_n) \leq \phi(z, x_n) \}, \\
Q_n &= \{ z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0 \}, \\
x_{n+1} &= \Pi_{C_n \cap Q_n}(x_0), n \geq 1. \quad (1.7)
\end{align*}

They proved that the sequence $\{x_n\}$ generated by (1.7) converges strongly to the point $\Pi_{F(T)}(x_0)$, where $\Pi_{F(T)}$ is the generalized projection from $C$ onto $F(T)$.

More recently, many authors have also considered the problem of finding a common element of the fixed point set of a relatively nonexpansive or a Bregman relatively nonexpansive mapping and the solution set of a variational inequality problem for $\gamma$--inverse strongly monotone mapping (see, e.g., \[7, 11, 12, 26, 27, 28, 32, 33, 34, 35\]). For other related results, we refer to \[22, 23, 36, 37\].
In 2009, Inoue et al. [8] proposed the following hybrid iteration method in a uniformly convex and uniformly smooth Banach space $E$ for a sequence $\{x_n\}$ as follows:

$$
\begin{align*}
  x_0 &\in C = C_1, \text{ chosen arbitrary,} \\
  u_n & = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_n x_n), \\
  C_n & = \{z \in C_n : \phi(z,u_n) \leq \phi(z,x_n)\}, \\
  Q_n & = \{z \in C : \langle x_n - z, Jx_n - Jx_0 \rangle \geq 0\}, \\
  x_{n+1} & = \Pi_{C_n \cap Q_n}(x_0), n \geq 1,
\end{align*}
$$

(1.8)

where $T : C \to C$ is a relatively nonexpansive mapping and $J_r = (I + rB)^{-1}J$, for $B : C \to E^*$ maximal monotone mapping and $r > 0$. They proved that the sequence $\{x_n\}$ converges strongly to the point $\Pi_{F(T) \cap B^{-1}(0)}(x_0)$, where $\Pi_{F(T)}$ is the generalized projection from $C$ onto $F(T)$.

In this paper, it is our purpose to investigate an iterative scheme for finding a common point of the fixed point set of a Bregman relatively nonexpansive mapping and the solution set of a variational inequality problem for a continuous monotone mapping in reflexive Banach spaces. We prove a strong convergence theorem for the sequence produced by the method. Our results improve and generalize various recent results (see, e.g., [3] [12]).

2. Preliminaries

Legendre function $f$ from a general Banach space $E$ into $(-\infty, +\infty]$ were defined in [2]. The Fenchel conjugate of $f$ is the function $f^* : E^* \to (-\infty, +\infty]$ defined by $f^*(y) = \sup\{(y,x) - f(x) : x \in E\}$. If $E$ is a reflexive Banach space and $f : E \to (-\infty, +\infty]$ is a Legendre function, then in view of [2],

$$
\nabla f = (\nabla f^*)^{-1}, \text{ ran } \nabla f = \text{dom}\nabla f^* = \text{int}(\text{dom} f^*) \text{ and ran}\nabla f^* = \text{int}(\text{dom} f),
$$

where ran$\nabla f$ denotes the range of $\nabla f$. When $E$ is a smooth and strictly convex Banach space, one important and interesting example of Legendre function is $f(x) := \frac{1}{p}||x||^p(1 < p < \infty)$. In this case the gradient $\nabla f$ of $f$ coincides with the generalized duality mapping of $E$, i.e., $\nabla f = J_2(1 < p < \infty)$. In particular, $\nabla f = I$, the identity mapping in Hilbert spaces.

**Lemma 2.1** ([24]). Let $f : E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent:

(i) $f$ is bounded on bounded subsets and uniformly smooth on bounded subsets of $E$;

(ii) $f^*$ is Fréchet differentiable and $\nabla f^*$ is uniformly norm-to-norm continuous on bounded subsets of $E^*$;

(iii) dom$f^* = E^*$, $f^*$ is strongly coercive and uniformly convex on bounded subsets of $E^*$.

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The modulus of total convexity of $f$ at $x \in \text{dom} f$ is the function $\nu_f(x,.) : [0, +\infty) \to [0, +\infty]$ defined by

$$
    \nu_f(x,t) := \inf\{D_f(y,x) : y \in \text{dom} f, \|y - x\| = t\}.
$$

The function $f$ is called totally convex at $x$ if $\nu_f(x,t) > 0$, whenever $t > 0$. The function $f$ is called totally convex if it is totally convex at any point $x \in \text{int}(\text{dom} f)$ and is said to be totally convex on bounded sets if $\nu_f(B,t) > 0$ for any nonempty bounded subset $B$ of $E$ and $t > 0$, where the modulus of total convexity of the function $f$ on the set $B$ is the function $\nu_f : \text{int}(\text{dom} f) \times [0, +\infty) \to [0, +\infty]$ defined by

$$
    \nu_f(B,t) := \inf\{\nu_f(x,t) : x \in B \cap \text{dom} f\}.
$$

We know that $f$ is totally convex on bounded sets if and only if $f$ is uniformly convex on bounded sets (see [5], Theorem 2.10).
Let $B_r := \{ z \in E : ||z|| \leq r \}$, for all $r > 0$ and $S_E = \{ x \in E : ||x|| = 1 \}$. Then a function $f : E \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of $E$ ([29], pp. 203) if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0, \infty) \to [0, \infty)$ is defined by

$$\rho_r(t) := \inf_{x,y \in B_r, ||x-y||=t, \alpha \in (0,1)} \frac{\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)}$$

for all $t \geq 0$.

In the sequel, we shall need the following lemmas.

**Lemma 2.2** ([15]). Let $E$ be a Banach space, let $r > 0$ be a constant and let $f : E \to \mathbb{R}$ be a uniformly convex on bounded subsets of $E$. Then

$$f\left(\sum_{k=0}^{n} \alpha_k x_k\right) \leq \sum_{k=0}^{n} \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r(||x_i - y_j||)$$

for all $i, j \in \{0, 1, 2, ..., n\}$, $x_k \in B_r, \alpha_k \in (0, 1)$ and $k = 0, 1, 2, ..., n$ with $\sum_{k=0}^{n} \alpha_k = 1$, where $\rho_r$ is the gauge of uniform convexity of $f$.

**Lemma 2.3** ([19]). Let $f : E \to (-\infty, +\infty]$ be uniformly Fréchet differentiable and bounded on bounded sets of $E$. Then $\nabla f$ is uniformly continuous on the strong topology of $E$ to the strong topology of $E^*$.

**Lemma 2.4** ([18]). Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Let $C$ be a nonempty closed convex subset of $\text{int}(\text{dom} f)$ and $T : C \to C$ be a quasi-Bregman nonexpansive mapping. Then $F(T)$ is closed and convex.

**Lemma 2.5** ([3]). The function $f : E \to (-\infty, +\infty)$ is totally convex on bounded subsets of $E$ if and only if for any two sequences $\{x_n\}$ and $\{y_n\} \subset \text{int}(\text{dom} f)$ and $\text{dom} f$, respectively, such that the first one is bounded,

$$\lim_{n \to \infty} D_f(y_n, x_n) = 0 \implies \lim_{n \to \infty} ||y_n - x_n|| = 0.$$

**Lemma 2.6** ([16]). Let $f : E \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper, weak* lower semi-continuous and convex function. Thus, for all $z \in E$, we have

$$D_f(z, \nabla f^* \left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)) \leq \sum_{i=1}^{N} t_i D_f(z, x_i).$$

**Lemma 2.7** ([13]). Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable on $\text{int}(\text{dom} f)$ such that $\nabla f^*$ is bounded on bounded subsets of $\text{dom} f^*$. Let $x \in E$ and $\{x_n\} \subset E$. If $\{D_f(x, x_n)\}$ is bounded, so is the sequence $\{x_n\}$.

**Lemma 2.8** ([5]). Let $C$ be a nonempty, closed and convex subset of $E$. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then

(i) $z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0, \forall y \in C$.

(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \forall y \in C$.

Let $f : E \to \mathbb{R}$ be a Legendre and Gâteaux differentiable function. Following [1] and [9], we make use of the function $V_f : E \times E^* \to [0, +\infty)$ associated with $f$, which is defined by

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \forall x \in E, x^* \in E^*.$$  \hspace{1cm} (2.1)$$

Then $V_f$ is nonnegative and

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)) \text{ for all } x \in E \text{ and } x^* \in E^*.$$  \hspace{1cm} (2.2)$$
Moreover, by the subdifferential inequality,
\begin{equation}
V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \leq V_f(x, x^* + y^*),
\end{equation}
\(\forall x \in E\) and \(x^*, y^* \in E^*\) (see [10]).

**Lemma 2.9** ([25]). Let \(\{a_n\}\) be a sequence of nonnegative real numbers satisfying the following relation:
\[ a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\delta_n, \quad n \geq n_0, \]
where \(\{\alpha_n\} \subset (0, 1)\) and \(\{\delta_n\} \subset \mathbb{R}\) satisfying the following conditions: \(\lim_{n \to \infty} \alpha_n = 0\), \(\sum_{n=1}^{\infty} \alpha_n = \infty\), and \(\limsup_{n \to \infty} \delta_n \leq 0\). Then, \(\lim_{n \to \infty} a_n = 0\).

**Lemma 2.10** ([12]). Let \(\{a_n\}\) be sequences of real numbers such that there exists a subsequence \(\{n_i\}\) of \(\{n\}\) such that \(a_{n_i} < a_{n_i+1}\) for all \(i \in \mathbb{N}\). Then there exists an increasing sequence \(\{m_k\} \subset \mathbb{N}\) such that \(m_k \to \infty\) and the following properties are satisfied by all (sufficiently large) numbers \(k \in \mathbb{N}\):
\[ a_{m_k} \leq a_{m_{k+1}} \text{ and } a_k \leq a_{m_{k+1}}. \]
In fact, \(m_k\) is the largest number \(n\) in the set \(\{1, 2, \ldots, k\}\) such that the condition \(a_n \leq a_{n+1}\) holds.

Following the agreement in [20] we have the following lemma.

**Lemma 2.11.** Let \(f : E \to (-\infty, +\infty]\) be a coercive Legendre function and \(C\) be a nonempty, closed and convex subset of \(E\). Let \(A : C \to E^*\) be a continuous monotone mapping. For \(r > 0\) and \(x \in E\), define the mapping \(F_r : E \to C\) as follows:
\[ F_r x := \{ z \in C : \langle Az, y - z \rangle + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \forall y \in C \} \]
for all \(x \in E\). Then the following hold:

1. \(F_r\) is single-valued;
2. \(F(F_r) = VI(C, A)\);
3. \(D_f(p, F_r x) + D_f(F_r x, x) \leq \phi(p, x), \forall p \in F(F_r)\);
4. \(VI(C, A)\) is closed and convex.

**3. Main Results**

Let \(C\) be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive real Banach space \(E\). Let \(A : C \to E^*\) be a continuous monotone mapping and let \(f : E \to \mathbb{R}\) be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of \(E\). Then in what follows, for each \(n\), let \(F_{r_n} : E \to C\) be defined by
\[ F_{r_n} x := \{ z \in C : \langle Az, y - z \rangle + \frac{1}{r_n}\langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \forall y \in C \}, \]
for all \(x \in E\), where \(\{r_n\} \subset (a, \infty)\) for some \(a > 0\).

We now prove the following theorem.
Similarly, we get that

\begin{equation}
\begin{aligned}
y_n &= \nabla f^*(a_n \nabla f(x_n) + b_n \nabla f(F_{r_n}(x_n)) + c_n \nabla f(T(x_n))), \\
x_{n+1} &= P_{F}^f \nabla f^*(a_n \nabla f(u) + (1 - a_n) \nabla f(y_n)), \forall n \geq 0,
\end{aligned}
\end{equation}

where \(\{a_n\}, \{b_n\}, \{c_n\} \subset [c, d] \subset (0, 1)\) such that \(a_n + b_n + c_n = 1\) and \(\{a_n\} \subset (0, 1)\) satisfies \(\lim_{n \to \infty} a_n = 0\), \(\sum_{n=1}^{\infty} a_n = \infty\). Then, \(\{x_n\}\) converges strongly to \(p = P_{f}^f(u)\).

**Proof.** From Lemmas 2.4 and 2.11 we get that \(\mathcal{F}\) is closed and convex. Thus, \(P_{\mathcal{F}}^f\) is well-defined. Let \(p = P_{\mathcal{F}}^f(u)\) and \(u_n = F_{r_n}(x_n)\). Now, since \(f\) is bounded and uniformly smooth on bounded subsets of \(E\) by Lemma 2.1 we get that \(f^*\) is uniformly convex on bounded subsets of \(E^*\). Then, from (3.1), (2.1), (2.2) and Lemmas 2.2, 2.11 together with the property of \(D_f\) we obtain

\[
D_f(p, y_n) = D_f(p, \nabla f(x_n)) = V_f(p, \nabla f(x_n)) + b_n \nabla f(y_n)) + c_n \nabla f(T(x_n)))
\]

\[
\leq f(p) - \langle p, a_n \nabla f(x_n) + b_n \nabla f(u_n) + c_n \nabla f(T(x_n)))
\]

\[
+ f^*(a_n \nabla f(x_n) + b_n \nabla f(u_n) + c_n \nabla f(T(x_n)))
\]

\[
\leq f(p) - a_n \langle p, \nabla f(x_n) - \nabla f(u_n)) - b_n \langle p, \nabla f(u_n) - \nabla f(T(x_n)\rangle),
\]

\[
+ a_n f^*(\nabla f(x_n)) + b_n f^*(\nabla f(u_n)) + c_n f^*(\nabla f(T(x_n)))
\]

\[
- a_n b_n \rho^*_f(\nabla f(x_n) - \nabla f(u_n)))
\]

and

\[
D_f(p, y_n) \leq a_n V_f(p, \nabla f(x_n)) + b_n V_f(p, \nabla f(u_n)) + c_n V_f(p, \nabla f(T(x_n)))
\]

\[
- a_n b_n \rho^*_f(\nabla f(x_n) - \nabla f(u_n))
\]

\[
= a_n D_f(p, x_n) + b_n D_f(p, u_n) + c_n D_f(p, T(x_n))
\]

\[
- a_n b_n \rho^*_f(\nabla f(x_n) - \nabla f(u_n))
\]

\[
\leq a_n D_f(p, x_n) + b_n D_f(p, u_n) + c_n D_f(p, x_n)
\]

\[
- a_n b_n \rho^*_f(\nabla f(x_n) - \nabla f(u_n))
\]

\[
\leq D_f(p, x_n) - a_n b_n \rho^*_f(\nabla f(x_n) - \nabla f(u_n)) \leq D_f(p, x_n).
\]

Similarly, we get that

\[
D_f(p, y_n) \leq D_f(p, x_n) - a_n c_n \rho^*_f(\nabla f(x_n) - \nabla f(T(x_n))) \leq D_f(p, x_n).
\]

In addition, from (3.1), (3.3) and Lemmas 2.6, 2.8 we have

\[
D_f(p, x_{n+1}) = D_f(p, F_{C}^f \nabla f^*(a_n \nabla f(u) + (1 - a_n) \nabla f(y_n)))
\]

\[
\leq D_f(p, \nabla f^*(a_n \nabla f(u) + (1 - a_n) \nabla f(y_n))
\]

\[
\leq a_n D_f(p, u) + (1 - a_n) D_f(p, y_n)
\]

\[
\leq a_n D_f(p, u) + (1 - a_n) \left[D_f(p, x_n) - a_n b_n \rho^*_f(\nabla f(x_n) - \nabla f(u_n)) \right]
\]

\[
\leq a_n D_f(p, x_n) + (1 - a_n) D_f(p, x_n).
\]

Thus, by induction,

\[
D_f(p, x_{n+1}) \leq \max\{D_f(p, u), D_f(p, x_0)\}, \forall n \geq 0,
\]
which implies that \( \{x_n\} \) is bounded. Now, let \( z_n = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(y_n)) \). Then we have that \( x_{n+1} = P_{C}z_n \), for all \( n \in \mathbb{N} \). Since \( f \) is strongly coercive, uniformly convex, uniformly Fréchet differentiable and bounded, by Lemmas 2.3 and \ref{2.1} we get that \( \nabla f \) and \( \nabla f^* \) are bounded and hence \( \{z_n\} \) and \( \{y_n\} \) are bounded. Furthermore, using \( \ref{2.2} \), \( \ref{2.3} \) and property of \( D_f \) we obtain that

\[
D_f(p, x_{n+1}) \leq D_f(p, z_n) = D_f(p, \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(y_n))) \\
= V_f(p, \alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(y_n)) \\
= V_f(p, \alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(y_n) - \alpha_n(\nabla f(u) - \nabla f(p)) - \langle \alpha_n(\nabla f(u) - \nabla f(p), z_n - p) \\
= V_f(p, \alpha_n \nabla f(p) + (1 - \alpha_n)\nabla f(y_n)) + \alpha_n(\nabla f(u) - \nabla f(p), z_n - p) \\
= D_f(p, \nabla f^*(\alpha_n \nabla f(p) + (1 - \alpha_n)\nabla f(y_n))) \\
+ \alpha_n(\nabla f(u) - \nabla f(p), z_n - p) \\
\leq D_f(p, p) + (1 - \alpha_n)D_f(p, y_n) + \alpha_n(\nabla f(u) - \nabla f(p), z_n - p) \\
\leq (1 - \alpha_n)D_f(p, y_n) + \alpha_n(\nabla f(u) - \nabla f(p), z_n - p).
\]

Thus, from \( \ref{3.3} \), \( \ref{3.4} \) and \( \ref{3.5} \) we get

\[
D_f(p, x_{n+1}) \leq (1 - \alpha_n)D_f(p, x_n) + \alpha_n(\nabla f(u) - \nabla f(p), z_n - p) \\
- \alpha_n b_n \rho^*_n(||\nabla f(x_n) - \nabla f(u_n)||) \\
\leq (1 - \alpha_n)D_f(p, x_n) + \alpha_n(\nabla f(u) - \nabla f(p), z_n - p),
\]

or

\[
D_f(p, x_{n+1}) \leq (1 - \alpha_n)D_f(p, x_n) + \alpha_n(\nabla f(u) - \nabla f(p), z_n - p) \\
- \alpha_n \delta_n \rho^*_n(||\nabla f(x_n) - \nabla f(T(x_n))||) \\
\leq (1 - \alpha_n)D_f(p, x_n) + \alpha_n(\nabla f(u) - \nabla f(p), z_n - p).
\]

The rest of the proof is divided into two cases:

**Case 1.** Suppose that there exists \( n_0 \in \mathbb{N} \) such that \( \{D_f(p, x_n)\} \) is non-increasing for all \( n \geq n_0 \). Thus, we get that \( \{D_f(p, x_n)\} \) is convergent. Now, from \( \ref{3.6} \) and \( \ref{3.8} \) we have that

\[
a_n b_n \rho^*_n(||\nabla f(x_n) - \nabla f(u_n)||) \to 0,
\]

and

\[
a_n \delta_n \rho^*_n(||\nabla f(x_n) - \nabla f(T(x_n))||) \to 0,
\]

which give by the property of \( \rho^*_n \) that

\[
\nabla f(x_n) - \nabla f(u_n) \to 0, \nabla f(x_n) - \nabla f(T(x_n)) \to 0 \text{ as } n \to \infty.
\]

Moreover, from \( \ref{3.1} \) and \( \ref{3.11} \) we have that

\[
||\nabla f(y_n) - \nabla f(x_n)|| \leq a_n ||\nabla f(x_n) - \nabla f(x_n)|| + b_n ||\nabla f(u_n) - \nabla f(x_n)|| \\
+ c_n ||\nabla f(T(x_n)) - \nabla f(x_n)|| \to 0 \text{ as } n \to \infty.
\]

In addition, since \( f \) is strongly coercive and uniformly convex on bounded subsets of \( E \) we have that \( f^* \) is uniformly Fréchet differentiable on bounded subsets of \( E^* \) and by Lemma 2.1 we get that \( \nabla f^* \) is uniformly continuous. Thus, this with \( \ref{3.11} \) and \( \ref{3.12} \) give that

\[
x_n - u_n \to 0, x_n - T(x_n) \to 0, x_n - y_n \to 0 \text{ as } n \to \infty.
\]
Furthermore, Lemma 2.6, property of \( D_f \) and the fact that \( \alpha_n \to 0 \) as \( n \to \infty \), imply that
\[
D_f(y_n, z_n) = D_f(y_n, \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(y_n))) \\
\leq \alpha_n D_f(x_n, u) + (1 - \alpha_n) D_f(y_n, y_n) \\
\leq \alpha_n D_f(x_n, u) + (1 - \alpha_n) D_f(y_n, y_n) \to 0 \quad \text{as} \quad n \to \infty,
\]
and hence by Lemma 2.5 we get that
\[
y_n - z_n \to 0 \quad \text{as} \quad n \to \infty.
\]

Now, since \( \{z_n\} \) is bounded and \( E \) is reflexive, we choose a subsequence \( \{z_{n_i}\} \) of \( \{z_n\} \) such that \( z_{n_i} \to z \) and \( \limsup_{n \to \infty} \langle \nabla f(u) - \nabla f(p), z_n - p \rangle = \lim_{i \to \infty} \langle \nabla f(u) - \nabla f(p), z_{n_i} - p \rangle \). Then, from (3.15) and (3.13) we get that
\[
x_{n_i} \to z, \quad \text{as} \quad i \to \infty.
\]
Thus, from (3.13) and the fact that \( T \) is Bregman relatively nonexpansive we obtain that \( z \in F(T) \).

Now, we show that \( z \in VI(C, A) \). By definition we have that
\[
\langle Au_n, y - u_n \rangle + \langle \nabla f(u_n) - \nabla f(x_n), y - u_n \rangle \geq 0, \quad \forall \ y \in C,
\]
and hence
\[
\langle Au_n, y - u_n \rangle + \langle \nabla f(u_n) - \nabla f(x_n), y - u_n \rangle \geq 0, \quad \forall \ y \in C.
\]

Set \( v_t = ty + (1 - t)z \) for all \( t \in (0, 1] \) and \( y \in C \). Consequently, we get that \( v_t \in C \). Now, from (3.18) it follows that
\[
\langle Av_t, v_t - u_n \rangle \geq \langle Av_t, v_t - u_{n_i} \rangle - \langle Av_{n_i}, v_t - u_{n_i} \rangle - \langle \nabla f(u_{n_i}) - \nabla f(x_{n_i}), v_t - u_{n_i} \rangle
\]
\[
= \langle Av_t - Av_{n_i}, v_t - u_{n_i} \rangle - \langle \nabla f(u_{n_i}) - \nabla f(x_{n_i}), v_t - u_{n_i} \rangle.
\]

But, from (3.13) have that
\[
\frac{\nabla f(u_{n_i}) - \nabla f(x_{n_i})}{r_{n_i}} \to 0 \quad \text{as} \quad i \to \infty \quad \text{and the monotonicity of} \ A \quad \text{implies that}
\]
\[
\langle Av_t - Av_{n_i}, v_t - u_{n_i} \rangle \geq 0. \quad \text{Thus, it follows that}
\]
\[
0 \leq \lim_{i \to \infty} \langle Av_t, v_t - u_{n_i} \rangle = \langle Av_t, v_t - z \rangle,
\]
and hence
\[
\langle Av_t, y - z \rangle \geq 0, \quad \forall \ y \in C.
\]

If \( t \to 0 \), the continuity of \( A \) implies that
\[
\langle Az, y - z \rangle \geq 0, \quad \forall \ y \in C.
\]

This implies that \( z \in VI(C, A) \) and hence \( z \in F = F(T) \cap VI(C, A) \).

Therefore, by Lemma 2.8, we immediately obtain that \( \limsup_{n \to \infty} \langle \nabla f(u) - \nabla f(p), z_n - p \rangle = \lim_{i \to \infty} \langle \nabla f(u) - \nabla f(p), z_{n_i} - p \rangle = \langle \nabla f(u) - \nabla f(p), z - p \rangle \leq 0 \). It follows from Lemma 2.9 and (3.7) that \( D_f(p, x_n) \to 0 \), as \( n \to \infty \). Consequently, by Lemma 2.5 we obtain that, \( x_n \to p \).

**Case 2.** Suppose that there exists a subsequence \( \{n_i\} \) of \( \{n\} \) such that
\[
D_f(p, x_{n_i}) < D_f(p, x_{n_{i+1}})
\]
for all \( i \in \mathbb{N} \). Then, by Lemma 2.10, there exists a nondecreasing sequence \( \{m_k\} \subset \mathbb{N} \) such that \( m_k \to \infty \), \( D_f(p, x_{m_k}) \leq D_f(p, x_{m_k+1}) \) and \( D_f(p, x_k) \leq D_f(p, x_{m_k+1}) \), for all \( k \in \mathbb{N} \). Then from (3.6), (3.8) and the fact that \( \alpha_n \to 0 \) we obtain that

\[
\rho^*_r(||\nabla f(x_{m_k}) - \nabla f(Tx_{m_k})||) \to 0 \text{ and } \rho^*_r(||\nabla f(x_{m_k}) - \nabla f(u_{m_k})||) \to 0,
\]

as \( k \to \infty \). Thus, following the method of proof in Case 1, we obtain that \( x_{m_k} - Tx_{m_k} \to 0 \), \( x_{m_k} - u_{m_k} \to 0 \), \( x_{m_k} - y_{m_k} \to 0 \), \( y_{m_k} - z_{m_k} \to 0 \) as \( k \to \infty \), and hence we obtain that

\[
\limsup_{k \to \infty} (\nabla f(u) - \nabla f(p), z_{m_k} - p) \leq 0.
\]

(3.19)

Now, from (3.7) we have that

\[
D_f(p, x_{m_k+1}) \leq (1 - \alpha_{m_k})D_f(p, x_{m_k}) + \alpha_{m_k} (\nabla f(u) - \nabla f(p), z_{m_k} - p),
\]

(3.20)

and since \( D_f(p, x_{m_k}) \leq D_f(p, x_{m_k+1}) \), inequality (3.20) implies

\[
\alpha_{m_k}D_f(p, x_{m_k}) \leq D_f(p, x_{m_k}) - D_f(p, x_{m_k+1}) + \alpha_{m_k} (\nabla f(u) - \nabla f(p), z_{m_k} - p)
\]

\[
\leq \alpha_{m_k} (\nabla f(u) - \nabla f(p), z_{m_k} - p).
\]

In particular, since \( \alpha_{m_k} > 0 \), we get

\[
D_f(p, x_{m_k}) \leq (\nabla f(u) - \nabla f(p), z_{m_k} - p).
\]

Hence, from (3.19) we get \( D_f(p, x_{m_k}) \to 0 \) as \( k \to \infty \). This together with (3.20) gives \( D_f(p, x_{m_k+1}) \to 0 \) as \( k \to \infty \). But \( D_f(p, x_k) \leq D_f(p, x_{m_k+1}) \) for all \( k \in \mathbb{N} \), thus we obtain that \( x_k \to p \). Therefore, from the above two cases, we can conclude that \( \{x_n\} \) converges strongly to \( p = P_F^f(u) \) and the proof is complete.

If, in Theorem 3.1, we assume that \( T = I \), the identity mapping on \( C \), we obtain the following corollary.

**Corollary 3.2.** Let \( C \) be a nonempty, closed and convex subset of \( \text{int}(\text{dom} f) \). Let \( A : C \to E^* \) be a continuous monotone mapping. Assume that \( V(C, A) \) is nonempty. For \( u, x_0 \in C \) let \( \{x_n\} \) be a sequence generated by

\[
\begin{align*}
\{y_n\} &= \nabla f^*(a_n \nabla f(x_n) + (1 - a_n) \nabla f(F_n(x_n))), \forall n \geq 0, \\
\{x_{n+1}\} &= \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(y_n)), \forall n \geq 0.
\end{align*}
\]

(3.21)

where \( \{a_n\} \subset [c, d] \subset (0, 1) \) and \( \{\alpha_n\} \subset (0, 1) \) satisfies \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \). Then, \( \{x_n\} \) converges strongly to \( p = P_{V(C, A)}^f(u) \).

If, in Theorem 3.1, we assume that \( C = E \), the projection mapping \( P_C^f \) is not required and \( VI(C, A) = A^{-1}(0) \) hence we get the following corollary.

**Corollary 3.3.** Let \( T : E \to E \) be a Bregman relatively nonexpansive mapping and \( A : E \to E^* \) be a continuous monotone mapping. Assume that \( F := F(T) \cap A^{-1}(0) \) is nonempty. For \( u, x_0 \in C \) let \( \{x_n\} \) be a sequence generated by

\[
\begin{align*}
\{y_n\} &= \nabla f^*(a_n \nabla f(x_n) + b_n \nabla f(F_n(x_n)) + c_n \nabla f(T(x_n))), \forall n \geq 0, \\
\{x_{n+1}\} &= \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(y_n)), \forall n \geq 0,
\end{align*}
\]

(3.22)

where \( \{a_n\}, \{b_n\}, \{c_n\} \subset [c, d] \subset (0, 1) \) such that \( a_n + b_n + c_n = 1 \) and \( \{\alpha_n\} \subset (0, 1) \) satisfies \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \). Then, \( \{x_n\} \) converges strongly to \( p = P_F^f(u) \).
We also note that the method of proof of Theorem 3.1 provides the following theorem for approximating the minimum-norm common point of the fixed point set of a Bregman relatively nonexpansive mapping and the solution set of a variational inequality problem for a continuous monotone mapping.

**Theorem 3.4.** Let $C$ be a nonempty, closed and convex subset of $\text{int}(\text{dom} f)$. Let $T : C \to E$ be a Bregman relatively nonexpansive mapping and $A : C \to E^*$ be a continuous monotone mapping. Assume that $F := F(T) \cap V(C, A)$ is nonempty. For $x_0 \in C$ let $\{x_n\}$ be a sequence generated by

$$
\begin{align*}
\left\{ \begin{array}{l}
y_n &= \nabla f^* (a_n \nabla f(x_n) + b_n \nabla f(F_{r_n}(x_n)) + c_n \nabla f(T(x_n))), \\
x_{n+1} &= P_C \nabla f^* ((1 - \alpha_n) \nabla f(y_n)), \forall n \geq 0,
\end{array} \right.
\end{align*}
$$

where $\{a_n\}, \{b_n\}, \{c_n\} \subset [c, d] \subset (0, 1)$ such that $a_n + b_n + c_n = 1$ and $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to the minimum-norm point $p$ of $F$ with respect to the Bregman distance.

**Remark 3.5.** Theorem 3.1 improves and extends the corresponding results of Inoue et al. [8] to the class of Bregman relatively nonexpansive mappings and to the class of continuous monotone mappings in reflexive Banach spaces.

**Acknowledgements**

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant no. (275-130-1436-G). The authors, therefore, acknowledge with thanks the DSR technical and financial support.

**References**


