Cyclic hybrid methods for finding common fixed points of a finite family of nonexpansive mappings

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Abstract

In this paper, we propose a cyclic hybrid method for computing a common fixed point of a finite family of nonexpansive mappings. The strong convergence of the method is established. Numerical examples illustrate that the proposed method has an advantage in computing. ©2016 All rights reserved.

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1. Introduction and Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and $C$ a nonempty closed convex subset of $H$. Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

holds for all $x, y \in C$. We denote by $\text{Fix}(T)$ the set of fixed points of $T$, i.e., $\text{Fix}(T) = \{x \in C : Tx = x\}$.

Construction of common fixed points for a finite family of nonexpansive mappings has received vast investigation, see \cite{3, 6, 13, 16}, since various problems of science and engineering, such as split feasibility problems and multiple-sets split feasibility problems with applications in intensity-modulated radiation therapy (IMRT) in the field of medical care, see \cite{11, 12}, can be reduced to a problem of finding a common fixed point of a family of nonexpansive mappings.

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In 2003, Nakajo and Takahashi [12] firstly introduced a hybrid algorithm for a nonexpansive mapping, thereafter, several researchers generalized the hybrid methods for computing common fixed points of a family of nonlinear mappings, see [7, 8, 14, 17, 18, 21]. For a finite family of relatively nonexpansive mappings \( \{T_i\}_{i=1}^{N} \), Anh and Chung [1] recently proposed a parallel hybrid algorithm as following:

**Algorithm AC**

\[
\begin{align*}
  x_0 &\in C \text{ chosen arbitrarily}, \\
  z_k &:= P_C(x_k), \\
  y_k^i &:= \alpha_k z_k + (1 - \alpha_k) T_i(z_k), \quad i = 1, 2, \cdots, N, \\
  i_k &:= \arg \max_{i=1,2,\cdots,N} \{\|y_k^i - x_k\|\}, \\
  C_k &:= \{v \in H : \|v - y_k^{i_k}\| \leq \|v - x_k\|\}, \\
  Q_k &:= \{u \in H : (x_0 - x_k, x_k - u) \geq 0\}, \\
  x_{k+1} &:= P_{C_k \cap Q_k}(x_0).
\end{align*}
\] (1.1)

Algorithm AC is inherently parallel and Anh and Chung showed their advantage in parallel computation in numerical examples.

Motivating by Anh and Chung’s work, we proposed a cyclic hybrid method which can be regarded as a counterpart of the parallel one. Our ideas consists of determining successively \( y_k \) for each operator \( T_i \), \( i = 1, 2, \ldots, N \) and constructing of \( y_k \) by using the value of \( y_k^{-1} \). Subsequent steps are the same with Algorithm AC. The benefit of our approach is using the newly-obtained \( y_k^{-1} (y_k^0 = x_k) \).

The remainder of this article is organized as follows. In the next section, some useful facts and tools are presented. In Section 3, convergence analysis of the cyclic algorithm is given while in Section 4 the numerical experiment is considered.

2. Preliminaries

We will use the notation:
1. \( \rightharpoonup \) for weak convergence and \( \rightarrow \) for strong convergence.
2. \( \omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\} \) denotes the weak \( \omega \)-limit set of \( \{x_n\} \).

We need some facts and tools in a real Hilbert space \( H \) which are listed as lemmas below.

**Lemma 2.1** ([2]). There holds the identity in a real Hilbert space \( H \):
\[
\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v \rangle, \quad u, v \in H.
\]

**Lemma 2.2** ([10]). Let \( C \) be a closed convex subset of a real Hilbert space \( H \) and let \( T : C \to C \) be a nonexpansive mapping such that \( \text{Fix}(T) \neq \emptyset \). If a sequence \( \{x_n\} \) in \( C \) is such that \( x_n \rightharpoonup z \) and \( x_n - T x_n \to 0 \), then \( z = Tz \).

**Lemma 2.3** ([2]). Let \( K \) be a closed convex subset of real Hilbert space \( H \) and let \( P_K \) be the (metric or nearest point) projection from \( H \) onto \( K \) (i.e., for \( x \in H \), \( P_K x \) is the only point in \( K \) such that \( \|x - P_K x\| = \inf \{\|x - z\| : z \in K\} \)). Given \( x \in H \) and \( z \in K \). Then \( z = P_K x \) if and only if there holds the relation:
\[
\langle x - z, y - z \rangle \leq 0, \quad \text{for all } y \in K.
\]

**Lemma 2.4** ([11]). Let \( K \) be a closed convex subset of \( H \). Let \( \{x_n\} \) be a sequence in \( H \) and \( u \in H \). Let \( q = P_K u \). If \( \{x_n\} \) is such that \( \omega_w \{x_n\} \subset K \) and satisfies the condition
\[
\|x_n - u\| \leq \|u - q\|, \quad \text{for all } n,
\]
then \( x_n \to q \).
3. A cyclic hybrid algorithm and its convergence

Let \( \{T_i\}_{i=1}^{N} \) be a family of nonexpansive mappings from \( C \) into itself and assume that the set \( F := \bigcap_{i=1}^{N} F(T_i) \) is not empty.

We consider the following algorithm.

**Algorithm 3.1.** Let \( x_0 \in C \) be an arbitrarily chosen element and \( \{\alpha_k\} \subset (0, \alpha] \) where \( \alpha < 1 \). For \( k \geq 0 \), assuming \( x_k \) is known, we

- Calculate
  
  \[
  \begin{align*}
  y_k^1 := & \alpha_k x_k + (1 - \alpha_k)T_1(x_k), \\
  y_k^{i+1} := & \alpha_k y_k^i + (1 - \alpha_k)T_{i+1}(y_k^i), \quad i = 0, 1, \ldots, N - 1.
  \end{align*}
  \tag{3.1}
  \]

- Find
  
  \[
  i_k := \arg\max_{i=1,2,\ldots,N} \{\|y_k^i - x_k\|\}. \tag{3.2}
  \]

- If \( \|y_k^{i_k} - x_k\| = 0 \) then stop. Else:
  - Define
    
    \[
    \begin{align*}
    C_k := & \{u \in C : \|u - y_k^{i_k}\| \leq \|u - x_k\|\}, \\
    Q_k := & \{v \in C : (x_0 - x_k, x_k - v) \geq 0\}.
    \end{align*}
    \tag{3.3}
    \]

- Compute
  
  \[
  x_{k+1} := P_{C_k \cap Q_k}(x_0). \tag{3.4}
  \]

- If \( x_{k+1} = x_k \) then stop. Else, set \( k := k + 1 \) and repeat.

**Lemma 3.2.** If Algorithm 3.1 finishes at a step \( k < \infty \), then \( x_k \) is a common fixed point of \( T_i, i = 1, 2, \ldots, N \), i.e., \( x_k \in \text{Fix}(T_i) \).

**Proof.** Using stopping rule \( x_k = x_{k+1} \), we have \( x_k \in C_k \). From the definition of \( C_k \), it follows

\[
\|x_k - y_k^{i_k}\| \leq \|x_k - x_k\| = 0.
\]

Applying the definition of \( i_k \), we get \( y_k^i = x_k \) for \( i = 1, 2, \ldots, N \). Taking into account (3.1), we have

\[
x_k = \alpha_k x_k + (1 - \alpha_k)T_1(x_k), \quad i = 1, 2, \cdots, N.
\]

Since \( \alpha_k < 1 \) we see \( x_k = T_1(x_k) \) for \( i = 1, 2, \cdots, N \), i.e., \( x_k \in \text{Fix}(T_i) \).

**Theorem 3.3.** Let \( \{x_k\} \) be the (infinite) sequence generated by Algorithm 3.1. \( T_1 \) be nonexpansive for \( i = 1, 2, \ldots, N \). Then \( x_k \to x^1 := P_{\text{Fix}(T_1)}(x_0) \) as \( k \to \infty \).

**Proof.** For each \( k \geq 0 \), it is easy to see that \( Q_k \) is a halfspace or \( Q_k = H \). Further, the relation \( \|u - y_k^{i_k}\| \leq \|u - x_k\| \) is equivalent to \( \langle u, x_k - y_k^{i_k} \rangle \leq \frac{1}{2} \{\|x_k\|^2 - \|y_k^{i_k}\|^2\} \) or \( \langle u - \frac{1}{2}(x_k + y_k^{i_k}), x_k - y_k^{i_k} \rangle \leq 0 \). Hence, for all \( k \geq 0 \), \( C_k \) is a halfspace in \( H \) or \( C_k = H \). An explicit formula for \( P_{C_k \cap Q_k}(x_0) \) can be obtained similarly as in [15]. Therefore, if \( C_k \cap Q_k \neq \emptyset \) then \( x_{k+1} \) is easily computed by (3.4).

Next we show that \( \text{Fix}(T_i) \subset C_k \cap Q_k \). Firstly we show that \( \text{Fix}(T_i) \subset C_k \) for all \( k \geq 0 \). To observe this, arbitrarily take \( p \in \text{Fix}(T_i) \), we have

\[
\|p - y_k^{i_k}\| = \|p - \{\alpha_k y_k^{i_k} + (1 - \alpha_k)T_1(y_k^{i_k})\}\|
\]
From (3.5) and (3.6) we obtain
\[
\leq \alpha_k \|p - y_k^{i-1}\| + (1 - \alpha_k)\|p - T_i(y_k^{i-1})\|
\]\
\leq \alpha_k \|p - y_k^{i-1}\| + (1 - \alpha_k)\|p - y_k^i\|\
= \|p - y_k^{i-1}\|
\leq \cdots \leq \|p - y_k^1\| \leq \|p - x_k\|.
\]

Therefore, \(p \in C_k\) and hence, \(Fix(T_i) \subset C_k\) for all \(k \geq 0\).

Next we show \(Fix(T_i) \subset Q_k\) for all \(k \geq 0\), by induction. For \(k = 0\), we have \(Fix(T_i) \subset C = Q_0\). Assume \(Fix(T_i) \subset Q_k\). Since \(x_{k+1}\) is the projection of \(x_0\) onto \(C_k \cap Q_k\), by Lemma 2.3, we have \(\langle x_0 - x_k, x_k - u \rangle \geq 0\) for all \(u \in C_k \cap Q_k\). As \(Fix(T_i) \subset C_k \cap Q_k\), by the induction assumption, the last inequality holds, in particular, for all \(u \in Fix(T_i)\). This together with the definition of \(Q_{k+1}\) implies that \(Fix(T_i) \subset Q_{k+1}\). Therefore we have \(Fix(T_i) \subset Q_k\) for all \(k \geq 0\). Hence \(Fix(T_i) \subset C_k \cap Q_k\).

Since \(Fix(T_i)\) is a nonempty closed convex subset of \(C\), there exists a unique element \(x^\dagger \in Fix(T_i)\) such that \(x^\dagger = P_{Fix(T_i)}x_0\). From \(x_k = P_{Q_k}x_0\) (by the definition of \(Q_k\)) and \(Fix(T_i) \subset Q_k\), we have \(\|x_k - x_0\| \leq \|p - x_0\|\) for all \(p \in Fix(T_i)\). Due to \(x^\dagger \in Fix(T_i)\), then we get
\[
\|x_k - x_0\| \leq \|x^\dagger - x_0\|,\tag{3.5}
\]
which implies that \(\{x_k\}\) is bounded.

The fact that \(x_{k+1} \in Q_k\) implies that \(\langle x_{k+1} - x_k, x_k - x_0 \rangle \geq 0\). This together with Lemma 2.1 imply
\[
\|x_{k+1} - x_k\|^2 \leq \|x_{k+1} - x_0\|^2 - \|x_k - x_0\|^2.\tag{3.6}
\]

From (3.5) and (3.6) we obtain
\[
\|x_k - x_{k+1}\| \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.\tag{3.7}
\]

Using the definition of \(C_k\) and the inclusion \(x_{k+1} \in C_k\), we also have
\[
\|x_{k+1} - y_k^i\| \leq \|x_{k+1} - x_k\|,
\]
which with (3.7) yields
\[
\|x_k - y_k^i\| \leq \|x_{k+1} - x_k\| + \|x_{k+1} - y_k^i\|
\leq 2\|x_{k+1} - x_k\| \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.\tag{3.8}
\]

From the definition of \(i_k\) in (3.2), it follows that
\[
\|x_k - y_k^i\| \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty \quad \text{for} \quad i = 1, 2, \cdots, N,\tag{3.9}
\]
which implies
\[
\|y_k^{i+1} - y_k^i\| \leq \|y_k^{i+1} - x_k\| + \|y_k^i - x_k\| \rightarrow 0, \quad \text{for} \quad i = 1, 2, \cdots, N - 1.
\]

From (3.1) it follows
\[
\|x_k - T_i(x_k)\| = \frac{1}{1 - \alpha_k}\|y_k^i - x_k\| \rightarrow 0,\tag{3.9}
\]
and
\[
\|y_k^i - T_{i+1}(y_k^i)\| = \frac{1}{1 - \alpha_k}\|y_k^i - y_k^{i+1}\| \rightarrow 0, \quad i = 1, 2, \ldots, N - 1.\tag{3.10}
\]

Using (3.8), (3.10) and nonexpansivity of \(\{T_i\}_{i=2}^N\), we get
\[
\|x_k - T_{i+1}(x_k)\| \leq \|x_k - y_k^i\| + \|y_k^i - T_{i+1}(y_k^i)\| + \|T_{i+1}(x_k) - T_{i+1}(y_k^i)\|
\leq 2\|x_k - y_k^i\| + \|y_k^i - T_{i+1}(y_k^i)\| \rightarrow 0, \quad i = 1, 2, \ldots, N - 1.\tag{3.11}
\]

Equations (3.9), (3.11) and Lemma 2.2 imply that \(\omega_w(x_k) \subset Fix(T_i), i = 1, 2, \ldots, N, i.e., \omega_w(x_k) \subset F\). This, together with (3.5) and Lemma 2.4 guarantee strong convergence of \(x_k\) to \(P_{Fix(T_i)}x_0\).
4. A numerical example

In this section, we perform Algorithm 3.1 and Algorithm AC for finding a common fixed point of two nonexpansive mappings and compare them through a numerical example.

We take $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ by $T_1 : v = (v_1, v_2)^\top \mapsto (\sin \frac{v_1 + v_2}{\sqrt{2}}, \cos \frac{v_1 + v_2}{\sqrt{2}})^\top$ (see [9]) and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ as $T_2 := P_C$ with $C = \{x \in \mathbb{R}^2 : \|x - c\| \leq r\}$ where $c \in [-1, 1]^2$ generated randomly, and $r = 3$. The terminal condition is $\|x - T(x)\| + \|x - S(x)\| \leq \epsilon$. In the numerical results listed in the following table, 'Iter.' and 'Sec.' denote the number of iterations and the cpu time in seconds, respectively.

For randomly chosen initial values, we compare Algorithm 3.1 and Algorithm AC with different terminal condition many times, and results in the Table 1 were the average values. From Table 1 we observe that Algorithm 3.1 is better than Algorithm AC in the sense of the average.

<table>
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<tr>
<th>$\epsilon$</th>
<th>Algorithm 3.1</th>
<th>Algorithm AC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iter.</td>
<td>Sec.</td>
</tr>
<tr>
<td>$\epsilon = 0.01$</td>
<td>86</td>
<td>0.0234</td>
</tr>
<tr>
<td>$\epsilon = 0.001$</td>
<td>265</td>
<td>0.05772</td>
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<tr>
<td>$\epsilon = 0.0001$</td>
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<td>0.11388</td>
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References


