Some results on asymptotically quasi-$\phi$-nonexpansive mappings in the intermediate sense and Ky Fan inequalities

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Abstract

In this paper, we study asymptotically quasi-$\phi$-nonexpansive mappings in the intermediate sense and Ky Fan inequalities. A convergence theorem is established in a strictly convex and uniformly smooth Banach space. The results presented in the paper improve and extend some recent results. ©2016 All rights reserved.

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1. Introduction and Preliminaries

Let $E$ be a real Banach space and let $C$ be nonempty closed and convex subset of $E$. Let $B : C \times C \to \mathbb{R}$ be a function. Recall the following equilibrium problem in the terminology of Blum and Oettli [4].

Find $\bar{x} \in C$ such that $B(\bar{x}, y) \geq 0, \forall y \in C$.

In this paper, we use $\text{Sol}(B)$ to denote the solution set of the equilibrium problem. That is, $\text{Sol}(B) = \{x \in C : B(x, y) \geq 0, \forall y \in C\}$. The following restrictions on function $B$ are essential in this paper.

(A-1) $B(a, a) = 0, \forall a \in C$;

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(A-2) \( 0 \geq B(b,a) + B(a,b), \forall a, b \in C; \)

(A-3) \( b \mapsto B(a,b) \) is convex and weakly lower semi-continuous, \( \forall a \in C; \)

(A-4) \( B(a,b) \geq \lim \sup_{t \downarrow 0} B(tc + (1-t)a,b), \forall a, b, c \in C. \)

The equilibrium problem has been extensively studied based on iterative methods because of its applications in nonlinear analysis, optimization, economics, game theory, mechanics, medicine and so forth, see \[3, 7, 11, 13, 17, 18, 25, 27-31\] and the references therein.

Let \( E^* \) be the dual space of \( E \). Let \( B_E \) be the unit sphere of \( E \). Recall that \( E \) is said to be uniformly convex if for any \( a \in (0, 2] \) there exists \( b > 0 \) such that for any \( x, y \in B_E, \)

\[
\|y - x\| \geq a \quad \text{implies} \quad \|y + x\| \leq 2 - 2b.
\]

\( E \) is said to be a strictly convex space if and only if \( \|y + x\| < 2 \) for all \( x, y \in B_E \) and \( x \neq y \). It is known that a uniformly convex Banach space is reflexive and strictly convex.

Recall that \( E \) is said to have a Gâteaux differentiable norm if and only if \( \lim_{t \to 0} \frac{\|ty + (1-t)x\| - \|x\|}{t} \) exists for each \( x, y \in B_E \). In this case, we also say that \( E \) is smooth. \( E \) is said to have a uniformly Gâteaux differentiable norm if for each \( y \in B_E, \lim_{t \to 0} \frac{\|ty + (1-t)x\| - \|x\|}{t} \) is attained uniformly for all \( x \in B_E \). \( E \) is also said to have a uniformly Fréchet differentiable norm iff \( \lim_{t \to 0} \frac{\|ty + (1-t)x\| - \|x\|}{t} \) is attained uniformly for \( x, y \in B_E \). In this case, we say that \( E \) is uniformly smooth. It is known that a uniformly smooth Banach space is reflexive and smooth.

Recall that \( E \) is said to have the Kadec-Klee property if \( \lim_{m \to \infty} \|x_m - x\| = 0 \), for any sequence \( \{x_m\} \subset E \), and \( x, y \in E \) with \( \{x_n\} \) converges weakly to \( x \), and \( \{\|x_n\|\} \) converges strongly to \( \|x\| \). It is known that every uniformly convex Banach space has the Kadec-Klee property; see [13] and the references therein.

Recall that the normalized duality mapping \( J \) from \( E \) to \( 2^{E^*} \) is defined by

\[
Jx = \{x^* \in E^* : \|x\|^2 = \langle x, x^* \rangle = \|x^*\|^2\}.
\]

It is known if \( E \) is uniformly smooth, then \( J \) is uniformly norm-to-norm continuous on every bounded subset of \( E \); if \( E \) is a smooth Banach space, then \( J \) is single-valued and demicontinuous, i.e., continuous from the strong topology of \( E \) to the weak star topology of \( E \); if \( E \) is smooth, strictly convex and reflexive Banach space, then \( J \) is single-valued, one-to-one and onto.

Let \( T \) be a mapping on \( C \). \( T \) is said to be closed if for any sequence \( \{x_n\} \subset C \) such that \( \lim_{m \to \infty} x_m = x' \) and \( \lim_{m \to \infty} Tx_m = y' \), then \( Tx' = y' \). Let \( W \) be a bounded subset of \( C \). Recall that \( T \) is said to be uniformly asymptotically regular on \( C \) if and only if \( \lim_{n \to \infty} \sup_{x \in W} \|T^n x - T^{n+1} x\| = 0 \). From now on, we use \( \to \) and \( \rightarrow \) to stand for the weak convergence and strong convergence, respectively and use \( Fix(T) \) to denote the fixed point set of mapping \( T \).

Next, we assume that \( E \) is a smooth Banach space which means mapping \( J \) is single-valued. Study the functional

\[
\phi(x, y) := \|x\|^2 + \|y\|^2 - 2 \langle x, Jy \rangle, \quad \forall x, y \in E.
\]

Let \( C \) be a closed convex subset of a real Hilbert space \( H \). For any \( x \in H \), there exists a unique nearest point in \( C \), denoted by \( P_C x \), such that \( \|x - P_C x\| \leq \|x - y\| \), for all \( y \in C \). The operator \( P_C \) is called the metric projection from \( H \) onto \( C \). It is known that \( P_C \) is firmly nonexpansive, that is, \( \|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle \). In [2], Alber studied a new mapping \( Proj_C \) in a Banach space \( E \) which is an analogue of \( P_C \), the metric projection, in Hilbert spaces. Recall that the generalized projection \( Proj_C : E \to C \) is a mapping that assigns to an arbitrary point \( x \in E \) the minimum point of \( \phi(x, y) \).

Recall that \( T \) is said to be asymptotically quasi-\( \phi \)-nonexpansive in the intermediate sense iff \( Fix(T) \neq \emptyset \) and

\[
\limsup_{n \to \infty} \sup_{p \in Fix(T), x \in C} (\phi(p, T^n x) - \phi(p, x)) \leq 0.
\]

Putting \( \xi_n = \max\{0, \sup_{p \in Fix(T), x \in C} (\phi(p, T^n x) - \phi(p, x))\} \), we see \( \xi_n \to 0 \) as \( n \to \infty \). Hence, we have

\[
\phi(p, T^n x) \leq \phi(p, x) + \xi_n, \quad \forall x \in C, \forall p \in Fix(T).
\]
$T$ is said to be asymptotically quasi-$\phi$-nonexpansive iff $\text{Fix}(T) \neq \emptyset$ and
$$\phi(p, T^n x) \leq (1 + u_n)\phi(p, x), \quad \forall x \in C, \forall p \in \text{Fix}(T), \forall n \geq 1,$$
where $\{u_n\}$ is a sequence $\{u_n\} \subset [0, \infty)$ with $u_n \to 0$ as $n \to \infty$.

$T$ is said to be quasi-$\phi$-nonexpansive iff $\text{Fix}(T) \neq \emptyset$ and
$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \forall p \in \text{Fix}(T).$$

Recall that $p$ is said to be an asymptotic fixed point of $T$ if and only if $C$ contains a sequence $\{x_n\}$, where $x_n \to p$ such that $x_n - Tx_n \to 0$. Here, we use $\widetilde{\text{Fix}}(T)$ to denote the asymptotic fixed point set of $T$.

$T$ is said to be asymptotically relatively quasi-$\phi$-nonexpansive iff $\text{Fix}(T) = \widetilde{\text{Fix}}(T) \neq \emptyset$ and
$$\phi(p, T^n x) \leq (1 + u_n)\phi(p, x), \quad \forall x \in C, \forall p \in \text{Fix}(T) = \widetilde{\text{Fix}}(T), \forall n \geq 1,$$
where $\{u_n\}$ is a sequence $\{u_n\} \subset [0, \infty)$ with $u_n \to 0$ as $n \to \infty$.

$T$ is said to be relatively nonexpansive iff $\text{Fix}(T) = \widetilde{\text{Fix}}(T) \neq \emptyset$ and
$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \forall p \in \text{Fix}(T) = \widetilde{\text{Fix}}(T).$$

Remark 1.1. The class of asymptotically quasi-$\phi$-nonexpansive mappings in the intermediate sense [24] is reduced to the class of asymptotically quasi-nonexpansive mappings in the intermediate sense, which was considered in [15] as a non-Lipschitz continuous mappings, in the framework of Hilbert spaces.

Remark 1.2. The class of quasi-$\phi$-nonexpansive mappings [21] is a generalization of relatively nonexpansive mappings [6]. The class of quasi-$\phi$-nonexpansive mappings do not require the strong restriction that the fixed point set equals the asymptotic fixed point set.

Remark 1.3. The class of asymptotically quasi-$\phi$-nonexpansive mappings [22] is more desirable than the class of asymptotically relatively nonexpansive [1] mappings. Asymptotically quasi-$\phi$-nonexpansive mappings are reduced to asymptotically quasi-nonexpansive mappings in the framework of Hilbert spaces.

In this paper, we study the equilibrium problem in the terminology of Blum and Oettli [4] and a finite family of asymptotically quasi-$\phi$-nonexpansive mappings in the intermediate sense. With the aid of generalization projections, we establish a strong theorem in a strictly convex and uniformly smooth Banach space. The results obtained in this paper mainly improve the corresponding results in [15, 16, 19, 20, 23, 30]. In order to prove our main results, we also need the following lemmas.

Lemma 1.4 ([2]). Let $E$ be a strictly convex, reflexive, and smooth Banach space and let $C$ be a nonempty, closed, and convex subset of $E$. Let $x \in E$. Then
$$\phi(y, x) - \phi(\Pi_C x, x) \geq \phi(y, \Pi_C x), \quad \forall y \in C;$$
$$0 \geq \langle y - x_0, Jx - Jx_0 \rangle, \forall y \in C \text{ if and only if } x_0 = \Pi_C x.$$

Lemma 1.5 ([26]). Let $r$ be a positive real number and let $E$ be uniformly convex. Then there exists a convex, strictly increasing and continuous function $\text{cog} : [0, 2r] \to \mathbb{R}$ such that $\text{cog}(0) = 0$ and
$$t\|a\|^2 + (1 - t)\|b\|^2 \geq \|(1 - t)b + ta\|^2 + t(1 - t)\text{cog}(\|b - a\|)$$
for all $a, b \in B_r := \{a \in E : \|a\| \leq r\}$ and $t \in [0, 1]$.

Lemma 1.6 ([4, 21]). Let $E$ be a strictly convex, smooth, and reflexive Banach space and let $C$ be a closed convex subset of $E$. Let $B$ be a function with restrictions (A-1), (A-2), (A-3) and (A-4), from $C \times C$
to ℝ. Let \( x \in E \) and let \( r > 0 \). Then there exists \( z \in C \) such that \( \langle z - y, Jz - Jx \rangle + rB(z, y) \leq 0, \forall y \in C \). Define a mapping \( K^{B,r} \) by

\[
K^{B,r}x = \{ z \in C : 0 \leq \langle y - z, Jz - Jx \rangle + rB(z, y), \quad \forall y \in C \}.
\]

The following conclusions hold:

1. \( K^{B,r} \) is single-valued quasi-\( \phi \)-nonexpansive;
2. \( \text{Sol}(B) = \text{Fix}(K^{B,r}) \) is convex and closed.

Lemma 1.7 \((24)\). Let \( E \) be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let \( C \) be a convex and closed subset of \( E \) and let \( T \) be an asymptotically quasi-\( \phi \)-nonexpansive mapping in the intermediate sense on \( C \). \( \text{Fix}(T) \) is convex.

2. Main results

**Theorem 2.1.** Let \( E \) be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let \( C \) be a convex and closed subset of \( E \) and let \( B \) be a function with restrictions \((A-1), (A-2), (A-3)\) and \((A-4)\). Let \( \{T_m\}_{m=1}^N \), where \( N \) is some positive integer, be a sequence of asymptotically quasi-\( \phi \)-nonexpansive mappings in the intermediate sense on \( C \). Assume that every \( T_m \) is uniformly asymptotically regular and closed and \( \text{Sol}(B) \cap \bigcap_{m=1}^N \text{Fix}(T_m) \) is nonempty. Let \( \{\alpha_{n,m}\}, \{\alpha_{n,1}\}, \cdots, \{\alpha_{n,N}\} \) be real sequences in \((0,1)\) such that \( \sum_{m=0}^N \alpha_{0,m} = 1 \) and \( \lim \inf_{n \to \infty} \alpha_{0,0} \alpha_{0,m} > 0 \) for any \( 1 \leq m \leq N \). Let \( \{x_n\} \) be a sequence generated by

\[
\begin{align*}
x_0 &\in E \text{ chosen arbitrarily,} \\
x_1 &= \text{proj}_{C_1}x_0, \\
r_nB(u_n, u) &\geq \langle u_n - u, Ju_n - Jx_n \rangle, \forall u \in C_n, \\
Jy_n &= \left( \sum_{m=1}^N \alpha_{n,m} J_{T_m}x_n + \alpha_{n,0} Ju_n \right), \\
C_{n+1} &= \{ z \in C_n : \phi(z, y_n) \leq (1 - \alpha_{n,0}) \xi_n + \phi(z, x_n) \}, \\
x_{n+1} &= \text{proj}_{C_{n+1}}x_n,
\end{align*}
\]

where \( \xi_n = \max \{ \max \{ \sup_{p \in \text{Fix}(T_m)} \phi(p, T_m x) - \phi(p, x), 0 \} : 1 \leq m \leq N \} \), and \( \{r_n\} \) is a real sequence such that \( \lim \inf_{n \to \infty} r_n > 0 \). Then \( \{x_n\} \) converges strongly to \( \text{Proj}_{\text{Sol}(B) \cap \bigcap_{m=1}^N \text{Fix}(T_m)}}x_1 \).

**Proof.** The proof is split into seven steps.

1. **Step 1.** Prove that \( \text{Sol}(B) \cap \bigcap_{m=1}^N \text{Fix}(T_m) \) is convex and closed.

Using Lemmas 1.6 and 1.7, we find that \( \text{Fix}(T_m) \) is convex and \( \text{Sol}(B) \) is convex and closed. Since \( T_m \) is closed, we find that \( \text{Fix}(T_m) \) is also closed. So, \( \text{Proj}_{\text{Sol}(B) \cap \bigcap_{m=1}^N \text{Fix}(T_m)}}x_1 \) is well defined, for any element \( x \) in \( E \).

2. **Step 2.** Prove that \( C_n \) is convex and closed.

It is obvious that \( C_1 = C \) is convex and closed. Assume that \( C_i \) is convex and closed for some \( i \geq 1 \). Let \( p_1, p_2 \in C_{i+1} \). It follows that \( p = sp_1 + (1 - s)p_2 \in C_i \), where \( s \in (0, 1) \). Since

\[
(1 - \alpha_{(i,0)}) \xi_i + \phi(p_1, x_i) \geq \phi(p_1, y_i),
\]

and

\[
(1 - \alpha_{(i,0)}) \xi_i + \phi(p_2, x_i) \geq \phi(p_2, y_i),
\]

the conclusion follows.
one has
\[(1 - \alpha_{(i,0)})\xi_i \geq 2 \langle p_1, Jx_i - Jy_i \rangle - \|x_i\|^2 + \|y_i\|^2,\]

and
\[(1 - \alpha_{(i,0)})\xi_i \geq 2 \langle p_2, Jx_i - Jy_i \rangle - \|x_i\|^2 + \|y_i\|^2.\]

Using the above two inequalities, one has
\[
\phi(p, y_i) - \phi(p, x_i) \leq (1 - \alpha_{(i,0)})\xi_i.
\]

This shows that $C_{i+1}$ is closed and convex. Hence, $C_n$ is a convex and closed set.

**Step 3.** Prove $\cap_{m=1}^N \text{Fix}(T_m) \cap \text{Sol}(B) \subset C_n$.

It is obvious
\[
\cap_{m=1}^N \text{Fix}(T_m) \cap \text{Sol}(B) \subset C_1 = C.
\]

Suppose that $\cap_{m=1}^N \text{Fix}(T_m) \cap \text{Sol}(B) \subset C_i$ for some positive integer $i$. For any $z \in \cap_{m=1}^N \text{Fix}(T_m) \cap \text{Sol}(B) \subset C_i$, we see that
\[
\phi(z, x_i) + (1 - \alpha_{(i,0)})\xi_i \\
\geq \sum_{m=1}^N \alpha_{(i,m)}\phi(z, T_m^i x_i) + \alpha_{(i,0)}\phi(z, u_i) \\
\geq \|z\|^2 + \sum_{m=1}^N \alpha_{(i,m)}\|T_m^i x_i\|^2 + \alpha_{(i,0)}\|J u_i\|^2 \\
- 2\alpha_{(i,0)}\langle z, J u_i \rangle - 2 \sum_{m=1}^N \alpha_{(i,m)}\langle z, J T_m^i x_i \rangle \\
\geq \|z\|^2 + \sum_{m=1}^N \alpha_{(i,m)}\|J T_m^i x_i\|^2 + \alpha_{(i,0)}\|J u_i\|^2 \\
- 2\langle z, \sum_{m=1}^N \alpha_{(i,m)}J T_m^i x_i + \alpha_{(i,0)}J u_i \rangle \\
= \phi(z, y_i),
\]

where
\[
\xi_i = \max \{ \max \{ \sup_{p \in \text{Fix}(T_m), x \in C} (\phi(p, T_m^i x) - \phi(p, x)), 0 \} : 1 \leq m \leq N \}.
\]

This shows that $z \in C_{i+1}$. This implies that $\cap_{m=1}^N \text{Fix}(T_m) \cap \text{Sol}(B) \subset C_n$.

**Step 4.** Prove that $\{x_n\}$ is bounded.

Now, we have $\langle x_n - z, Jx_n - Jx \rangle \geq 0$, for any $z \in C_n$. It follows that
\[
0 \leq \langle x_n - z, Jx_n - Jx \rangle, \quad \forall z \in \cap_{m=1}^N \text{Fix}(T_m) \cap \text{Sol}(B) \subset C_n.
\]

On the other hand, we find from Lemma 1.4
\[
\phi(\text{Proj}_{\cap_{m=1}^N \text{Fix}(T_m) \cap \text{Sol}(B)} x_1, x_1) \\
\geq \phi(\text{Proj}_{\cap_{m=1}^N \text{Fix}(T_m) \cap \text{Sol}(B)} x_1, x_1) - \phi(\text{Proj}_{\cap_{m=1}^N \text{Fix}(T_m) \cap \text{Sol}(B)} x_1, x_n) \\
\geq \phi(x_n, x_1),
\]

which shows that $\{\phi(x_n, x_1)\}$ is bounded. Hence, $\{x_n\}$ is also bounded. Without loss of generality, we assume $x_n \rightarrow \hat{x}$. Since every $C_n$ is convex and closed. So $\hat{x} \in C_n$.

**Step 5.** Prove $\hat{x} \in \cap_{m=1}^N \text{Fix}(T_m)$. 


Since $\bar{x} \in C_n$, one has $\phi(x_n, x_1) \leq \phi(\bar{x}, x_1)$. This implies that
\[
\phi(\bar{x}, x_1) \leq \liminf_{n \to \infty} (\|x_n\|^2 + \|x_1\|^2 - 2\langle x_n, Jx_1 \rangle) = \limsup_{n \to \infty} \phi(x_n, x_1) \leq \phi(\bar{x}, x_1).
\]
Hence, one has
\[
\lim_{n \to \infty} \phi(x_n, x_1) = \phi(\bar{x}, x_1).
\]
It follows that
\[
\lim_{n \to \infty} \|x_n\| = \|\bar{x}\|.
\]
Using the Kadec-Klee property, one obtains that $\{x_n\}$ converges strongly to $\bar{x}$ as $n \to \infty$. Since $x_{n+1} \in C_{n+1} \subset C_n$, we find that
\[
\phi(x_{n+1}, x_1) \geq \phi(x_n, x_1),
\]
which shows that $\{\phi(x_n, x_1)\}$ is nondecreasing. It follows that $\lim_{n \to \infty} \phi(x_n, x_1)$ exists. Since
\[
\phi(x_{n+1}, x_1) - \phi(x_n, x_1) \geq \phi(x_{n+1}, x_n) \geq 0,
\]
one has $\lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0$. Using the fact $x_{n+1} \in C_{n+1}$, one sees
\[
\phi(x_{n+1}, y_n) - \phi(x_{n+1}, x_n) \leq (1 - \alpha(n, 0))\xi_n.
\]
Since
\[
\lim_{n \to \infty} \phi(x_{n+1}, x_n) = \lim_{n \to \infty} \xi_n = 0,
\]
one has
\[
\lim_{n \to \infty} \phi(x_{n+1}, y_n) = 0.
\]
Therefore, one has
\[
\lim_{n \to \infty} (\|y_n\| - \|x_{n+1}\|) = 0.
\]
This implies that
\[
\lim_{n \to \infty} \|Jy_n\| = \lim_{n \to \infty} \|y_n\| = \|\bar{x}\| = \|J\bar{x}\|.
\]
This implies that $\{Jy_n\}$ is bounded. Without loss of generality, we assume that $\{Jy_n\}$ converges weakly to $y^* \in E^*$. In view of the reflexivity of $E$, we see that $J(E) = E^*$. This shows that there exists an element $y \in E$ such that $Jy = y^*$. It follows that
\[
\phi(x_{n+1}, y_n) + 2\langle x_{n+1}, Jy_n \rangle = \|x_{n+1}\|^2 + \|Jy_n\|^2.
\]
Taking $\liminf_{n \to \infty}$, one has $0 \geq \|\bar{x}\|^2 - 2\langle \bar{x}, y^* \rangle + \|y^*\|^2 = \|\bar{x}\|^2 + \|Jy\|^2 - 2\langle \bar{x}, Jy \rangle = \phi(\bar{x}, y) \geq 0$. That is, $\bar{x} = y$, which in turn implies that $J\bar{x} = y^*$. Hence, $Jy_n \rightharpoonup \bar{x} \in E^*$. Since $E$ is uniformly smooth. Hence, $E^*$ is uniformly convex and it has the Kadec-Klee property, we obtain
\[
\lim_{n \to \infty} Jy_n = J\bar{x}.
\]
Since $J^{-1} : E^* \to E$ is demi-continuous and $E$ has the Kadec-Klee property, one gets that $y_n \to \bar{x}$, as $n \to \infty$. Using the fact
\[
(\|x_n\| + \|y_n\|)\|y_n - x_n\| + 2\langle z, Jy_n - Jx_n \rangle \geq \phi(z, x_n) - \phi(z, y_n),
\]
we find
\[
\lim_{n \to \infty} (\phi(z, x_n) - \phi(z, y_n)) = 0.
\]
It follows from Lemma 1.5 that
Hence, one has
\[
\phi(z, x_n) + (1 - \alpha(n,0))\xi_n - \alpha(n,0)\alpha(n,m)g(\|JT^m_n x_n - Ju_n\|)
\]
\[
\geq \sum_{m=1}^N \alpha(n,m)\phi(z, T^m_n x_n) + \alpha(n,0)\phi(z, u_n) - \alpha(n,0)\alpha(n,m)g(\|JT^m_n x_n - Ju_n\|)
\]
\[
\geq \sum_{m=0}^N \alpha(n,m)\|z\|^2 + \sum_{m=1}^N \alpha(n,m)\|T^m_n x_n\|^2 + \alpha(n,0)\|Ju_n\|^2
\]
\[
- 2\alpha(n,0)\langle z, Ju_n \rangle - 2\sum_{m=1}^N \alpha(n,m)\langle z, JT^m_n x_n \rangle
\]
\[
- \alpha(n,0)\alpha(n,m)g(\|JT^m_n x_n - Ju_n\|)
\]
\[
geq \phi(z, y_n).
\]
This implies
\[
0 \leq \alpha(n,0)\alpha(n,m)g(\|JT^m_n x_n - Ju_n\|) \leq (\phi(z, x_n) - \phi(z, y_n)) + (1 - \alpha(n,0))\xi_n.
\]
Since \(\lim\inf_{n \to \infty} \alpha(n,0)\alpha(n,m) > 0\), one sees from 2.1
\[
\lim_{n \to \infty} \|Ju_n - JT^m_n x_n\| = 0
\]
for any \(1 \leq m \leq N\). Using the fact
\[
\sum_{m=1}^N \alpha(n,m)(JT^m_n x_n - Ju_n) = Jy_n - Ju_n,
\]
one has \(\{Ju_n\}\) converges strongly to \(J\bar{x}\). It follows that \(JT^m_n x_n \to J\bar{x}\) as \(n \to \infty\). Since \(J^{-1} : E^* \to E\) is demi-continuous, one has \(T^m_n x_n \to \bar{x}\). Using the fact
\[
\|T^m_n x_n\| - \|\bar{x}\| = \|JT^m_n x_n\| - \|J\bar{x}\| \leq \|JT^m_n x_n - J\bar{x}\|
\]
one has \(\|T^m_n x_n\| \to \|\bar{x}\|\) as \(n \to \infty\). Since \(E\) has the Kadec-Klee property, one has
\[
\lim_{n \to \infty} \|\bar{x} - T^m_n x_n\| = 0.
\]
Since \(T_m\) is also uniformly asymptotically regular, one has
\[
\lim_{n \to \infty} \|\bar{x} - T^{n+1}_m x_n\| = 0.
\]
That is, \(T_m(T^m_n x_n) \to \bar{x}\). Using the closedness of \(T_m\), we find \(T_m \bar{x} = \bar{x}\). This proves \(\bar{x} \in Fix(T_m)\), that is, \(\bar{x} \in \bigcap_{m=1}^N Fix(T_m)\).

Step 6. Prove \(\bar{x} \in Sol(B)\).

Since \(B\) is a monotone bifunction, one has
\[
r_nB(u, u_n) \leq \|u - u_n\||Ju_n - J\bar{x}_n||.
\]
Since \(\lim\inf_{n \to \infty} r_n > 0\), we may assume there exists \(\lambda > 0\) such that \(r_n \geq \lambda\). It follows that
\[
B(u, u_n) \leq \frac{\|u - u_n\||Ju_n - J\bar{x}_n||}{\lambda}.
\]
Hence, one has \(B(u, \bar{x}) \leq 0\). For \(0 < s < 1\), define \(u_s = (1 - s)\bar{x} + su\). This implies that \(0 \geq B(u_s, \bar{x})\). Hence, we have
\[
sB(u_s, u) \geq B(u_s, u_s) = 0.
\]
It follows that $B(\bar{x},u) \geq 0, \forall u \in C$. This implies that $\bar{x} \in \text{Sol}(B)$.

Step 7. Prove $\bar{x} = \text{Proj}_{\bigcap_{m=1}^{N} \text{Fix}(T) \cap \text{Sol}(B)} x_1$.

Using Lemma 1.5, we find

$$0 \leq \langle x_n - z, Jx_1 - Jx \rangle, \forall z \in \bigcap_{m=1}^{N} \text{Fix}(T_m) \cap \text{Sol}(B).$$

Let $n \to \infty$, one has

$$0 \leq \langle \bar{x} - z, Jx_1 - J\bar{x} \rangle.$$

It follows that $\bar{x} = \text{Proj}_{\bigcap_{m=1}^{N} \text{Fix}(T) \cap \text{Sol}(B)} x_1$. This completes the proof. \hfill \Box

If $N = 1$, we have the following result.

**Corollary 2.2.** Let $E$ be a strictly convex and uniformly smooth Banach space which also has the KKP. Let $C$ be a convex and closed subset of $E$ and let $B$ be a bifunction with (A-1), (A-2), (A-3) and (A-4). Let $T$ be an asymptotically quasi-$\phi$-nonexpansive mapping in the intermediate sense on $C$. Assume that $T$ is uniformly asymptotically regular and closed and $\text{Sol}(B) \bigcap \text{Fix}(T)$ is nonempty. Let $\{\alpha(n,0)\}$ be a real sequence in $(0,1)$ such that $\liminf_{n \to \infty} \alpha(n,0)(1 - \alpha(n,0)) > 0$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases}
    x_0 \in E \text{ chosen arbitrarily}, \\
    C_1 = C, x_1 = \text{Proj}_{C_1} x_0, \\
    r_n B(u_n, u) \geq \langle u_n - u, J u_n - J x_n \rangle, \forall u \in C_n, \\
    y_n = J^{-1}(1 - \alpha(n,0)) J T^n x_n + \alpha(n,0) Ju_n, \\
    C_{n+1} = \{z \in C_n : \phi(z,y_n) \leq (1 - \alpha(n,0)) \xi_n + \phi(z,x_n)\}, \\
    x_{n+1} = \text{Proj}_{C_{n+1}} x_1,
\end{cases}$$

where $\xi_n = \max\{\sup_{p \in \text{Fix}(T), x \in C} (\phi(p,T^n x) - \phi(p,x)), 0\}$, and $\{r_n\}$ is a real sequence such that $\liminf_{n \to \infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\text{Proj}_{\text{Sol}(B) \cap \text{Fix}(T)} x_1$.

If $T$ is the identity mapping, we have the following results on the equilibrium problem.

**Corollary 2.3.** Let $E$ be a strictly convex and uniformly smooth Banach space which also has the KKP. Let $C$ be a convex and closed subset of $E$ and let $B$ be a bifunction with (A-1), (A-2), (A-3) and (A-4). Let $N \geq 1$ be some positive integer and assume $\text{Sol}(B) \neq \emptyset$. Let $\{\alpha(n,0)\}$, $\{\alpha(n,1)\}, \cdots, \{\alpha(n,N)\}$ be real sequences in $(0,1)$ such that $\sum_{m=0}^{N} \alpha(n,m) = 1$ and $\liminf_{n \to \infty} \alpha(n,0)\alpha(n,m) > 0$ for any $1 \leq m \leq N$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases}
    x_0 \in E \text{ chosen arbitrarily}, \\
    C_1 = C, x_1 = \text{Proj}_{C_1} x_0, \\
    r_n B(u_n, u) \geq \langle u_n - u, J u_n - J x_n \rangle, \forall u \in C_n, \\
    y_n = J^{-1}\left(\sum_{m=1}^{N} \alpha(n,m) J x_n + \alpha(n,0) Ju_n\right), \\
    C_{n+1} = \{z \in C_n : \phi(z,y_n) \leq \phi(z,x_n)\}, \\
    x_{n+1} = \text{Proj}_{C_{n+1}} x_1,
\end{cases}$$

where $\{r_n\}$ is a real sequence such that $\liminf_{n \to \infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\text{Proj}_{\text{Sol}(B)} x_1$.

In the framework of Hilbert spaces, $\sqrt{\phi(x,y)} = ||x - y||$, $\forall x, y \in E$. The generalized projection is reduced to the metric projection and the class of asymptotically-$\phi$-nonexpansive mappings in the intermediate sense is reduced to the class of asymptotically quasi-nonexpansive mappings in the intermediate sense.
Corollary 2.4. Let $E$ be a Hilbert space. Let $C$ be a convex and closed subset of $E$ and let $B$ be a function with (A-1), (A-2), (A-3) and (A-4). Let $\{T_m\}_{m=1}^N$, where $N$ is some positive integer, be a sequence of asymptotically quasi-nonexpansive mappings in the intermediate sense on $C$. Assume that every $T_m$ is uniformly asymptotically regular and closed and $\text{Sol}(B) \cap \bigcap_{m=1}^N \text{Fix}(T_m)$ is nonempty. Let $\{\alpha_{(n,0)}\}, \{\alpha_{(n,1)}\}, \cdots, \{\alpha_{(n,N)}\}$ be real sequences in $(0,1)$ such that $\sum_{m=0}^N \alpha_{(n,m)} = 1$ and

$$\liminf_{n \to \infty} \alpha_{(n,0)} \alpha_{(n,m)} > 0$$

for any $1 \leq m \leq N$. Let $\{x_n\}$ be a sequence generated by

$$
\begin{align*}
    x_0 & \in E \text{ chosen arbitrarily,} \\
    C_1 & = C, \quad x_1 = P_{C_1} x_0, \\
    r_n & B(u_n, u) \geq \langle u_n - u, u_n - x_n \rangle, \forall u \in C_n, \\
    y_n & = \sum_{m=1}^N \alpha_{(n,m)} T_m x_n + \alpha_{(n,0)} u_n, \\
    C_{n+1} & = \{ z \in C_n : \|z - y_n\|^2 \leq (1 - \alpha_{(n,0)}) \xi_n + \|z - x_n\|^2 \}, \\
    x_{n+1} & = \text{Proj}_{C_{n+1}} x_1,
\end{align*}
$$

where $\xi_n = \max \left\{ \max \{ \sup_{p \in \text{Fix}(T_m) \cap CC \} \left( \|p - T_m x\|^2 - \|p - x\|^2 \right), 0 \} : 1 \leq m \leq N \right\}$, and $\{r_n\}$ is a real sequence such that $\liminf_{n \to \infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $P_{\text{Sol}(B) \cap \bigcap_{m=1}^N \text{Fix}(T_m)} x_1$.

References


