Remark on fundamentally non-expansive mappings in hyperbolic spaces

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Abstract

In this paper, we prove some properties of fixed point set of fundamentally non-expansive mappings and derive the existence of fixed point theorems as follows results of Salahifard et al. [H. Salahifard, S. M. Vaezpour, S. Dhompongsa, J. Nonlinear Anal. Optim., 4 (2013), 241–248] in hyperbolic spaces. ©2016 All rights reserved.

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1. Introduction and Preliminaries

There are many nonlinear mappings which are more general than the non-expansive ones. The existence problem of fixed point of those mappings is very useful in studying the theory of equations in science and applied science. Let \( X \) be a real Banach space and let \( K \) be a nonempty closed convex subset of \( X \). A mapping \( T : K \to K \) is said to be nonexpansive, if \( ||Tx - Ty|| \leq ||x - y|| \), for all \( x, y \in K \). In 2008, Suzuki [8] introduced condition \( C \) as follows.

Let \( T \) be a mapping on a subset \( K \) of a Banach space \( X \). Then \( T \) is said to satisfy condition \( C \) (or Suzuki’s generalized non-expansive) if

\[
\frac{1}{2} ||x - Tx|| \leq ||x - y|| \quad \text{implies} \quad ||Tx - Ty|| \leq ||x - y||
\]

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for all \( x, y \in K \).

It is obvious that every non-expansive mapping satisfies condition \( C \), but the converse is not true. The next simple example can show this fact.

**Example 1.1** (\([2]\)). Define a mapping \( T \) on \([0, 3]\) by

\[
T_x = \begin{cases} 
0 & x \neq 3, \\
1 & x = 3.
\end{cases}
\]

Then \( T \) satisfies condition \( C \), but \( T \) is not non-expansive.

In 2014, Ghoncheh and Razani \([2]\), introduced the following definition and recalled some other conditions which generalize the Suzuki and studied fixed point for some generalized non-expansive mappings in Ptolemy spaces as follows.

Let \( X \) be a metric space and \( K \) be a subset of \( X \). A mapping \( T : K \rightarrow K \) is said to be fundamentally non-expansive if

\[
d(T^2x, Ty) \leq d(Tx, y) \tag{1.1}
\]

for all \( x, y \in K \).

**Proposition 1.2.** Every mapping which satisfies condition \( C \) is fundamentally non-expansive, but the converse is not true.

**Example 1.3.** Suppose \( X = \{(0, 0), (0, 1), (1, 1), (1, 2)\} \). Define

\[
d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.
\]

Define \( T \) on \( X \) by \( T(0, 0) = (1, 2), T(0, 1) = (0, 0), T(1, 1) = (1, 1), T(1, 2) = (0, 1) \).

Then \( T \) is fundamentally nonexpansive, but \( T \) dose not satisfy condition \( C \).

In 2013, Salahifard et al. \([7]\), introduced the fundamentally non-expansive mappings in complete \( CAT(0) \) space and proved for some theorems as follows,

**Theorem 1.4.** Let \( K \) be a bounded closed convex subset of complete \( CAT(0) \) space \( X \). Let \( T : K \rightarrow K \) be fundamentally non-expansive and \( F(T) \neq \emptyset \), then \( F(T) \) is \( \triangle \)-closed and convex.

Throughout this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach \([3]\). A hyperbolic space is a metric space \((X, d)\) with a mapping \( W : X^2 \times [0, 1] \rightarrow X \) satisfying the following conditions.

\[
\begin{align*}
(i) & \quad d(u, W(x, y, \alpha)) \leq (1 - \alpha)d(u, x) + \alpha d(u, y); \\
(ii) & \quad d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|d(x, y); \\
(iii) & \quad W(x, x, \alpha) = W(y, x, 1 - \alpha); \\
(iv) & \quad d(W(x, z, \alpha), W(y, w, \alpha)) \leq |\alpha - \beta|d(x, y) + \alpha d(z, w)
\end{align*}
\]

for all \( x, y, z, w \in X \) and \( \alpha, \beta \in [0, 1] \).

**Example 1.5.** Let \( X \) be a real Banach space which is equipped with norm \( || \cdot || \). Define the function \( d : X^2 \rightarrow [0, \infty) \) by

\[
d(x, y) = ||x - y||
\]

as a meter on \( X \). Let \( K \) be a nonempty bounded closed convex subset of Banach space. We see that \((X, d)\) is a hyperbolic space with mapping \( W : X^2 \times [0, 1] \rightarrow X \) which is defined by

\[
W(x, y, \alpha) = (1 - \alpha)x + \alpha y.
\]
Definition 1.6 ([3],[4],[6]). Let $X$ be a hyperbolic space with a mapping $W : X^2 \times [0, 1] \to X$.

(i) A nonempty subset $K \subseteq X$ is said to be convex, if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$.

(ii) A hyperbolic space is said to be uniformly convex if for any $r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $u, x, y \in X$
\[
d(W(x, y, \frac{1}{2}), u) \leq (1 - \delta)r,
\]
provided $d(x, u) \leq r, d(y, u) \leq r$ and $d(x, y) \geq \epsilon r$.

(iii) A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\epsilon \in (0, 2]$, is known as a modulus of uniform convexity of $X$. $\eta$ is said to be monotone, if it decreases with $r$ (for a fixed $\epsilon$), i.e., $\forall \epsilon > 0$, $\forall r_1 \geq r_2 > 0$ $[\eta(r_2, \epsilon) \leq \eta(r_1, \epsilon)]$.

Definition 1.7. Let $(X, d)$ be a metric space and let $K$ be a nonempty subset of $X$. We shall denote the fixed point set of a mapping $T$ by $F(T) = \{x \in K : Tx = x\}$.

Definition 1.8. Let $\{x_n\}$ be a bounded sequence in a hyperbolic space $(X, d)$. For $x \in X$, we define a continuous functional $r(\cdot, x_n) : X \to [0, \infty)$ by
\[
r(x, x_n) = \limsup_{n \to \infty} d(x, x_n).
\]
The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by
\[
r(\{x_n\}) = \inf \{r(x, x_n) : x \in X\}.
\]
The asymptotic center $A_K(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to $K \subseteq X$ is the set
\[
A_K(\{x_n\}) = \{x \in X : r(x, x_n) \leq r(y, x_n), \forall y \in K\}.
\]
This implies that the asymptotic center is the set of minimizer of the functional $r(\cdot, x_n)$ in $K$. If the asymptotic center is taken with respect to $X$, then it is simply denoted by $A_K(\{x_n\})$. It is known that uniformly convex Banach spaces and CAT(0) spaces enjoy the property that bounded sequences have unique asymptotic centers with respect to closed convex subsets.

Lemma 1.9 ([1],[5]). Let $(X, d, W)$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity $\eta$. Then every bounded sequence $\{x_n\}$ in $K$ has a unique asymptotic center in $K$.

Lemma 1.10 ([1]). Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying
\[
a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \forall n > 1.
\]  
If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \to \infty} a_n$ exists.
If there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $\lim_{n \to \infty} a_n = 0$.

Lemma 1.11 ([1]). Let $(X, d, W)$ be a uniformly convex hyperbolic space with monotone modulus of uniform convexity $\eta$. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that
\[
\limsup_{n \to \infty} d(x_n, x) \leq c, \quad \limsup_{n \to \infty} d(y_n, x) \leq c \quad \text{and} \quad \limsup_{n \to \infty} d(W(x_n, y_n, \alpha_n), x) = c
\]
for some $c \geq 0$. Then $\lim_{n \to \infty} d(x_n, y_n) = 0$.

In this paper, we prove some properties of the fixed point set of fundamentally non-expansive mappings and derive the existence of fixed point theorems as follows results of Salahifard et al. [7] in hyperbolic spaces.
2. Main results

In this section, we shall prove some lemmas for fundamentally non-expansive mappings in a hyperbolic space.

**Definition 2.1.** Let $X$ be a hyperbolic space and $K$ be a nonempty bounded closed strictly convex subset of $X$. A mapping $T : K \to K$ is said to be fundamentally non-expansive if
\[
  d(T^2x, Ty) \leq d(Tx, y)
\]
for all $x, y \in K$.

**Lemma 2.2.** Let $K$ be a nonempty bounded closed strictly convex subset of complete hyperbolic space $X$. Let $T : K \to K$ be fundamentally non-expansive and $F(T) \neq \emptyset$, then $F(T)$ is $\triangle$-closed and convex.

*Proof.* Suppose that $\{x_n\}$ is a sequence in $F(T)$ which $\triangle$-converges to some $y \in K$. To show that $y \in F(T)$, we write
\[
d(x_n, Ty) = d(T^2x_n, Ty) \leq d(Tx_n, y) = d(x_n, y),
\]
thus
\[
\limsup_{n \to \infty} d(x_n, Ty) \leq \limsup_{n \to \infty} d(x_n, y).
\]
By the uniqueness of asymptotic center, we get $Ty = y$. Hence $F(T)$ is closed.

Next, we will show that $F(T)$ is convex. Let $x, y \in F(T)$ and each $\alpha \in [0, 1]$. Then,
\[
d(x, Tz) = d(T^2x, Tz) \leq d(Tx, z) = d(x, z)
\]
and
\[
d(y, Tz) = d(T^2y, Tz) \leq d(Ty, z) = d(y, z).
\]
For $z = W(x, y, \alpha)$, we have
\[
d(x, y) \leq d(x, Tz) + d(Tz, y)
\]
\[
\leq d(x, z) + d(z, y)
\]
\[
= d(x, W(x, y, \alpha)) + d(W(x, y, \alpha), y)
\]
\[
\leq (1 - \alpha)d(x, z) + \alpha d(x, y) + (1 - \alpha)d(x, y) + \alpha d(y, y)
\]
\[
= d(x, y).
\]
Thus $d(x, Tz) = d(x, z)$ and $d(Tz, y) = d(z, y)$, because if $d(x, Tz) < d(x, z)$ or $d(Tz, y) < d(z, y)$, then which the contradiction to $d(x, y) < d(x, y)$, therefore $Tz = W(x, y, \alpha)$ and $Tz = z$, and then $W(x, y, \alpha) \in F(T)$. Hence $F(T)$ is convex.

\[\square\]

**Lemma 2.3.** Let $K$ be a nonempty bounded closed subset of complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$. Let $T : K \to K$ be fundamentally non-expansive, then $F(T)$ is nonempty.

*Proof.* By Lemma 1.9, the asymptotic center of any bounded sequence is in $K$, particularly, the asymptotic center of approximate fixed point sequence for $T$ is in $K$. Let $A(\{x_n\}) = \{y\}$, we want to show that $y$ is a fixed point of $T$. We can consider
\[
d(x_n, Ty) \leq d(T^2x_n, Ty) \leq d(Tx_n, y) = d(x_n, y),
\]

\[
\limsup_{n \to \infty} d(x_n, Ty) \leq \limsup_{n \to \infty} d(x_n, y).
\]
By the uniqueness of the asymptotic center $Ty = y$.

\[\square\]
Theorem 2.4. Let $K$ be a nonempty bounded closed strictly convex subset of complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$. Let $T : K \to K$ be fundamentally non-expansive, then $F(T)$ is nonempty $\triangle$-closed and convex.

Proof. By Lemmas 2.2 and 2.3 we get that $F(T)$ is nonempty $\triangle$-closed and convex. \hfill \Box

Acknowledgements

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