Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 1463–1474
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Dynamics and behavior of a higher order rational difference equation

E. M. Elsayed

Mathematics Department, Faculty of Science, King AbdulAziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia.

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

Communicated by R. Saadati

Abstract

We study the global result, boundedness, and periodicity of solutions of the difference equation

$$x_{n+1} = a + \frac{bx_{n-l} + cx_{n-k}}{dx_{n-l} + ex_{n-k}}, \quad n = 0, 1, \ldots,$$

where the parameters a, b, c, d, and e are positive real numbers and the initial conditions $x_{-t}, x_{-t+1}, \ldots, x_{-1}$ and x_0 are positive real numbers where $t = \max\{l, k\},\ l \neq k$.

Keywords: Rational difference equations, rational systems, periodicity.
2010 MSC: 39A10, 39A11, 39A20.

1. Introduction

The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations. However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Difference equations appear as natural descriptions of observed evolution phenomena because most measurements of time evolving variables are discrete and as such these equations are in their own right

Email address: `emmelsayed@yahoo.com`, `emelsayed@mans.edu.eg`. (E. M. Elsayed)

Received 2015-08-05

important mathematical models. More importantly, difference equations also appear in the study of discretization methods for differential equations. Several results in the theory of difference equations have been obtained as more or less natural discrete analogues of corresponding results of differential equations.

Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. Some results in this area are, for example: Agarwal et al.[1] studied the global stability, periodicity character and gave a solution form of some special cases of the recursive sequence

$$x_{n+1} = a + \frac{dx_{n-l}x_{n-k}}{b - cx_{n-s}}.$$

Aloqeili [2] obtained a form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Elabbasy et al.[7] investigated the global stability character, boundedness and the periodicity of solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{Ax_n + Bx_{n-1} + Cx_{n-2}}.$$

In [6], Elabbasy et al. got the dynamics such that the global stability, periodicity character and gave a solution of a special case of the following recursive sequence

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}.$$

Elabbasy et al.[8] investigated the behavior of the difference equation, especially global stability, boundedness, periodicity character and gave a solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Saleh et al.[36] and [35] investigated the difference equations

$$y_{n+1} = A + \frac{y_n}{y_{n-k}} \quad \textit{with} \quad A < 0, \qquad x_{n+1} = A + \frac{x_n}{x_{n-k}}.$$

Simsek et al.[37] obtained a solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

Yalçinkaya et al.[42], [43] considered the dynamics of the difference equations

$$x_{n+1} = \frac{ax_{n-k}}{b + cx_n^p} \quad \text{and} \quad x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Zayed et al. [44], [45] studied the behavior of the following rational recursive sequences

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-k}} \quad \text{and} \quad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}.$$

Other related results on rational difference equations can be found in [1]–[46].

Our goal in this paper is to investigate the global stability character and the periodicity of solutions of the recursive sequence

$$x_{n+1} = a + \frac{bx_{n-l} + cx_{n-k}}{dx_{n-l} + ex_{n-k}}, \tag{1.1}$$

where the parameters a, b, c, d and e are positive real numbers and the initial conditions $x_{-t}, x_{-t+1}, \ldots, x_{-1}$ and x_0 are positive real numbers where $t = \max\{l, k\}$and $l \neq k$.

2. Definitions and Some Basic Properties

Here, we recall some basic definitions and some theorems that we need in the sequel; see [32].

Let I be an interval of real numbers and let

$$F : I^{k+1} \to I$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{2.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 2.1 (Equilibrium Point)**.** A point $\overline{x} \in I$ is called an equilibrium point of (2.1) if

$$\overline{x} = F(\overline{x}, \overline{x}, \ldots, \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of (2.1), or equivalently, $\overline{x}$ is a fixed point of F.

Definition 2.2 (Periodicity)**.** A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 2.3 (Stability)**.**

(i) The equilibrium point $\overline{x}$ of (2.1) is locally stable if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{ for all } \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of (2.1) is locally asymptotically stable if $\overline{x}$ is locally stable solution of (2.1) and there exists a $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of (2.1) is a global attractor if for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of (2.1) is globally asymptotically stable if $\overline{x}$ is locally stable and $\overline{x}$ is also a global attractor of (2.1).

(v) The equilibrium point $\overline{x}$ of (2.1) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of (2.1) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, \ldots, \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{2.2}$$

Theorem 2.4 ([32])**.** *Assume that* $p_i \in R$, $i = 1, 2, \ldots$ *and* $k \in \{0, 1, 2, \ldots\}$. *Then*

$$\sum_{i=1}^{k} |p_i| < 1, \tag{2.3}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+k} + p_1 y_{n+k-1} + \cdots + p_k y_n = 0, \quad n = 0, 1, \ldots.$$

Consider the following equation

$$x_{n+1} = g(x_n, x_{n-1}). \tag{2.4}$$

The following two theorems will be useful for the proofs of our results.

Theorem 2.5 ([33])**.** *Let* $[\alpha, \beta]$ *be an interval of real numbers and assume that*

$$g : [\alpha, \beta]^2 \to [\alpha, \beta],$$

is a continuous function satisfying the following properties:

(a) $g(x, y)$ *is nondecreasing in* x *in* $[\alpha, \beta]$ *for each* $y \in [\alpha, \beta]$, *and is nonincreasing in* $y \in [\alpha, \beta]$ *for each* x *in* $[\alpha, \beta]$;

(b) If $(m, M) \in [\alpha, \beta] \times [\alpha, \beta]$ *is a solution of the system*

$$M = g(M, m) \quad \text{and} \quad m = g(m, M),$$

then

$$m = M.$$

Then (2.4) *has a unique equilibrium* $\overline{x} \in [\alpha, \beta]$ *and every solution of* (2.4) *converges to* $\overline{x}$.

Theorem 2.6 ([33])**.** *Let* $[\alpha, \beta]$ *be an interval of real numbers and assume that*

$$g : [\alpha, \beta]^2 \to [\alpha, \beta],$$

is a continuous function satisfying:

(a) $g(x, y)$ *is nonincreasing in* x *in* $[\alpha, \beta]$ *for each* $y \in [\alpha, \beta]$, *and is nondecreasing in* $y \in [\alpha, \beta]$ *for each* x *in* $[\alpha, \beta]$;

(b) If $(m, M) \in [\alpha, \beta] \times [\alpha, \beta]$ *is a solution of the system*

$$M = g(m, M) \quad \text{and} \quad m = g(M, m),$$

then

$$m = M.$$

Then (2.4) *has a unique equilibrium* $\overline{x} \in [\alpha, \beta]$ *and every solution of* (2.4) *converges to* $\overline{x}$.

The paper proceeds as follows. In Section 3 we show that the equilibrium point of (1.1) is locally asymptotically stable when $2\,|be - dc| < (d + e)\,(a(d + e) + b + c)$. In Section 4 we prove that the solution is bounded and persists. In Section 5 we prove that the equilibrium point of (1.1) is a global attractor. In Section 6 we prove that there exists a periodic solution of prime period two of (1.1). Finally, we give numerical examples of some special cases of (1.1) and draw it by using Matlab.

3. Local Stability of the Equilibrium Point of (1.1)

This section deals with study of the local stability character of the equilibrium point of (1.1).

Theorem 3.1. *Assume that*

$$2\left|be-dc\right| < (d+e)\left(a(d+e)+b+c\right).$$

Then the positive equilibrium point of (1.1) *is locally asymptotically stable.*

Proof. The only positive equilibrium point of (1.1) is given by

$$\overline{x} = a + \frac{b+c}{d+e}.$$

Let $f : (0,\infty)^2 \longrightarrow (0,\infty)$ be a continuous function defined by

$$f(u,v) = a + \frac{bu+cv}{du+ev}. \tag{3.1}$$

We have

$$\frac{\partial f(u,v)}{\partial u} = \frac{(be-dc)v}{(du+ev)^2} \quad \text{and} \quad \frac{\partial f(u,v)}{\partial v} = \frac{(dc-be)u}{(du+ev)^2}.$$

Then we see that

$$\frac{\partial f(\overline{x},\overline{x})}{\partial u} = \frac{(be-dc)}{(d+e)^2\overline{x}} = \frac{(be-dc)}{(d+e)\left(a(d+e)+b+c\right)} = -a_1,$$
$$\frac{\partial f(\overline{x},\overline{x})}{\partial v} = \frac{(dc-be)}{(d+e)\left(a(d+e)+b+c\right)} = -a_0.$$

Then the linearized equation of (1.1) about $\overline{x}$ is

$$y_{n+1} + a_1 y_{n-l} + a_0 y_{n-k} = 0, \tag{3.2}$$

whose characteristic equation is

$$\lambda^{k+1} + a_1\lambda^{k-1} + a_0 = 0. \tag{3.3}$$

It follows by Theorem 2.4 that (3.2) is asymptotically stable if all the roots of (3.3) lie in the open disc $|\lambda| < 1$, that is if

$$|a_1| + |a_0| < 1,$$
$$\left|\frac{(be-dc)}{(d+e)\left(a(d+e)+b+c\right)}\right| + \left|\frac{(dc-be)}{(d+e)\left(a(d+e)+b+c\right)}\right| < 1,$$

and so

$$2\left|\frac{be-dc}{(d+e)\left(a(d+e)+b+c\right)}\right| < 1,$$

or

$$2\left|be-dc\right| < (d+e)\left(a(d+e)+b+c\right).$$

The proof is complete. □

4. Boundedness of Solutions of (1.1)

Here we study the boundedness nature of the solutions of (1.1).

Theorem 4.1. *Every solution of* (1.1) *is bounded and persists.*

Proof. Let $\{x_n\}_{n=-t}^{\infty}$ be a solution of (1.1). It follows from (1.1) that

$$x_{n+1} = a + \frac{bx_{n-l} + cx_{n-k}}{dx_{n-l} + ex_{n-k}} \leq a + \frac{\max\{b,c\}}{\min\{d,e\}}.$$

Then

$$x_{n+1} \leq a + \frac{\max\{b,c\}}{\min\{d,e\}} = M \quad \text{for all} \quad n \geq 1. \tag{4.1}$$

Also, we see from (1.1) that

$$x_{n+1} = a + \frac{bx_{n-l} + cx_{n-k}}{dx_{n-l} + ex_{n-k}} \geq a + \frac{\min\{b,c\}}{\max\{d,e\}}.$$

Then

$$x_{n+1} \geq a + \frac{\min\{b,c\}}{\max\{d,e\}} = m \quad \text{for all} \quad n \geq 1. \tag{4.2}$$

Thus we get from (4.1) and (4.2) that

$$a + \frac{\min\{b,c\}}{\max\{d,e\}} \leq x_{n+1} \leq a + \frac{\max\{b,c\}}{\min\{d,e\}} \quad \text{for all} \quad n \geq 1.$$

Thus, the solution is bounded and persists. □

5. Global Attractivity of the Equilibrium Point of (1.1)

In this section we investigate the global asymptotic stability of (1.1).

Lemma 5.1. *For any values of the quotients* $\frac{b}{d}$ *and* $\frac{c}{e}$, *the function* $f(u,v)$, *defined by* (3.1), *behaves monotonically in both arguments.*

Proof. The proof follows from some computations and will be omitted. □

Theorem 5.2. *The equilibrium point* $\overline{x}$ *is a global attractor of* (1.1) *if one of the following statements holds*

$$(1) \quad be \geq dc \ \ and \ \ c \geq b. \tag{5.1}$$

$$(2) \quad be \leq dc \ \ and \ \ c \leq b. \tag{5.2}$$

Proof. Let α and β be real numbers and assume that $g : [\alpha,\beta]^2 \longrightarrow [\alpha,\beta]$ is a function defined by

$$g(u,v) = a + \frac{bu + cv}{du + ev}.$$

Then

$$\frac{\partial g(u,v)}{\partial u} = \frac{(be - dc)v}{(du + ev)^2} \quad \text{and} \quad \frac{\partial g(u,v)}{\partial v} = \frac{(dc - be)u}{(du + ev)^2}.$$

We consider two cases:

Case (1). Assume that (5.1) is true. Then we can easily see that the function $g(u,v)$ is increasing in u and decreasing in v.

Suppose that (m,M) is a solution of the system $M = g(M,m)$ and $m = g(m,M)$. Then from (1.1), we see that

$$M = a + \frac{bM + cm}{dM + em} \quad \text{and} \quad m = a + \frac{bm + cM}{dm + eM},$$

or

$$dM^2 + emM - adM - aem = bM + cm \quad \text{and} \quad dm^2 + emM - adm - aeM = bm + cM.$$

Then

$$dM^2 + emM - (ad + b)M - (ae + c)m = 0 \quad \text{and} \quad dm^2 + emM - (ad + b)m - (ae + c)M = 0.$$

Subtracting these two equations we obtain

$$(M - m)\{d(M + m - a) + ea + (c - b)\} = 0,$$

under the condition $c \geq b$, we see that

$$M = m.$$

It follows by Theorem 2.5 that $\overline{x}$ is a global attractor of (1.1) and then the proof is complete.

Case (2) Assume that (5.2) is true. Let α and β be real numbers and assume that $g : [\alpha, \beta]^2 \longrightarrow [\alpha, \beta]$ is a function defined by $g(u, v) = a + \dfrac{bu + cv}{du + ev}$. Then we can easily see that the function $g(u, v)$ is decreasing in u and increasing in v.

Suppose that (m, M) is a solution of the system $M = g(m, M)$ and $m = g(M, m)$. Then from (1.1), we see that

$$M = a + \frac{bm + cM}{dm + eM} \quad \text{and} \quad m = a + \frac{bM + cm}{dM + em},$$

or

$$eM^2 + dmM - adm - aeM = bm + cM \quad \text{and} \quad em^2 + dmM - adM - aem = bM + cm.$$

Subtracting these two equations we obtain

$$(M - m)\{e(M + m - a) + da + (b - c)\} = 0.$$

under the condition $c \leq b$, we see that

$$M = m.$$

It follows by Theorem 2.6 that $\overline{x}$ is a global attractor of (1.1) and then the proof is complete. □

6. Existence of Periodic Solutions

In this section we study the existence of periodic solutions of (1.1). The following theorem states the necessary and sufficient conditions that this equation has periodic solutions of prime period two.

Theorem 6.1. *Equation* (1.1) *has a positive periodic solutions of prime period two if and only if*

(*i*) $\{(ad + b) - (ae + c)\}(e - d) - 4d(ae + c)) > 0$ *and* l *– odd,* k *– even*;

(*ii*) $\{(ae + c) - (ad + b)\}(d - e) - 4e(ad + b)) > 0$ *and* k *– odd,* l *– even.*

Proof. We prove that when l is odd and k even (the other case is similar and will be omitted.)

First suppose that there exists a periodic solution of prime period two

$$\ldots, p, q, p, q, \ldots,$$

of (1.1). We will prove that condition (i) holds.

We see from (1.1) when l is odd and k even that

$$p = a + \frac{bp + cq}{dp + eq}$$

and

$$q = a + \frac{bq + cp}{dq + ep}.$$

Then

$$dp^2 + epq = adp + aeq + bp + cq \tag{6.1}$$

and

$$dq^2 + epq = adq + aep + bq + cp. \tag{6.2}$$

Subtracting (6.1) from (6.2) gives

$$d(p^2 - q^2) = (ad + b)(p - q) - (ae + c)(p - q).$$

Since $p \neq q$, it follows that

$$p + q = \frac{ad + b - ae - c}{d} = \frac{\alpha - \beta}{d}, \qquad \alpha = ad + b, \quad \beta = ae + c. \tag{6.3}$$

It is known that p, q are positive. Then it should be $(ad + b) > (ae + c)$ {i.e., $\alpha > \beta$}.

Again, summing (6.1) and (6.2) yields

$$d(p^2 + q^2) + 2epq = (p + q)(\alpha + \beta). \tag{6.4}$$

It follows by (6.3), (6.4) and the relation

$$p^2 + q^2 = (p + q)^2 - 2pq \quad \text{for all} \quad p, q \in R,$$

that

$$2(e - d)pq = \frac{2\beta(\alpha - \beta)}{d}.$$

Thus

$$pq = \frac{\beta(\alpha - \beta)}{d(e - d)}. \tag{6.5}$$

Again, since p and q are positive and $(ad + b) > (ae + c)$, we see that $e > d$.

Now it is clear from (6.3) and (6.5) that p and q are two distinct roots of the quadratic equation

$$\begin{aligned} t^2 - \left(\frac{\alpha - \beta}{d}\right) t + \left(\frac{\beta(\alpha - \beta)}{d(e - d)}\right) &= 0, \\ dt^2 - (\alpha - \beta)t + \left(\frac{\beta(\alpha - \beta)}{(e - d)}\right) &= 0, \end{aligned} \tag{6.6}$$

and so

$$\begin{aligned} [\alpha - \beta]^2 - 4\left[\frac{d\beta(\alpha - \beta)}{(e - d)}\right] &> 0, \\ [\alpha - \beta]\left\{(\alpha - \beta) - \frac{4d\beta}{(e - d)}\right\} &> 0, \\ [\alpha - \beta]\left\{(\alpha - \beta)(e - d) - 4d\beta\right\} &> 0, \end{aligned}$$

or

$$\{(ad + b) - (ae + c)\}(e - d) - 4d(ae + c)) > 0.$$

Therefore inequalities (i) holds.

Second, suppose that inequalities (i) are true. We will show that (1.1) has a periodic solution of prime period two.

Assume that

$$p=\frac{(ad+b-ae-c)+\zeta}{2d}=\frac{\alpha-\beta+\zeta}{2d}$$

and

$$q=\frac{(ad+b-ae-c)-\zeta}{2d}=\frac{\alpha-\beta-\zeta}{2d},$$

where $\zeta=\sqrt{[ad+b-ae-c]^2-\frac{4d(ae+c)(ad+b-ae-c)}{(e-d)}}=\sqrt{[\alpha-\beta]^2-\frac{4d\beta(\alpha-\beta)}{(e-d)}}$.

We see from inequalities (i) that

$$\{(ad+b)-(ad+b)\}(e-d)-4d(ad+b)>0,$$

or

$$\{(\alpha-\beta)(e-d)-4d\beta\}>0,$$

which is equivalent to

$$[\alpha-\beta]\left\{(\alpha-\beta)-\frac{4d\beta}{(e-d)}\right\}>0,$$

$$[\alpha-\beta]^2>\frac{4d\beta(\alpha-\beta)}{(e-d)}.$$

Therefore p and q are distinct real numbers.

Set

$$x_{-l}=p,\ x_{-k}=q,\ldots,x_{-2}=q,\ \ x_{-1}=p \text{ and } x_0=q.$$

We wish to show that

$$x_1=x_{-1}=p \quad\text{and}\quad x_2=x_0=q.$$

It follows from (1.1) that

$$x_1=a+\frac{bp+cq}{dp+eq}=a+\frac{b\left(\frac{\alpha-\beta+\zeta}{2d}\right)+c\left(\frac{\alpha-\beta-\zeta}{2d}\right)}{d\left(\frac{\alpha-\beta+\zeta}{2d}\right)+e\left(\frac{\alpha-\beta-\zeta}{2d}\right)}.$$

Dividing the denominator and numerator by $2d$ gives

$$\begin{aligned}x_1&=a+\frac{b(\alpha-\beta+\zeta)+c(\alpha-\beta-\zeta)}{d(\alpha-\beta+\zeta)+e(\alpha-\beta-\zeta)}\\&=a+\frac{(b+c)(\alpha-\beta)+\zeta(b-c)}{(d+e)(\alpha-\beta)+\zeta(d-e)}.\end{aligned}$$

Multiplying the denominator and numerator of the right-hand side by $(d+e)(\alpha-\beta)-\zeta(d-e)$ gives, after some computation, that

$$x_1=\frac{\alpha-\beta+\zeta}{2d}=\frac{(ad+b-ae-c)+\zeta}{2d}=p.$$

Similarly as before, one can easily show that

$$x_2=q.$$

Then it follows by induction that

$$x_{2n}=q \quad\text{and}\quad x_{2n+1}=p \quad\text{for all}\quad n\geq -1.$$

Thus, (1.1) has the periodic solution of prime period two

$$\ldots,p,q,p,q,\ldots,$$

where p and q are the distinct roots of the quadratic equation (6.6) and the proof is complete. □

7. Numerical examples

For confirming our results, we consider numerical examples which represent different types of solutions to (1.1).

Example 7.1. We assume $l = 2$, $k = 3$, $x_{-3} = 2$, $x_{-2} = 11$, $x_{-1} = 4$, $x_0 = 7$, $a = 1$, $b = 5$, $c = 6$, $d = 3$, $e = 4$. See Fig. 1.

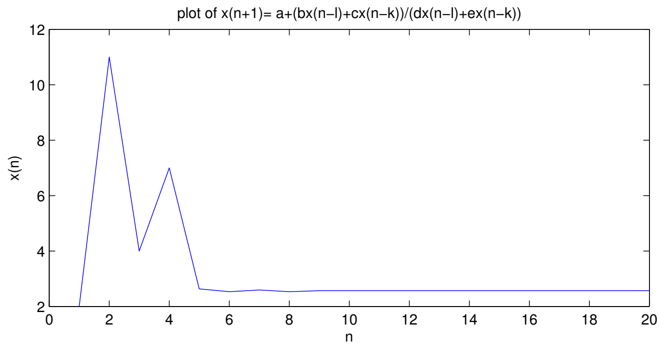

Fig. 1

Example 7.2. See Fig. 2, since $l = 2$, $k = 1$, $x_{-2} = 11$, $x_{-1} = 1$, $x_0 = 14$, $a = 0.4$, $b = 5$, $c = 0.6$, $d = 0.3$, $e = 4$.

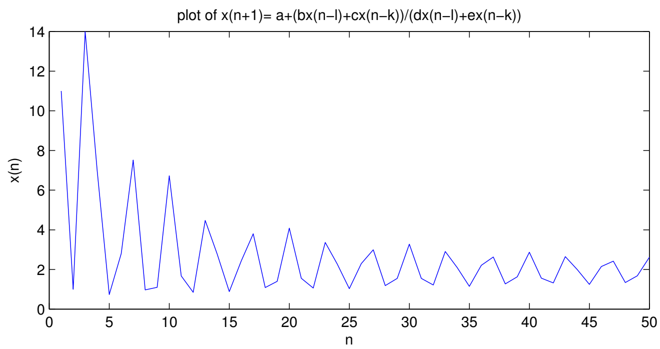

Fig. 2

Example 7.3. We consider $l = 3$, $k = 1$, $x_{-3} = 4$, $x_{-3} = 1$, $x_{-2} = 8$, $x_{-1} = 17$, $x_0 = 3$, $a = 0.1$, $b = 5$, $c = 6$, $d = 3$, $e = 1$. See Fig. 3.

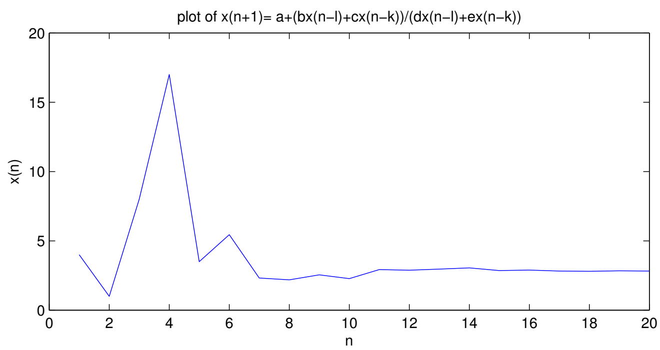

Fig. 3

Example 7.4. Fig. 4. shows the solutions when $l = 1$, $k = 2$, $a = 0.8$, $b = 0.5$, $c = 0.2$, $d = 5$, $e = 0.6$, $x_{-2} = q$, $x_{-1} = p$, $x_0 = q$.

$$\left(\text{Since } p, q = \frac{(ad + b - ae - c) \pm \zeta}{2d}, \quad \zeta = \sqrt{[ad + b - ae - c]^2 - \frac{4d(ae + c)(ad + b - ae - c)}{(e - d)}}\right).$$

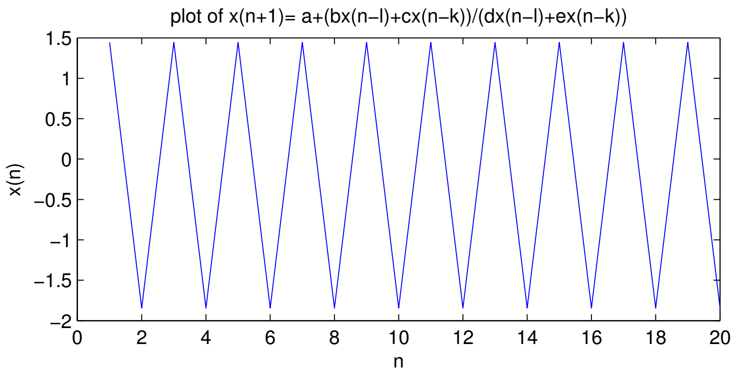

Fig. 4

Acknowledgement

This article was supported by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

References

[1] R. P. Agarwal, E. M. Elsayed, *Periodicity and stability of solutions of higher order rational difference equation*, Adv. Stud. Contemp. Math., **17** (2008), 181–201. 1

[2] M. Aloqeili, *Dynamics of a rational difference equation*, Appl. Math. Comput., **176** (2006), 768–774. 1

[3] C. Çinar, *On the positive solutions of the difference equation* $x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}$, Appl. Math. Comput., **156** (2004), 587–590.

[4] M. Dehghan, R. Mazrooei-sebdani, *Dynamics of* $x_{n+1} = \frac{x_{n-2k+1}}{x_{n-2k+1} + \alpha x_{n-2l}}$, Appl. Math. Comput., **185** (2007), 464–472.

[5] Q. Din, E. M. Elsayed, *Stability analysis of a discrete ecological model*, Comput. Ecol. Software, **4** (2014), 89–103.

[6] E. M. Elabbasy, H. El-Metwally, E. M. Elsayed, *On the difference equation* $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}$, Adv. Differ. Equ., **2006** (2006), 10 pages. 1

[7] E. M. Elabbasy, H. El-Metwally, E. M. Elsayed, *Global attractivity and periodic character of a fractional difference equation of order three*, Yokohama Math. J., **53** (2007), 89–100. 1

[8] E. M. Elabbasy, H. El-Metwally, E. M. Elsayed, *On the difference equations* $x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}$, J. Concr. Appl. Math., **5** (2007), 101–113. 1

[9] E. M. Elabbasy, H. El-Metwally, E. M. Elsayed, *Qualitative behavior of higher order difference equation*, Soochow J. Math., **33** (2007), 861–873.

[10] E. M. Elabbasy, H. El-Metwally, E. M. Elsayed, *Some Properties and Expressions of Solutions for a Class of Nonlinear Difference Equation*, Util. Math., **87** (2012), 93–110.

[11] H. El-Metwally, M. M. El-Afifi, *On the behavior of some extension forms of some population models*, Chaos Solitons Fractals, **36** (2008), 104–114.

[12] H. El-Metwally, E. M. Elsayed, *Solution and behavior of a third rational difference equation*, Util. Math., **88** (2012), 27–42.

[13] H. El-Metwally, E. A. Grove, G. Ladas, *On the difference equation* $y_{n+1} = \frac{y_{n-(2k+1)} + p}{y_{n-(2k+1)} + qy_{n-2l}}$, Proceedings of the 6th ICDE, Taylor and Francis, London, (2004).

[14] M. A. El-Moneam, *On the dynamics of the higher order nonlinear rational difference equation*, Math. Sci. Lett., **3** (2014), 121–129.

[15] M. A. El-Moneam, *On the dynamics of the solutions of the rational recursive sequences*, Br. J. Math. Comput. Sci., **5** (2015), 654–665.

[16] E. M. Elsayed, *On the solution of recursive sequence of order two*, Fasc. Math., **40** (2008), 5–13.

[17] E. M. Elsayed, *A solution form of a class of rational difference equations*, Int. J. Nonlinear Sci., **8** (2009), 402–411.

[18] E. M. Elsayed, *Dynamics of recursive sequence of order two*, Kyungpook Math. J., **50** (2010), 483–497.

[19] E. M. Elsayed, *Solutions of rational difference system of order two*, Math. Comput. Modelling, **55** (2012), 378–384.

[20] E. M. Elsayed, *Behavior and expression of the solutions of some rational difference equations*, J. Comput. Anal. Appl., **15** (2013), 73–81.

[21] E. M. Elsayed, *Solution for systems of difference equations of rational form of order two*, Comput. Appl. Math., **33** (2014), 751–765.

[22] E. M. Elsayed, *On the solutions and periodic nature of some systems of difference equations*, Int. J. Biomath., **7** (2014), 26 pages.

[23] E. M. Elsayed, *New method to obtain periodic solutions of period two and three of a rational difference equation*, Nonlinear Dyn., **79** (2015), 241–250.

[24] E. M. Elsayed, *The expressions of solutions and periodicity for some nonlinear systems of rational difference equations*, Adv. Stud. Contemp. Math., **25** (2015), 341–367.

[25] E. M. Elsayed, M. M. El-Dessoky, *Dynamics and global behavior for a fourth-order rational difference equation*, Hacet. J. Math. Stat., **42** (2013), 479–494.

[26] E. M. Elsayed, H. El-Metwally, *Stability and solutions for rational recursive sequence of order three*, J. Comput. Anal. Appl., **17** (2014), 305–315.

[27] E. M. Elsayed, H. El-Metwally, *Global behavior and periodicity of some difference equations*, J. Comput. Anal. Appl., **19** (2015), 298–309.

[28] A. E. Hamza, S. G. Barbary, *Attractivity of the recursive sequence* $x_{n+1} = (\alpha - \beta x_n)F(x_{n-1}, \dots, x_{n-k})$, Math. Comput. Modelling, **48** (2008), 1744–1749.

[29] D. Jana, E. M. Elsayed, *Interplay between strong Allee effect, harvesting and hydra effect of a single population discrete-time system*, Int. J. Biomath., **9** (2016), 25 pages.

[30] A. Khaliq, E. M. Elsayed, *The dynamics and solution of some difference equations*, J. Nonlinear Sci. Appl., **9** (2016), 1052–1063.

[31] A. Q. Khan, M. N. Qureshi, Q. Din, *Asymptotic behavior of an anti-competitive system of rational difference equations*, Life Sci. J., **11** (2014), 16–20.

[32] V. L. Kocić, G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers Group, Dordrecht, (1993). 2, 2.4

[33] M. R. Kulenovic, G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall/CRC Press, Florida, (2002). 2.5, 2.6

[34] A. Rafiq, *Convergence of an iterative scheme due to Agarwal et al.*, Rostock. Math. Kolloq., **61** (2006), 95–105.

[35] M. Saleh, M. Aloqeili, *On the difference equation* $x_{n+1} = A + \dfrac{x_n}{x_{n-k}}$, Appl. Math. Comput., **171** (2005), 862–869. 1

[36] M. Saleh, M. Aloqeili, *On the difference equation* $y_{n+1} = A + \dfrac{y_n}{y_{n-k}}$, *with* $A < 0$, Appl. Math. Comput., **176** (2006), 359–363. 1

[37] D. Simsek, C. Cinar, I. Yalcinkaya, *On the recursive sequence* $x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}}$, Int. J. Contemp. Math. Sci., **1** (2006), 475–480. 1

[38] N. Touafek, *On a second order rational difference equation*, Hacet. J. Math. Stat., **41** (2012), 867–874.

[39] N. Touafek, E. M. Elsayed, *On the solutions of systems of rational difference equations*, Math. Comput. Modelling, **55** (2012), 1987–1997.

[40] C. Y. Wang, F. Gong, S. Wang, L. R. LI, Q. H. Shi, *Asymptotic behavior of equilibrium point for a class of nonlinear difference equation*, Adv. Difference Equ., **2009** (2009), 8 pages.

[41] C. Y. Wang, S. Wang, Z. Wang, F. Gong, R. Wang, *Asymptotic stability for a class of nonlinear difference equation*, Discrete Dyn. Nat. Soc., **2010** (2010), 10 pages.

[42] I. Yalçinkaya, *On the difference equation* $x_{n+1} = \alpha + \dfrac{x_{n-m}}{x_n^k}$, Discrete Dyn. Nat. Soc., **2008** (2008), 8 pages. 1

[43] I. Yalçinkaya, C. Cinar, *On the dynamics of the difference equation* $x_{n+1} = \dfrac{a x_{n-k}}{b + c x_n^p}$, Fasc. Math., **42** (2009), 141–148. 1

[44] E. M. E. Zayed, M. A. El-Moneam, *On the rational recursive sequence* $x_{n+1} = \dfrac{\alpha + \beta x_n + \gamma x_{n-1}}{A + B x_n + C x_{n-1}}$, Comm. Appl. Nonlinear Anal., **12** (2005), 15–28. 1

[45] E. M. E. Zayed, M. A. El-Moneam, *On the rational recursive sequence* $x_{n+1} = a x_n - \dfrac{b x_n}{c x_n - d x_{n-k}}$, Comm. Appl. Nonlinear Anal., **15** (2008), 47–57. 1

[46] E. M. E. Zayed, M. A. El-Moneam, *On the global asymptotic stability for a rational recursive sequence*, Iran. J. Sci. Technol. Trans. A Sci., **35** (2011), 333–339. 1