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A new viscosity approximation method for common fixed points of a sequence of nonexpansive mappings with weakly contractive mappings in Banach spaces

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Abstract

By use of a new viscosity approximation method, we construct an explicit iterative algorithm for finding common fixed points of a sequence of nonexpansive mappings with weakly contractive mappings in the framework of Banach spaces. A strong convergence theorem is obtained for solving a kind of variational inequality problems. Our results improve and extend the corresponding ones of other authors with related interest.

Keywords: Viscosity approximation, nonexpansive mappings with weakly contractive mappings, common fixed points, variational inequalities.
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1. Introduction

Let C be a nonempty closed convex subset of a Banach space X. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\| \quad \forall x, y \in C.$$

Alber and Guerre-Delabriere [1] defined the weakly contractive maps in Hilbert spaces, and Rhoades [5] showed that the result of [1] is also valid in the complete metric spaces as follows.

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Definition 1.1. Let (X, d) be a complete metric space. A mapping $f : X \to X$ is called weakly contractive if

$$d(f(x), f(y)) \leq d(x, y) - \psi(d(x, y)) \ \forall x, y \in X,$$

where $x, y \in X$ and $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\psi(0) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$.

Definition 1.2 ([6])**.** Let C be a nonempty closed convex subset of a Banach space X and $T_n : C \to C$, where $n \in \{1, 2, \cdots\}$. Then the mapping sequence $\{T_n\}$ is called uniformly asymptotically regular on C, if for all $m \in \{1, 2, \cdots\}$ and any bounded subset K of C we have

$$\lim_{n\to\infty} \sup_{x\in K} \|T_m(T_n x) - T_n x\| = 0. \tag{1.1}$$

Theorem 1.3 ([6])**.** *Let $f : X \to X$ be a weakly contractive mapping, where (X, d) is a complete metric space, then f has a unique fixed point.*

In 2010, Razani and Homaeipour [4] considered the iterative sequence $\{x_m\}$ generated by

$$x_m = t_m f(x_m) + (1 - t_m)T_m x_m \ \forall m \geq 1 \tag{1.2}$$

and proved the following strong convergence theorem for $\{x_m\}$, where f is a weakly contractive and $\{T_m\}$ is a uniformly asymptotically regular sequence of nonexpansive mappings in a reflexive Banach space X.

Theorem 1.4 ([4])**.** *Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from X to X^*. Suppose that C is a nonempty closed convex subset of X and $\{T_m\} : C \to C$ is a uniformly asymptotically regular sequence of nonexpansive mappings with $F := \cap_{m=1}^{\infty} F(T_m) \neq \emptyset$. Let $f : C \to C$ be a weakly contractive mapping. Suppose $\{x_m\}$ is defined by* (1.1)*, where $\{t_m\}$ is a sequence of positive numbers in $(0, 1)$ satisfying $\lim_{m\to\infty} t_m = 0$. Then $\{x_m\}$ converges strongly to a common fixed point $p \in F$ which is the unique solution to the following variational inequality:*

$$\langle f(p) - p, J(y - p)\rangle \leq 0 \ \forall y \in F.$$

Remark 1.5. Note that the iteration sequence $\{x_m\}$ generated by (1.2) is an implicit one that will lead to complicated computations. Additionally, a stronger condition was imposed on the involved mappings, that is, $\{T_m\}$ was assumed to be a uniformly asymptotically regular sequence of nonexpansive mappings, and hence the corresponding result was less applicable.

Inspired and motivated by the study mentioned above, in this paper, by use of a new viscosity approximation method, we construct an explicit iteration scheme for finding common fixed points of a sequence of nonexpansive mappings. A strong convergence theorem for solving some variational inequality problems is established in the framework of Banach spaces.

2. Preliminaries

Throughout the paper, let X be a real Banach space. We say that X is *strictly convex* if the following implication holds for $x, y \in X$:

$$\|x\| = \|y\| = 1, x \neq y \Rightarrow \left\|\frac{x + y}{2}\right\| < 1.$$

X is also said to be *uniformly convex* if for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \|x-y\| \geq \epsilon \Rightarrow \left\|\frac{x+y}{2}\right\| \leq 1-\delta.$$

The following results are well known, which can be founded in [7].

(i) A uniformly convex Banach space X is reflexive and strictly convex.

(ii) If C is a nonempty convex subset of a strictly convex Banach space X and $T : C \to C$ is a nonexpansive mapping, then the fixed point set $F(T)$ of T is a closed convex subset of C.

By a gauge function we mean a continuous and strictly increasing function φ defined on $[0,\infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. The mapping J_φ from X to 2^{X^*}, defined by

$$J_\varphi x = \{f \in X^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \varphi(\|x\|)\} \ \forall x \in X, \tag{2.1}$$

is called the duality mapping with the gauge function φ. In the case where $\varphi(t) = t$, then $J_\varphi = J$, which is the normalized duality mapping.

Proposition 2.1 ([8])**.**

(i) *$J = I$ if and only if X is a Hilbert space.*

(ii) *J is surjective if and only if X is reflexive.*

(iii) *$J_\varphi(\lambda x) = sign\lambda \left(\frac{\varphi(|\lambda|\|x\|)}{\|x\|}\right) Jx$ for all $x \in X \setminus \{0\}, \lambda \in \mathbb{R}$; particularly, $J(-x) = -J(x)$ for all $x \in X$.*

We say that a Banach space X has a weakly sequentially continuous duality mapping if there exists a gauge function φ such that the duality mapping J_φ is single-valued and continuous from the weak topology to the weak* topology of X.

In what follows we shall make use of the following definitions and lemmas.

Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from X to X^*. The function $\phi : X \times X \to \mathbb{R}^+ \cup \{0\}$ is defined by

$$\phi(x,y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2.$$

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2.$$

The function ϕ also has the following property:

$$\phi(y,x) = \phi(z,x) + \phi(y,z) + 2\langle z-y, J(x-z)\rangle. \tag{2.2}$$

Lemma 2.2. *Let X be a Banach space. Then for all $x, y \in X$ and $\alpha_i \in [0,1]$ for $i = 1, 2, \cdots, n$ such that $\Sigma_{i=1}^n \alpha_i = 1$ the following inequality holds:*

$$\left\|\sum_{i=1}^n \alpha_i x_i\right\|^2 \leq \sum_{i=1}^n \alpha_i \|x_i\|^2. \tag{2.3}$$

Lemma 2.3 ([3])**.** *Let $\{a_n\}, \{\delta_n\}$, and $\{b_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \le (1+\delta_n)a_n + b_n, \forall n \ge 1. \tag{2.4}$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Definition 2.4 ([10])**.** Let $\{A_n\} : C \to C$ be a sequence of mappings and $A : C \to C$ be a mapping. $\{A_n\}$ is said to be graph convergent to A if $\{graph(A_n)\}$ (the sequence of graph of A_n) converges to graph A in the sense of Kuratowski-Painleve, that is,

$$\limsup_{n\to\infty} graph(A_n) \subset graph(A_n) \subset \liminf_{n\to\infty} graph(A_n).$$

Definition 2.5.

(i) A multi-valued mapping $A : X \to X$ is said to be accretive if $\langle Ax - Ay, J(x-y)\rangle \ge 0 \; \forall x, y \in X$. A mapping $A : X \to X$ is said to be maximal accretive if it is accretive, and for any $x, u \in X$ when

$$\langle u - v, J(x-y)\rangle \ge 0 \; \forall (y, v) \in graph(A),$$

we have $u \in Ax$.

(ii) A mapping $A : X \to X$ is said to be strongly accretive if there exists a strictly increasing function $\tilde{\varphi} : [0,\infty) \to [0,\infty)$ with $\tilde{\varphi}(0) = 0$ such that

$$\langle Ax - Ay, J(x-y)\rangle \ge \tilde{\varphi}(\|x-y\|)\|x-y\| \; \forall x, y \in X.$$

Definition 2.6. The normal cone $N_{F(T)}$ to $F(T)$ is defined by

$$N_{F(T)}(x) = \begin{cases} \{u \in X : \langle y - x, Ju\rangle \le 0 \; \forall y \in F\}, \; x \in F(T); \\ \emptyset, \; x \in F(T)^c. \end{cases}$$

Finding an $x^* \in F(T)$ such that

$$\langle (I-f)x^*, J(x^* - x)\rangle \le 0 \; (\forall x \in F(T))$$

is equivalent to the following variational inclusion problem: finding an $x^* \in C$ such that

$$\theta \in (I-f)x^* + N_{F(T)}(x^*).$$

Lemma 2.7 ([2])**.**

(i) *Let $A : X \to X$ be a maximal accretive operator. Then $(t^{-1}A)$ graph converges to $N_{A^{-1}(0)}$ as $t \to 0$ provided that $A^{-1}(0) \ne \emptyset$.*

(ii) *Let $\{B_n : X \to X\}$ be a sequence of maximal accretive operators, which graph converges to an operator B. If A is a strongly accretive operator, then $\{A + B_n\}$ graph converges to $A + B$, and $A + B$ is maximal accretive.*

Lemma 2.8. *Let $f : X \to X$ be a weakly contractive mapping and $T : X \to X$ be a nonexpansive mapping. Then, the following results are obtained:*

(i) *the mapping $(I - f) : X \to X$ is strongly accretive;*

(ii) *the mapping $(I - T) : X \to X$ is accretive, so it is maximal accretive.*

Remark 2.9. This conclusion results directly from Lemma 1.6 in [10].

Lemma 2.10. *The unique solutions to the positive integer equation*

$$n = i_n + \frac{(m_n-1)m_n}{2},\ \ m_n \geq i_n, n = 1, 2, \cdots \tag{2.5}$$

are

$$i_n = n - \frac{(m_n-1)m_n}{2},\ \ m_n = -\left[\frac{1}{2} - \sqrt{2n+\frac{1}{4}}\right],\ \ n = 1, 2, \cdots,$$

where $[x]$ *denotes the maximal integer that is not larger than* x.

Proof. It follows from (2.5) that

$$i_n = n - \frac{(m_n-1)m_n}{2},\ \ i_n \leq m_n,\ \ n = 1, 2, 3, \cdots,$$

and hence

$$1 \leq i_n = n - \frac{(m_n-1)m_n}{2} \leq m_n,\ \ n = 1, 2, 3, \cdots, \tag{2.6}$$

that is,

$$\frac{(m_n-1)m_n}{2} + 1 \leq n \leq \frac{(m_n+1)m_n}{2},\ \ n = 1, 2, 3, \cdots,$$

which implies that

$$\left(m_n - \frac{1}{2}\right)^2 \leq 2n - \frac{7}{4},\ \ \left(m_n + \frac{1}{2}\right)^2 \geq 2n + \frac{1}{4},\ \ n = 1, 2, 3, \cdots.$$

Thus

$$\sqrt{2n+\frac{1}{4}} - \frac{1}{2} \leq m_n \leq \frac{1}{2} + \sqrt{2n-\frac{7}{4}},\ \ n = 1, 2, 3, \cdots,$$

that is,

$$-\sqrt{2n-\frac{7}{4}} - \frac{1}{2} \leq -m_n \leq \frac{1}{2} - \sqrt{2n+\frac{1}{4}},\ \ n = 1, 2, 3, \cdots, \tag{2.7}$$

while the difference of the two sides of the inequality above is

$$1 - \left(\sqrt{2n+\frac{1}{4}} - \sqrt{2n-\frac{7}{4}}\right) = 1 - \frac{2}{\sqrt{2n+\frac{1}{4}} + \sqrt{2n-\frac{7}{4}}} \in [0, 1),\ \ n = 1, 2, 3, \cdots.$$

Then, it follows from (2.7) that (2.6) holds obviously. □

3. Main results

Theorem 3.1. *Let* X *be a reflexive Banach space which admits a weakly sequentially continuous duality mapping* J *from* X *to* X^*. *Suppose that* C *is a nonempty closed convex subset of* X *and* $\{T_i\}_{i=1}^{\infty} : C \to C$ *is a sequence of nonexpansive mappings with the interior of* $F := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. *Let* $f : C \to C$ *be a weakly contractive mapping. Starting from an arbitrary* $x_1 \in C$, *define* $\{x_n\}$ *by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_n^* x_n\ \forall n \geq 1, \tag{3.1}$$

where $\{\alpha_n\}$ *is a decreasing sequence in* $(0, 1)$ *satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n < \infty$;
(ii) $\sum_{n=1}^{\infty} \left(\alpha_{n-1}^2/\alpha_n^2 - 1\right) < \infty$;
(iii) $\sum_{n=1}^{\infty} (\alpha_{n-1} - \alpha_n)/\alpha_n^2 < \infty$;

and $T_n^* = T_{i_n}$ *with* i_n *being the solution to the positive integer equation:* $n = i_n + \frac{(m_n-1)m_n}{2}$ $(m_n \geq i_n, n = 1, 2, \cdots)$, *that is, for each* $n \geq 1$, *there exists a unique* i_n *such that*

$$i_1 = 1, i_2 = 1, i_3 = 2, i_4 = 1, i_5 = 2, i_6 = 3, i_7 = 1, i_8 = 2, i_9 = 3, i_{10} = 4, i_{11} = 1, \cdots.$$

If $f \neq 0$, *then* $\{x_n\}$ *converges strongly to a point* $x^* \in F$ *which is the unique solution to the following variational inequality:*

$$\langle (I - f)x^*, J(x - x^*)\rangle \geq 0 \ \forall x \in F. \tag{3.2}$$

Proof. We divide the proof into several steps.

(I) $\lim_{n\to\infty} \|x_n - p^*\|$ exists $\forall p^* \in F$.

For any $p^* \in F$, from (3.1), we have

$$\begin{aligned}
\|x_{n+1} - p^*\| =& \|\alpha_n(f(x_n) - p^*) + (1 - \alpha_n)T_n^*(x_n - p^*)\| \\
\leq& \alpha_n\|f(x_n) - p^*\| + (1 - \alpha_n)\|x_n - p^*\| \\
\leq& \alpha_n\|f(x_n) - f(p^*)\| + \alpha_n\|f(p^*) - p^*\| + (1 - \alpha_n)\|x_n - p^*\| \\
\leq& \alpha_n\|x_n - p^*\| - \alpha_n\psi(\|x_n - p^*\|) + \alpha_n\|f(p^*) - p^*\| + (1 - \alpha_n)\|x_n - p^*\| \\
\leq& \|x_n - p^*\| + \mu_n,
\end{aligned}$$

where $\mu_n = \alpha_n\|f(p^*) - p^*\|$, and so $\sum_{n=1}^{\infty} \mu_n < \infty$. So by Lemma 2.3 we conclude that $\lim_{n\to\infty} \|x_n - p^*\|$ exists and hence $\{x_n\}, \{f(x_n)\}$, and $\{T_n^* x_n\}$ are bounded.

(II) $x_n \to x^* \in C$ as $n \to \infty$.

From (3.1) and Lemma 2.2, we also have

$$\begin{aligned}
\|x_{n+1} - p^*\|^2 =& \|\alpha_n(f(x_n) - p^*) + (1 - \alpha_n)T_n^*(x_n - p^*)\|^2 \\
=& \alpha_n\|f(x_n) - p^*\|^2 + (1 - \alpha_n)\|T_n^*(x_n - p^*)\|^2 \\
&- \alpha_n(1 - \alpha_n)\|f(x_n) - T_n^* x_n\|^2 \\
\leq& \alpha_n(\|f(x_n) - f(p^*)\| + \|f(p^*) - p^*\|)^2 + (1 - \alpha_n)\|x_n - p^*\|^2 \\
\leq& \alpha_n[(\|x_n - p^*\| - \psi(\|x_n - p^*\|)) + \|f(p^*) - p^*\|]^2 + (1 - \alpha_n)\|x_n - p^*\|^2 \\
\leq& \alpha_n\|x_n - p^*\|^2 + (1 - \alpha_n)\|x_n - p^*\|^2 \\
&+ \alpha_n(2\|f(p^*) - p^*\| \cdot \|x_n - p^*\| + \|f(p^*) - p^*\|^2) \\
\leq& \|x_n - p^*\|^2 + \nu_n,
\end{aligned} \tag{3.3}$$

where $\nu_n := \alpha_n(2\|f(p^*) - p^*\| \cdot \|x_n - p^*\| + \|f(p^*) - p^*\|^2)$ and $\sum_{n=1}^{\infty} \nu_n < \infty$, since $\{x_n\}$ is bounded and $\sum_{n=1}^{\infty} \alpha_n < \infty$.

Furthermore, it follows from (2.2) that

$$\phi(p, x_n) = \phi(x_{n+1}, x_n) + \phi(p, x_{n+1}) + 2\langle x_{n+1} - p, J(x_n - x_{n+1})\rangle \ \forall p \in X.$$

This implies that

$$\langle x_{n+1} - p, J(x_n - x_{n+1})\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) = \frac{1}{2}(\phi(p, x_n) - \phi(p, x_{n+1})). \tag{3.4}$$

Moreover, since the interior of F is nonempty, there exists a $p^* \in F$ and $r > 0$ such that $(p^*+rh) \in F$ whenever $\|h\| \leq 1$. Thus, from (3.3) and (3.4) we obtain

$$0 \leq \langle x_{n+1} - (p^* + rh), J(x_n - x_{n+1})\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) + \frac{1}{2}\nu_n. \tag{3.5}$$

Then from (3.4) and (3.5) we obtain

$$\begin{aligned} r\langle h, J(x_n - x_{n+1})\rangle \leq& \langle x_{n+1} - p^*, J(x_n - x_{n+1})\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) + \frac{1}{2}\nu_n \\ =& \frac{1}{2}(\phi(p^*, x_n) - \phi(p^*, x_{n+1})) + \frac{1}{2}\nu_n \end{aligned}$$

and hence,

$$\langle h, J(x_n - x_{n+1})\rangle \leq \frac{1}{2r}(\phi(p^*, x_n) - \phi(p^*, x_{n+1})) + \frac{1}{2r}\nu_n. \tag{3.6}$$

Since h with $\|h\| \leq 1$ is arbitrary, we have, by taking $h = \frac{x_n - x_{n+1}}{\|x_n - x_{n+1}\|}$,

$$\|x_n - x_{n+1}\| \leq \frac{1}{2r}(\phi(p^*, x_n) - \phi(p^*, x_{n+1})) + \frac{1}{2r}\nu_n. \tag{3.7}$$

So, if $n > m$, then we have

$$\begin{aligned} \|x_m - x_n\| \leq& \sum_{j=m}^{n-1}\|x_j - x_{j+1}\| \\ \leq& \frac{1}{2r}\sum_{j=m}^{n-1}(\phi(p^*, x_j) - \phi(p^*, x_{j+1})) + \frac{1}{2r}\sum_{j=m}^{n-1}\nu_j \\ =& \frac{1}{2r}(\phi(p^*, x_m) - \phi(p^*, x_n)) + \frac{1}{2r}\sum_{j=m}^{n-1}\nu_j. \end{aligned} \tag{3.8}$$

But we know that $\{\phi(p^*, x_n)\}$ converges, and $\sum_{n=1}^{\infty}\nu_n < \infty$. Therefore, we obtain from (3.8) that $\{x_n\}$ is a Cauchy sequence. Since X is complete there exists an $x^* \in X$ such that $x_n \to x^* \in X$ as $n \to \infty$. Thus, since $\{x_n\} \subset C$ and C is closed and convex, then $x^* \in C$, that is,

$$x_n \to x^* \in C \ (n \to \infty). \tag{3.9}$$

(III) $\|x_n - T_i x_n\| \to 0$ for each $i \geq 1$ as $n \to \infty$.

It follows from (3.1) and (3.7) that, as $n \to \infty$,

$$\|x_{n+1} - T_n^* x_n\| = \alpha_n\|f(x_n) - T_n^* x_n\| \to 0$$

and

$$\|x_{n+1} - x_n\| \to 0,$$

which implies that, by induction, for any nonnegative integer j,

$$\lim_{n\to\infty}\|x_{n+j} - x_n\| = 0. \tag{3.10}$$

We then have, as $n \to \infty$,

$$\|x_n - T_n^* x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_n^* x_n\| \to 0. \tag{3.11}$$

For each $i \geq 1$, since

$$\begin{aligned}\left\|x_n - T^*_{n+i}x_n\right\| \leq& \|x_n - x_{n+i}\| + \left\|x_{n+i} - T^*_{n+i}x_n\right\| \\ \leq& \|x_n - x_{n+i}\| + \left\|x_{n+i} - T^*_{n+i}x_{n+i}\right\| + \left\|T^*_{n+i}x_{n+i} - T^*_{n+i}x_n\right\| \\ \leq& 2\|x_n - x_{n+i}\| + \left\|x_{n+i} - T^*_{n+i}x_{n+i}\right\|,\end{aligned}$$

it follows from (3.10) and (3.11) that

$$\lim_{n\to\infty} \left\|x_n - T^*_{n+i}x_n\right\| = 0. \tag{3.12}$$

Now, for each $i \geq 1$, we claim that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0. \tag{3.13}$$

As a matter of fact, setting

$$n = N_m + i,$$

where $N_m = \frac{(m-1)m}{2}$, $m \geq i$, we obtain that

$$\begin{aligned}\|x_n - T_i x_n\| \leq& \|x_n - x_{N_m}\| + \|x_{N_m} - T_i x_n\| \\ \leq& \|x_n - x_{N_m}\| + \left\|x_{N_m} - T^*_{N_m+i}x_{N_m}\right\| + \left\|T^*_{N_m+i}x_{N_m} - T_i x_n\right\| \\ =& \|x_n - x_{N_m}\| + \left\|x_{N_m} - T^*_{N_m+i}x_{N_m}\right\| + \|T_i x_{N_m} - T_i x_n\| \\ \leq& 2\|x_n - x_{N_m}\| + \left\|x_{N_m} - T^*_{N_m+i}x_{N_m}\right\| \\ =& 2\|x_n - x_{n-i}\| + \left\|x_{N_m} - T^*_{N_m+i}x_{N_m}\right\|.\end{aligned}$$

Then, since $N_m \to \infty$ as $n \to \infty$, it follows from (3.10) and (3.12) that (3.13) holds obviously.

(IV) $x_n \to x^* \in F$ as $n \to \infty$, which is the unique solution to the following variational inequality:

$$\langle (I - f)x^*, J(x - x^*)\rangle \geq 0 \ \forall x \in F.$$

It immediately follows from (3.9) and (3.13) that, as $n \to \infty$,

$$x_n \to x^* \in F. \tag{3.14}$$

Next, for any $i \geq 1$, we consider the corresponding subsequence $\left\{x^{(i)}_k\right\}_{k\in\mathbb{K}_i}$ of $\{x_n\}$, where $\mathbb{K}_i := \{k \in \mathbb{N} : k = i + (m-1)m/2, m \geq i, m \in \mathbb{N}\}$. For example, by Lemma 2.10 and the definition of $\mathbb{K}_1$, we have $\mathbb{K}_1 = \{1, 2, 4, 7, 11, 16, \cdots\}$ and $i_1 = i_2 = i_4 = i_7 = i_{11} = i_{16} = \cdots = 1$. Since $(T^*_k)^{(i)} = T_i$ whenever $k \in \mathbb{K}_i$, it follows from (3.1) that

$$\begin{aligned}\left\|x^{(i)}_{k+1} - x^{(i)}_k\right\| =& \|\alpha^{(i)}_k(f(x^{(i)}_k) - f(x^{(i)}_{k-1})) + (1 - \alpha^{(i)}_k)T_i(x^{(i)}_k - x^{(i)}_{k-1}) \\ &+ (\alpha^{(i)}_k - \alpha^{(i)}_{k-1})(f(x^{(i)}_{k-1}) - T_i x^{(i)}_{k-1})\| \\ \leq& \alpha^{(i)}_k \left(\left\|x^{(i)}_k - x^{(i)}_{k-1}\right\| - \psi\left(\left\|x^{(i)}_k - x^{(i)}_{k-1}\right\|\right)\right) \\ &+ \left(1 - \alpha^{(i)}_k\right)\left\|x^{(i)}_k - x^{(i)}_{k-1}\right\| + M\left|\alpha^{(i)}_k - \alpha^{(i)}_{k-1}\right| \\ \leq& \left\|x^{(i)}_k - x^{(i)}_{k-1}\right\| + M\left|\alpha^{(i)}_k - \alpha^{(i)}_{k-1}\right|,\end{aligned}$$

where $M := \sup_{k\in\mathbb{K}_i} \left\| f(x_{k-1}^{(i)}) - T_i x_{k-1}^{(i)} \right\| < \infty$.

Thus, we have

$$\begin{aligned}\frac{\left\|x_{k+1}^{(i)} - x_k^{(i)}\right\|}{\left(\alpha_k^{(i)}\right)^2} &\leq \frac{\left(\alpha_{k-1}^{(i)}\right)^2}{\left(\alpha_k^{(i)}\right)^2} \frac{\left\|x_k^{(i)} - x_{k-1}^{(i)}\right\|}{\left(\alpha_{k-1}^{(i)}\right)^2} + \frac{M\left|\alpha_k^{(i)} - \alpha_{k-1}^{(i)}\right|}{\left(\alpha_{k-1}^{(i)}\right)^2} \\ &= \left(1+\eta_k^{(i)}\right) \frac{\left\|x_k^{(i)} - x_{k-1}^{(i)}\right\|}{\left(\alpha_{k-1}^{(i)}\right)^2} + \gamma_k^{(i)},\end{aligned}$$

where $\eta_k^{(i)} := \left(\alpha_{k-1}^{(i)}/\alpha_k^{(i)}\right)^2 - 1$, $\gamma_k^{(i)} := M\left|\alpha_k^{(i)} - \alpha_{k-1}^{(i)}\right| / \left(\alpha_k^{(i)}\right)^2$, $\sum_{k\in\mathbb{K}_i} \eta_k^{(i)} < \infty$, and $\sum_{k\in\mathbb{K}_i} \gamma_k^{(i)} < \infty$.

It follows from Lemma 2.3 that $\lim_{\mathbb{K}_i \ni k\to\infty} \left\|x_{k+1}^{(i)} - x_k^{(i)}\right\| / \left(\alpha_k^{(i)}\right)^2$ exists and hence $\left\{y_k^{(i)}\right\} := \left\{\left(x_{k+1}^{(i)} - x_k^{(i)}\right) / \left(\alpha_k^{(i)}\right)^2\right\}$ is bounded. Then there exists an $M_i > 0$ such that

$$\frac{\left\|x_{k+1}^{(i)} - x_k^{(i)}\right\|}{M_i\left(\alpha_k^{(i)}\right)^2} \leq 1 \ \forall k \in \mathbb{K}_i.$$

Taking $h = \left(x_k^{(i)} - x_{k+1}^{(i)}\right) / M_i\left(\alpha_k^{(i)}\right)^2$, we have, from (3.6),

$$\frac{\left\|x_k^{(i)} - x_{k+1}^{(i)}\right\|^2}{\left(\alpha_k^{(i)}\right)^2} \leq \frac{M_i}{2r}\left(\phi\left(p^*, x_k^{(i)}\right) - \phi\left(p^*, x_{k+1}^{(i)}\right)\right) + \frac{M_i}{2r}\nu_k^{(i)}.$$

This implies that, as $\mathbb{K}_i \ni k \to \infty$,

$$\frac{x_k^{(i)} - x_{k+1}^{(i)}}{\alpha_k^{(i)}} \to \theta. \tag{3.15}$$

Furthermore, from (3.1), we have

$$\frac{x_k^{(i)} - x_{k+1}^{(i)}}{\alpha_k^{(i)}} = ((I-f) + \frac{1-\alpha_k^{(i)}}{\alpha_k^{(i)}}(I-T_i))x_k^{(i)}.$$

In addition, by Lemmas 2.7 and 2.8, $(I-f) + \left(1-\alpha_k^{(i)}\right)/\alpha_k^{(i)}(I-T_i)$ graph converges to $(I-f) + N_{F(T_i)}$. Since the graph of $(I-f) + N_{F(T_i)}$ is weakly-strongly closed, we obtain that, by taking into (3.15) and (3.14),

$$\theta \in (I-f)x^* + N_{F(T_i)}(x^*).$$

This implies that $\langle (I-f)x^*, x^* - x\rangle \leq 0 \ \ \forall x \in F(T_i)$, that is,

$$\langle (I-f)x^*, x - x^*\rangle \geq 0 \ \forall x \in F$$

since $F \subset F(T_i)$. The proof is completed. □

4. Applications

The so-called *convex feasibility problem* for a family of mappings $\{T_i\}_{i=1}^{\infty}$ is to find a point in the nonempty intersection $\cap_{i=1}^{\infty} F(T_i)$, which exactly illustrates the importance of finding common fixed points of infinite families. The following example also clarifies the same thing.

Example 4.1. Let X be a smooth, strictly convex, and reflexive Banach space, C be a nonempty and closed convex subset of X, and $\{f_i\}_{i=1}^{\infty} : C \times C \to \mathbb{R}$ be a sequence of bifunctions satisfying the conditions: for each $i \geq 1$,

(A_1) $f_i(x, x) = 0$;
(A_2) f_i is monotone, *i.e.*, $f_i(x, y) + f_i(y, x) \leq 0$;
(A_3) $\limsup_{t \downarrow 0} f_i(x + t(z - x), y) \leq f_i(x, y)$;
(A_4) The mapping $y \mapsto f_i(x, y)$ is convex and lower semicontinuous.

A *system of equilibrium problems* for $\{f_i\}_{i=1}^{\infty}$ is to find an $x^* \in C$ such that

$$f_i(x^*, y) \geq 0 \quad \forall y \in C, i \geq 1,$$

whose set of common solutions is denoted by $EP := \cap_{i=1}^{\infty} EP(f_i)$, where $EP(f_i)$ denotes the set of solutions to the equilibrium problem for f_i $(i = 1, 2, \cdots)$. It is shown in Theorem 4.3 in [10] that such a system of problems can be reduced to the approximation of some fixed point of a sequence of nonexpansive mappings.

Example 4.2. *Application to monotone variational inequalities.*

Let H be a real Hilbert space. Set $f = I - \gamma G$, where $G : H \to H$ is a η-Lipschitzian and κ-strongly monotone mapping and $\gamma \in (0, \frac{2\kappa}{\eta^2}]$. Now, we show that $f : H \to H$ is a nonexpansive mapping. In fact, by the assumptions, we have

$$\begin{aligned}
\|f(x) - f(y)\|^2 =& \|(x - y) - (\gamma Gx - \gamma Gy)\|^2 \\
=& \|x - y\|^2 - 2\gamma\langle x - y, Gx - Gy\rangle + \gamma^2\|Gx - Gy\|^2 \\
\leq& \|x - y\|^2 - 2\gamma\kappa\|x - y\|^2 + \gamma^2\eta^2\|x - y\|^2 \\
=& (1 - 2\gamma\kappa + \gamma^2\eta^2)\|x - y\|^2 \\
\leq& \|x - y\|^2
\end{aligned}$$

for all $x, y \in H$. Hence, (3.2) is reduced to finding an $x^* \in F$ such that

$$\langle Gx^*, x - x^*\rangle \geq 0 \ \forall x \in F,$$

where $\{T_n\}$ is a sequence of nonexpansive mappings, whose common fixed points set is denoted by F. This problem was first considered by Yamada and Ogura [9].

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