\(\beta_1\)-paracompact spaces

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Abstract

We introduce the class of \(\beta_1\)-paracompact spaces in topological spaces and give characterizations of such spaces. We study subsets and subspaces of \(\beta_1\)-paracompact spaces and discuss their properties. Also, we investigate the invariants of \(\beta_1\)-paracompact spaces by functions. ©2016 All rights reserved.

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1. Introduction and preliminaries

Throughout this work a space will always mean a topological space with no separation axioms assumed, unless otherwise stated. If \((X, \tau)\) is a given space, then \(\text{Int}(A)\) and \(\text{Cl}(A)\) denotes the interior of \(A\) and the closure of \(A\), respectively in \((X, \tau)\). Let \((X, \tau)\) be a space and \(A\) a subset of \(X\). A subset \(A\) is said to be preopen [16] (resp., semi-open [13], \(\alpha\)-open [18], regular open [21]) if \(A \subset \text{Int} \left( \text{Cl} \left( \text{Int} \left( A \right) \right) \right)\) (resp., \(A \subset \text{Cl} \left( \text{Int} \left( A \right) \right)\), \(A \subset \text{Int} \left( \text{Cl} \left( \text{Int} \left( A \right) \right) \right)\), \(A = \text{Int} \left( \text{Cl} \left( \text{Int} \left( A \right) \right) \right)\)). The family of \(\alpha\)-sets of a space \((X, \tau)\), denoted by \(\tau^\alpha\), forms a topology on \(X\), finer than \(\tau\) [13]. For a space \((X, \tau)\), if \((X, \tau^\alpha)\) is normal, then \(\tau = \tau^\alpha\) [10].

In 1983, Abd El-Monsef et al. [1] introduced and studied the concept of \(\beta\)-open sets in topological spaces. They define a subset \(A\) of a space \((X, \tau)\) is said to be \(\beta\)-open if \(A \subset \text{Cl} \left( \text{Int} \left( \text{Cl} \left( A \right) \right) \right)\). The complement of a \(\beta\)-open set is said to be \(\beta\)-closed [1]. The collection of all \(\beta\)-open (resp., \(\beta\)-closed) subsets of \(X\) is denoted by \(\beta O(X, \tau)\) (resp., \(\beta C(X, \tau)\)). The union of all \(\beta\)-open sets of \(X\) contained in \(A\) is called \(\beta\)-interior of \(A\) and is denoted by \(\beta \text{Int}(A)\) and the intersection of all \(\beta\)-closed sets of \(X\) containing \(A\) is called the \(\beta\)-closure of \(A\) and is denoted by \(\beta \text{Cl}(A)\). A set \(A\) is called \(\beta\)-regular [20] if it is both \(\beta\)-open and \(\beta\)-closed. A space \((X, \tau)\) is said to be \(\beta\)-regular [2] if for each \(\beta\)-open set \(U\) and each \(x \in U\), there exists a \(\beta\)-open set \(V\) such that \(x \in V \subset \beta \text{Cl}(V) \subset U\). For any space, one has \(\beta O(X, \tau^\alpha) = \beta O(X, \tau)\) [4]. A collection \(\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}\) of
subsets of a space \((X, \tau)\) is said to be locally finite if for each \(x \in X\), there exists an open set \(U\) containing \(x\) and \(U\) intersects at most finitely many members of \(\mathcal{G}\).

A space \((X, \tau)\) is said to be paracompact if every open cover of \(X\) has a locally finite open refinement. \(\alpha\)-paracompact \(^{5}\) (resp., \(P_1\)-paracompact \(^{15}\), \(S_1\)-paracompact \(^{3}\)) spaces are defined by replacing the open cover in original definition by \(\alpha\)-open (resp., preopen, semiopen) cover. A subset \(A\) of space \(X\) is said to be \(N\)-closed relative to \(X\) (briefly, \(N\)-closed) \(^{9}\) if for every cover \(\{U_\alpha : \alpha \in \Delta\}\) of \(A\) by open sets of \(X\), there exists a finite subfamily \(\Delta_0\) of \(\Delta\) such that \(A \subset \bigcup \{\text{Int}(\text{Cl}(U_\alpha)) : \alpha \in \Delta_0\}\). In \(^{11}\), it was shown that every compact \(T_2\)-space is regular.

In this paper, we follow a similar line and introduce \(\beta_1\)-paracompact spaces by utilizing the \(\beta\)-open cover. We provide several characterizations of \(\beta_1\)-paracompact spaces and study subsets and subspaces of \(\beta_1\)-paracompact spaces and discuss their properties. Finally, we investigate the invariants of \(\beta_1\)-paracompact spaces by functions.

Now we recall some known definitions, lemmas, and theorems, which will be used in the work.

**Theorem 1.1** \(^{17}\). Let \((X, \tau)\) be a space, \(A \subset Y \subset X\) and \(Y \beta\)-open in \((X, \tau)\). Then \(A\) is \(\beta\)-open in \((X, \tau)\) if and only if \(A\) is \(\beta\)-open in the subspace \((Y, \tau_Y)\).

**Definition 1.2.** A space \((X, \tau)\) is said to be \(\alpha\)-paracompact \(^{5}\) (resp., \(P_1\)-paracompact \(^{15}\), \(S_1\)-paracompact \(^{3}\)), if every \(\alpha\)-open (resp., preopen, semiopen) cover of \(X\) has a locally finite open refinement.

**Lemma 1.3** \(^{15}\). The union of a finite family of locally finite collection of sets in a space is a locally finite family of sets.

**Theorem 1.4** \(^{7}\). If \(\{U_\alpha : \alpha \in \Delta\}\) is a locally finite family of subsets in a space \(X\) and if \(V_\alpha \subset U_\alpha\) for each \(\alpha \in \Delta\), then the family \(\{V_\alpha : \alpha \in \Delta\}\) is a locally finite in \(X\).

**Lemma 1.5** \(^{12}\). If \(f : (X, \tau) \to (Y, \sigma)\) is a continuous surjective function and \(\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}\) is locally finite in \(Y\), then \(f^{-1}(\mathcal{U}) = \{f^{-1}(U_\alpha) : \alpha \in \Delta\}\) is locally finite in \(X\).

**Lemma 1.6** \(^{19}\). Let \(f : (X, \tau) \to (Y, \sigma)\) be almost closed surjection with \(N\)-closed point inverse. If \(\{U_\alpha : \alpha \in \Delta\}\) is a locally finite open cover of \(X\), then \(\{f(U_\alpha) : \alpha \in \Delta\}\) is a locally finite cover of \(Y\).

2. \(\beta_1\)-paracompact spaces

In this section we introduce and study a new class of spaces, namely \(\beta_1\)-paracompact spaces, and we provide several characterizations of them.

**Definition 2.1.** A space \((X, \tau)\) is called \(\beta_1\)-paracompact if every \(\beta\)-open cover of \(X\) has a locally finite open refinement.

The following diagram shows the relations among the mentioned properties.

\[
\beta_1\text{-paracompact} \rightarrow P_1\text{-paracompact} \rightarrow \alpha\text{-paracompact} \rightarrow \text{paracompact} \\
\downarrow \quad \quad \quad \uparrow\\n\quad \quad \quad S_1\text{-paracompact}
\]

The converses need not be true as shown by the following examples.

**Example 2.2.** Let \(X = \mathbb{R}\) with the topology \(\tau = \{\phi, X, \{1\}\}\). Then \((X, \tau)\) is paracompact but it is not \(\beta_1\)-paracompact, since \(\{\{1, x\} : x \in X\}\) is a \(\beta\)-open cover of \(X\) which admits no locally finite open refinement.

**Example 2.3.** Let \(X = \{1, 2, 3\}\) with the topology \(\tau = \{\phi, X, \{1\}, \{2, 3\}\}\). Then \((X, \tau)\) is \(S_1\)-paracompact since \(SO(X, \tau) = \tau\), but it is not \(\beta_1\)-paracompact since \(\{\{1\}, \{2\}, \{3\}\}\) is a \(\beta\)-open cover of \(X\) which admits no locally finite open refinement.
Example 2.4. Let \( X = \{1, 2, 3\} \) with the topology \( \tau = \{\varnothing, X, \{1\}, \{2, 1\}\} \). Then \((X, \tau)\) is \(P\)-paracompact since \(PO(X, \tau) = \tau\) but it is not \(\beta\)-paracompact since \(\{\{1, 2\}, \{2, 3\}\}\) is a \(\beta\)-open cover of \(X\) which admits no locally finite open refinements.

Theorem 2.5. If \((X, \tau)\) is a \(\beta\)-paracompact \(T_1\)-space, then \(\tau = \beta O(X, \tau) = \tau^\beta\).

Proof. Let \(U\) be a \(\beta\)-open set in \((X, \tau)\). For each \(x \in U\), we have \(U = \{U\} \cup \{X - \{x\}\}\) is a \(\beta\)-open cover for \((X, \tau)\) and so it has a locally finite open refinement \(V = \{V_\alpha: \alpha \in \Delta\}\). Since \(V\) is a refinement of \(U\) and \(x \in U\), there exist an \(\alpha_0 \in \Delta\) such that \(x \in V_{\alpha_0} \subseteq U\) where \(V_{\alpha_0}\) is open and so \(U\) is open. Now, we know that \(\tau \subseteq \tau^\beta \subseteq \beta O(X, \tau)\) and we show \(\tau = \beta O(X, \tau)\), so \(\tau = \tau^\beta\).

The proof of the following corollary follows immediately from Definition 2.1 and Theorem 2.5.

Corollary 2.6. Let \((X, \tau)\) be a \(T_1\)-space. Then \((X, \tau)\) is \(\beta\)-paracompact if and only if \((X, \tau)\) is paracompact and \(\tau = \beta O(X, \tau)\).

Recall that, a space \((X, \tau)\) is said to be extremally disconnected (briefly e.d.) if the closure of every open set in \((X, \tau)\) is open.

Proposition 2.7. Let \((X, \tau)\) be a \(\beta\)-paracompact space. Then:

i. If \((X, \tau)\) is \(T_1\), then it is extremally disconnected.

ii. If \((X, \tau)\) is \(T_2\), then it is \(\beta\)-regular.

Proof. i) Let \(U\) be an open set in \((X, \tau)\); then \(Cl(U)\) is a \(\beta\)-open set and by Theorem 2.5 \(Cl(U)\) is an open set in \((X, \tau)\).

ii) Let \(U\) be a \(\beta\)-open set in \((X, \tau)\) and \(x \in U\). By Theorem 2.5 \(U\) is an open set. Since \((X, \tau)\) is regular, there exists an open set \(V\) such that \(x \in V \subseteq Cl(V) \subseteq U\). Thus \(x \in V \subseteq \beta Cl(V) \subseteq U\). It follows that \((X, \tau)\) is \(\beta\)-regular.

Theorem 2.8. Let \((X, \tau)\) be a space. Then:

i. If \((X, \tau^\alpha)\) is \(\beta\)-paracompact, then \((X, \tau)\) is paracompact.

ii. If \((X, \tau)\) is \(\beta\)-paracompact, then \((X, \tau^\alpha)\) is \(\beta\)-paracompact; the converse true if the space is \(T_2\).

Proof. i) Let \(U\) be an open cover of \((X, \tau)\). Then \(U\) is an open cover of the \(\beta\)-paracompact space \((X, \tau^\alpha)\) and so it has a locally finite open refinement \(V\) in \((X, \tau^\alpha)\). Now for every \(V \in V\), choose \(U_V \in U\) such that \(V \subseteq U_V\). One can easily show that the collection \(\{U_V \cap Int(Cl(\text{Int}(V))) : V \in V\}\) is a locally finite open refinement of \(U\) in \((X, \tau)\).

ii) Let \(U\) be a \(\beta\)-open cover of \((X, \tau^\alpha)\). Then \(U\) is a \(\beta\)-open cover of the \(\beta\)-paracompact space \((X, \tau)\) and so it has a locally finite open refinement \(V\) in \((X, \tau)\). Since \(\tau \subseteq \tau^\alpha\), then \(V\) is a locally finite open refinement of \(U\) in \((X, \tau^\alpha)\) and so \((X, \tau^\alpha)\) is \(\beta\)-paracompact. To prove the converse, let \((X, \tau^\alpha)\) be \(\beta\)-paracompact. Then \((X, \tau^\alpha)\) is a paracompact \(T_2\)-space and so it is normal [11]. Therefore, \(\tau = \tau^\alpha\).

The following examples show that the converse of (i) in the above theorem need not be true in general and the condition \(T_2\) on the space \((X, \tau)\) in (ii) is essential.

Example 2.9. Let \((X, \tau)\) be as in Example 2.4. Then \((X, \tau)\) is paracompact, but \((X, \tau^\alpha)\) is not \(\beta\)-paracompact.

Example 2.10. Let \(X = \{1, 2, 3\}\) and \(\tau = \{\varnothing, X, \{1\}\}\). Then \(\tau^\alpha = \{\varnothing, X, \{1\}, \{1, 2\}, \{1, 3\}\} = \beta O(X, \tau^\alpha)\). Therefore \((X, \tau^\alpha)\) is a \(\beta\)-paracompact space. On the other hand, \((X, \tau)\) is not \(\beta\)-paracompact since \(\{\{1, 2\}, \{1, 3\}\}\) is a \(\beta\)-open cover of \((X, \tau)\) which admits no locally finite open refinement.

Theorem 2.11. If each \(\beta\)-open cover of a space \((X, \tau)\) has an open \(\sigma\)-locally finite refinement, then each \(\beta\)-open cover of \(X\) has a locally finite refinement.
Proof. Let $U$ be a $\beta$-open cover of $X$. Let $V = \bigcup_{n \in \mathbb{N}} V_n$ be an open $\sigma$-locally finite refinement of $U$ where $V_n$ is locally finite. For each $n \in \mathbb{N}$ and each $V \in V_n$, let $V'_n = V - \bigcup_{k<n} V^*_k$ where $V^*_k = \bigcup\{V : V \in V_k\}$ and put $V'_n = \{V'_n : V \in V_n\}$. Now, put $W = \{V'_n : n \in \mathbb{N}, V \in V_n\} = \bigcup\{V'_n : n \in \mathbb{N}\}$. We show that $W$ is a locally finite refinement of $U$. Let $x \in X$ and let $n$ be the first positive integer such that $x \in V'_n$. Therefore $x \in V'$ for some $V' \in V'_n$, Thus $W$ is a cover of $X$. To show that $W$ is locally finite, let $x \in X$ and $n$ be the first positive integer such that $x \in V'_n$. Then $x \in V$ for some $V \in V_n$. Now, $V \cap V' = \emptyset$ for each $V' \in V'_k$ and for each $k > n$. Therefore, $V$ can intersect at most the elements of $V'_n$ for $n \leq n$. Since $V'_n$ is locally finite for each $k \leq n$, so we choose an open set $O_{x(k)}$ containing $x$ such that $O_{x(k)}$ meets at most finitely many members of $V'_k$. Finally, put $O_x = V \cap (\bigcap_{k=1}^{\infty} O_{x(k)})$. Then $O_x$ is an open set containing $x$ such that $O_x$ meets at most finitely many members of $W$. □

Theorem 2.12. Let $(X, \tau)$ be a $\beta$-regular space. If each $\beta$-open cover of the space $X$ has a locally finite refinement, then each $\beta$-open cover of $X$ has a locally finite $\beta$-closed refinement.

Proof. Let $U$ be a $\beta$-open cover of $X$. For each $x \in X$, pick a $U_x \in U$ such that $x \in U_x$. Since $(X, \tau)$ is $\beta$-regular, there exists a $\beta$-open set $V_x$ such that $x \in V_x \subset \beta Cl(V_x) \subset U_x$. The family $V = \{V_x : x \in X\}$ is a $\beta$-open cover of $X$ and so has a locally finite refinement $W = \{W_\alpha : \alpha \in \Delta\}$. The collection $\beta Cl(W) = \{\beta Cl(W_\alpha) : \alpha \in \Delta\}$ is locally finite for each $\alpha \in \Delta$; if $W_\alpha \subset V_\alpha$, then $\beta Cl(W_\alpha) \subset U^*$ for some $U^* \in U$. Thus $\beta Cl(W)$ is a $\beta$-closed locally finite refinement of $U$. □

Theorem 2.13. If $(X, \tau)$ is $\beta$-regular space, then the following are equivalent:

i. $(X, \tau)$ is $\beta_1$-paracompact.

ii. Each $\beta$-open cover of $X$ has a $\sigma$-locally finite open refinement.

iii. Each $\beta$-open cover of $X$ has a locally finite refinement.

iv. Each $\beta$-open cover of $X$ has a locally finite $\beta$-closed refinement, provided that the space $(X, \tau)$ is $e.d.$

Proof. The proof follows from Theorems 2.11 and 2.12. □

3. properties of $\beta_1$-paracompact spaces

In this section we study some basic properties of $\beta_1$-paracompact spaces related to their subsets, subspaces, sums, images and inverse images under some types of functions.

Definition 3.1. A subset $A$ of a space $(X, \tau)$ is called a $\beta_1$-paracompact set in $(X, \tau)$ if every cover of $A$ by $\beta$-open subset of $(X, \tau)$ has a locally finite open refinement in $(X, \tau)$, and $A$ is called $\beta_1$-paracompact if $A \cup \tau_A$ is a $\beta_1$-paracompact space.

Theorem 3.2. If $A$ and $B$ are $\beta_1$-paracompact relative to a space $(X, \tau)$, then $A \cup B$ is $\beta_1$-paracompact relative to $X$.

Proof. Let $U = \{U_\alpha : \alpha \in \Delta\}$ be a $\beta$-open cover of $A \cup B$. Then $U = \{U_\alpha : \alpha \in \Delta\}$ is a $\beta$-open cover of $A$ and $B$. So, there exist open locally finite families $V_A = \{V_\alpha' : \alpha' \in \Delta_1\}$ of $A$ and $V_B = \{V_\alpha'' : \alpha'' \in \Delta_1\}$ of $B$ which refines $U$ such that $A \subset \bigcup_{\alpha' \in \Delta_1} V_\alpha'$ and $B \subset \bigcup_{\alpha'' \in \Delta_1} V_\alpha''$. Now $A \cup B \subset \bigcup_{\alpha', \alpha'' \in \Delta} (V_\alpha' \cup V_\alpha'') = V$. By Lemma 1.3, $V$ is a locally finite open refinement of $U$. Therefore, $A \cup B$ is $\beta_1$-paracompact relative to $X$. □

Theorem 3.3. Let $A$ and $B$ be subsets of a space $(X, \tau)$. If $A$ is $\beta_1$-paracompact relative to $X$ and $B$ is $\beta$-closed in $X$, then $A \cap B$ is $\beta_1$-paracompact relative to $X$.

Proof. Let $U = \{U_\alpha : \alpha \in \Delta_3\}$ be a cover of $A \cap B$ such that $U_\alpha$ is $\beta$-open in $(X, \tau)$. Since $X - B$ is $\beta$-open in $X$, $U_\alpha = \{U_\alpha' : \alpha' \in \Delta_2\} \cup \{X - B\}$ is a $\beta$-open cover of $A$. So, there exists a locally finite open family $V_1 = \{V_\alpha' : \alpha' \in \Delta_1\} \cup V (V_\alpha' \subset U_\alpha$ and $V \subset X - B$) which refines $U_\alpha$ such that $A \subset \bigcup_{\alpha' \in \Delta_1} V_\alpha' \cup V$. Now $A \subset \bigcup_{\alpha' \in \Delta_1} V_\alpha'$ implies that $A \cap B \subset \bigcup_{\alpha' \in \Delta_1} (V_\alpha' \cup V) \cap B \subset \bigcup_{\alpha' \in \Delta_1} V_\alpha' \cup V$. Take $V = \{V_\alpha' : \alpha' \in \Delta_1\}$. Then $V$ is a locally finite open family which refines $U$. Hence $A \cap B$ is $\beta_1$-paracompact relative to $X$. □
Definition 3.4 ([8]). A subset $A$ of a space $(X,\tau)$ is called $\beta$g-closed if $\beta Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is any $\beta$-open set in $(X,\tau)$.

Theorem 3.5. Let $(X,\tau)$ be a $\beta_1$-paracompact space and $A \subseteq X$. Then:

i. If $A$ is regular open, then $(A,\tau_A)$ is $\beta_1$-paracompact.

ii. If $A$ is a $g\beta$-closed set, then $A$ is a $\beta_1$-paracompact set in $(X,\tau)$.

Proof. i) Let $V = \{V_\alpha : \alpha \in \Delta\}$ be a $\beta$-open cover of $A$ in $(A,\tau_A)$. Since $A$ is open in $(X,\tau)$, by Theorem 3.3, $V_\alpha$ is a $\beta$-open set in $(X,\tau)$ for each $\alpha \in \Delta$. Therefore, the collection $U = \{V_\alpha : \alpha \in \Delta\} \cup \{X - A\}$ is a $\beta$-open cover of the $\beta_1$-paracompact space $(X,\tau)$ and so it has a locally finite open refinement in $(X,\tau)$, say $W = \{W_{\alpha'} : \alpha' \in \Delta_1\}$. Now the collection $\{A \cap W_{\alpha'} : \alpha' \in \Delta_1\}$ is an open refinement of $V$ in $(A,\tau_A)$. Therefore, $(A,\tau_A)$ is $\beta_1$-paracompact.

ii. Let $U = \{U_\alpha : \alpha \in \Delta\}$ be a cover of $A$ by $\beta$-open subsets of $(X,\tau)$. Since $A \subseteq \cup\{U_\alpha : \alpha \in \Delta\}$ and $A$ is $\beta g$-closed, we have $\beta Cl(A) \subseteq \cup U_\alpha : \alpha \in \Delta\}$. For each $x \notin \omega \beta Cl(A)$, there exists a $\beta$-open set $W_x$ of $(X,\tau)$ such that $A \cap W_x = \phi$. Put $U' = \{U_\alpha : \alpha \in \Delta\} \cup \{W_x : x \notin \beta Cl(A)\}$. Then $U'$ is a $\beta$-open cover of the $\beta_1$-paracompact space $(X,\tau)$. Let $\mathcal{H} = \{H_{\alpha'} : \alpha' \in \Delta_1\}$ be a locally finite open refinement of $U'$ and put $\mathcal{H}_u = \{H_{\alpha'} : H_{\alpha'} \subseteq U_{\alpha(\alpha')}, \alpha' \in \Delta_1\}$ and $\alpha(\alpha') \in \Delta\}$. Then $\mathcal{H}_u$ is a locally finite open refinement of $U$. Therefore $A$ is a $\beta_1$-paracompact set.

Theorem 3.6. Let $A$ and $B$ be subsets of a space $(X,\tau)$ such that $A \subseteq B \subseteq X$:

i. If $A$ is $\beta_1$-paracompact relative to $X$ and $B$ is $\beta$-open in $(X,\tau)$, then $A$ is $\beta_1$-paracompact relative to $B$.

ii. If $A$ is $\beta_1$-paracompact relative to $B$ and $B$ is open in $(X,\tau)$, then $A$ is $\beta_1$-paracompact relative to $X$.

Proof. i) Let $U = \{U_\alpha : \alpha \in \Delta_\alpha\}$ be a cover of $A$ such that $U_\alpha$ is $\beta$-open in $(B,\tau_B)$. Since $B$ is $\beta$-open in $(X,\tau)$, by Theorem 1.1, $U$ is a $\beta$-open cover of $A$ in $(X,\tau)$. So, there exist a locally finite open family $V_{\alpha'} = \{V_{\alpha'} : \alpha' \in \Delta_1\}$ which refines $U$ such that $A \subseteq \cup \{V_{\alpha'} : \alpha' \in \Delta_1\}$. Then $A \cap B \subseteq \{V_{\alpha'} \cap B : \alpha' \in \Delta_1\}$. Let $x \in B$. Since $V = \{V_{\alpha'} : \alpha' \in \Delta_1\}$ is locally finite in $X$, there exists an open set $W$ in $(X,\tau)$ such that $W \cap V_{\alpha'} = \phi$ for each $\alpha' \neq \alpha_1', \alpha_2', \ldots, \alpha_n'$, which implies $(W \cap V_{\alpha'}) \cap B = \phi$ for $\alpha' \neq \alpha_1', \alpha_2', \ldots, \alpha_n'$. Therefore, the family $V_1 = \{V_{\alpha'} \cap B : \alpha' \in \Delta_1\}$ is a locally finite open refinement of $U$ in $(B,\tau_B)$. Therefore, $A$ is $\beta_1$-paracompact relative to $B$.

ii) Let $U = \{U_\alpha : \alpha \in \Delta\}$ be a cover of $A$ by $\beta$-open subsets of $(X,\tau)$. Then the collection $W = \{B \cap U_\alpha : \alpha \in \Delta\}$ is a $\beta$-open cover of $A$ in $(B,\tau_B)$. But $A$ is a $\beta_1$-paracompact set in $(B,\tau_B)$, so $W$ has a locally finite open refinement $V$ in $(B,\tau_B)$. Since $B$ is open in $(X,\tau)$, by Theorem 1.1, $V$ is a locally finite open refinement in $(X,\tau)$ and so $A$ is a $\beta_1$-paracompact set in $(X,\tau)$.

Corollary 3.7. Let $A$ and $B$ be subsets of a space $(X,\tau)$. If $A$ is $\beta_1$-paracompact relative to $(X,\tau)$ and $B$ is $\beta$-regular, then the following hold:

i. $A \cap B$ is $\beta_1$-paracompact relative to $B$.

ii. $B$ is $\beta_1$-paracompact relative to $X$, provided that $B \subseteq A$.

Proof. i) Let $A$ be $\beta_1$-paracompact relative to $X$ and $B$ a $\beta$-regular set in $(X,\tau)$. By Theorem 3.3, $A \cap B$ is $\beta_1$-paracompact relative to $X$. Since $A \cap B \subseteq B$ and $B$ is $\beta$-open in $(X,\tau)$, by Theorem 3.6, $A \cap B$ is $\beta_1$-paracompact relative to $B$.

ii) Since $B \subseteq A$ and $B$ is a $\beta$-regular set, by Theorem 3.3, $B$ is $\beta_1$-paracompact relative to $X$.

Theorem 3.8. Let $A$ be a clopen subspace of a space $(X,\tau)$. Then $A$ is a $\beta_1$-paracompact set if and only if it is $\beta_1$-paracompact.

Proof. To prove necessity, let $A$ be an open $\beta_1$-paracompact subset of $(X,\tau)$. Let $V = \{V_\alpha : \alpha \in \Delta\}$ be a cover of $A$ by $\beta$-open subsets of the subspace $(A,\tau_A)$. Since $A$ is open, $V$ is a cover of $A$ by $\beta$-open subsets of $(X,\tau)$ and so it has a locally finite open refinement, say $W$, in $(X,\tau)$. Then $W_A = \{W \cap A : W \in W\}$ is
a locally finite open refinement of \(V\) in \((A, \tau_A)\) and the result follows.

To prove sufficiency, let \(U = \{U_\alpha : \alpha \in \Delta\}\) be a cover of \(A\) by \(\beta\)-open subsets of \((X, \tau)\). Then \(U' = \{A \cap U_\alpha : \alpha \in \Delta\}\) is a \(\beta\)-open cover of the \(\beta_1\)-paracompact subspace \((A, \tau_A)\) and so it has a locally finite open refinement \(W\) in \((A, \tau_A)\). But \(A\) is an open set in \((X, \tau)\), so \(W\) is an open set for every \(W \in W\). Now \(\tau_A \subseteq \tau\) and \(X - A\) is an open set in \((X, \tau)\) which intersects no member of \(W\). Therefore \(W\) is locally finite in \((X, \tau)\). Thus \(A\) is a \(\beta_1\)-paracompact set. \(\square\)

**Corollary 3.9.** Every clopen subspace of a \(\beta_1\)-paracompact space is \(\beta_1\)-paracompact.

**Definition 3.10 (11).** Let \(\{X_\alpha, \tau_\alpha\} : \alpha \in \Delta\) be a collection of topological spaces such that \(X_\alpha \cap X_\beta = \emptyset\) for each \(\alpha \neq \beta\). Let \(X = \bigcup_{\alpha \in \Delta} X_\alpha\) be covered by \(\tau = \{G \subseteq X : G \cap X_\alpha \in \tau_\alpha, \alpha \in \Delta\}\). Then \((X, \tau)\) is called the sum of space \((X_\alpha, \tau_\alpha)\) for each \(\alpha \in \Delta\) and we write \(X = \bigoplus_{\alpha \in \Delta} X_\alpha\).

**Theorem 3.11.** The topological sum \(X = \bigoplus_{\alpha \in \Delta} X_\alpha\) is \(\beta_1\)-paracompact if and only if the space \((X_\alpha, \tau_\alpha)\) is \(\beta_1\)-paracompact, for each \(\alpha \in \Delta\).

**Proof.** Necessity follows from Corollary 3.9 since \((X_\alpha, \tau_\alpha)\) is a clopen subspace of the space \(\bigoplus_{\alpha \in \Delta} X_\alpha\), for each \(\alpha \in \Delta\).

To prove sufficiency, let \(U\) be a \(\beta\)-open cover of \(\bigoplus_{\alpha \in \Delta} X_\alpha\). For each \(\alpha \in \Delta\) the family \(U_\alpha = \{U \cap X_\alpha : U \in U\}\) is a \(\beta\)-open cover of the \(\beta\)-paracompact space \((X_\alpha, \tau_\alpha)\). Therefore \(U_\alpha\) has a locally finite open refinement \(V_\alpha\) in \((X_\alpha, \tau_\alpha)\). Put \(V = \bigcup_{\alpha \in \Delta} V_\alpha\). It is clear that \(V\) is a locally finite open refinement of \(U\). Thus \(\bigoplus_{\alpha \in \Delta} X_\alpha\) is \(\beta_1\)-paracompact. \(\square\)

Recall that a function \(f : (X, \tau) \to (Y, \sigma)\) is said to be \(\beta\)-continuous [11] (resp., \(\beta\)-irresolute [13]) if \(f^{-1}(V)\) is \(\beta\)-open in \((X, \tau)\) for each \(\beta\)-open set \(V\) in \((Y, \sigma)\).

**Theorem 3.12.** Let \(f : (X, \tau) \to (Y, \sigma)\) be an open, \(\beta\)-irresolute and almost closed surjective function with \(N\)-closed point inverse. If \((X, \tau)\) is \(\beta_1\)-paracompact, then \((Y, \sigma)\) is also \(\beta_1\)-paracompact.

**Proof.** Let \(U = \{U_\alpha : \alpha \in \Delta\}\) be a \(\beta\)-open cover of \((Y, \sigma)\). Since \(f\) is \(\beta\)-irresolute, \(U_1 = \{f^{-1}(U_\alpha) : \alpha \in \Delta\}\) is a \(\beta\)-open cover of \((X, \tau)\). So, there exist a locally finite open refinement, say \(W\). Since \(f\) is open and by Lemma 1.6 \(f(W)\) is a locally finite open refinement of \(V\) in \((Y, \sigma)\). \(\square\)

Since compact sets are \(N\)-closed and closed maps are almost closed, the following corollary follows from Theorem 3.12.

**Corollary 3.13.** Let \(f : (X, \tau) \to (Y, \sigma)\) be an open, \(\beta\)-continuous, closed surjective function with compact point inverse. If \((X, \tau)\) is \(\beta_1\)-paracompact, then \((Y, \sigma)\) is also \(\beta_1\)-paracompact.

A function \(f : (X, \tau) \to (Y, \sigma)\) is said to be strongly \(\beta\)-continuous if the inverse image of each \(\beta\)-open set in \((Y, \sigma)\) is an open set in \((X, \tau)\).

**Theorem 3.14.** Let \(f : (X, \tau) \to (Y, \sigma)\) be an open, strongly \(\beta\)-continuous, almost closed, surjective function with \(N\)-closed point inverse. If \((X, \tau)\) is \(\beta\)-paracompact, then \((Y, \sigma)\) is also \(\beta_1\)-paracompact.

**Proof.** Let \(U = \{U_\alpha : \alpha \in \Delta\}\) be a \(\beta\)-open cover of \(Y\). Since \(f\) is strongly \(\beta\)-continuous, \(U_1 = \{f^{-1}(U_\alpha) : \alpha \in \Delta\}\) is an open cover of \(X\). Hence, there exists a locally finite open refinement \(W\) of \(U_1\). Since \(f\) is open and by Lemma 1.6 \(f(W)\) is a locally finite open refinement of \(U\). Therefore, \((Y, \sigma)\) is \(\beta_1\)-paracompact. \(\square\)

Recall that a function \(f : (X, \tau) \to (Y, \sigma)\) is said to be \(\beta\)-open [11] (resp., \(\beta\)-closed [11]) if \(f(V)\) is a \(\beta\)-open (resp., \(\beta\)-closed) set in \((Y, \sigma)\) for each \(\beta\)-open (resp., \(\beta\)-closed) set \(V\) in \((X, \tau)\).

**Proposition 3.15 (11).** A function \(f : (X, \tau) \to (Y, \sigma)\) is \(\beta\)-closed if and only if for each \(x \in X\) and each \(\beta\)-open set \(U\) in \((X, \tau)\) containing \(x\), there exists a \(\beta\)-open set \(V\) in \((Y, \sigma)\) containing \(f(x)\) such that \(f(x) \in V\) and \(f^{-1}(V) \subseteq U\).
Theorem 3.16. Let \( f : (X, \tau) \to (Y, \sigma) \) be a continuous \( \beta \)-closed surjective function with compact point inverse. If \((Y, \sigma)\) is a \( \beta_1 \)-paracompact space, then \((X, \tau)\) is \( \beta_1 \)-paracompact.

Proof. Let \( \mathcal{U} = \{ U_\alpha : \alpha \in \Delta \}\) be a \( \beta \)-open cover of \( X \). For each \( y \in Y \) and for each \( x \in f^{-1}(y) \), choose an \( \alpha(x) \in \Delta \) such that \( x \in U_{\alpha(x)} \). Therefore the collection \( \{ U_{\alpha(x)} : x \in f^{-1}(y) \} \) is a \( \beta \)-open cover of \( f^{-1}(y) \) and so there exists a finite subset \( \Delta(y) \) of \( \Delta \) such that \( f^{-1}(y) \subseteq \bigcup_{\alpha \in \Delta(y)} U_{\alpha(x)} = U_y \).

But \( f \) is \( \beta \)-closed, so by Proposition 3.15 there exists a \( \beta \)-open set \( V_y \) in \((Y, \sigma)\) such that \( y \in V_y \) and \( f^{-1}(V_y) \subseteq U_y \). Thus \( \mathcal{V} = \{ V_y : y \in Y \} \) is a \( \beta \)-open cover of \( Y \) and so it has a locally finite open refinement, say, \( \mathcal{W} = \{ W_{\alpha'} : \alpha' \in \Delta_0 \} \). Since \( f \) is continuous, the family \( \{ f^{-1}(W_{\alpha'}) : \alpha' \in \Delta_0 \} \) is an open locally finite cover of \( X \) such that for every \( \alpha' \in \Delta_0 \), we have \( f^{-1}(W_{\alpha'}) \subseteq U_y \) for some \( y \in Y \). Now, the family \( \{ f^{-1}(W_{\alpha'}) \cap U_{\alpha(x)} : \alpha' \in \Delta_0, \alpha(x) \in \Delta(y) \} \) is an open locally finite refinement of \( \mathcal{U} \). Therefore \((X, \tau)\) is \( \beta_1 \)-paracompact.

Theorem 3.17. Let \( f : (X, \tau) \to (Y, \sigma) \) be a \( \beta \)-open, continuous, bijective function. If \( A \) is \( \beta_1 \)-paracompact relative to \( Y \), then \( f^{-1}(A) \) is \( \beta_1 \)-paracompact relative to \( X \).

Proof. Let \( \mathcal{U} = \{ U_\alpha : \alpha \in \Delta_0 \} \) be a \( \beta \)-open cover of \( A \) in \((X, \tau)\). Since \( f \) is \( \beta \)-open, \( \mathcal{U}_1 = \{ f(U_\alpha) : \alpha \in \Delta_0 \} \) is a \( \beta \)-open cover of \( A \) in \((Y, \sigma)\). So, there exists a locally finite open refinement of \( \mathcal{U}_1 \), say \( \mathcal{V}_1 \). Since \( f \) is continuous, by Lemma 1.5 \( \mathcal{V} = f^{-1}(\mathcal{V}_1) \) is an open locally finite refinement of \( \mathcal{U} \). Therefore, \( f^{-1}(A) \) is \( \beta_1 \)-paracompact relative to \( X \).

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References


