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Numerical solution of fractional bioheat equation by quadratic spline collocation method

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Abstract

Based on the quadratic spline function, a quadratic spline collocation method is presented for the time fractional bioheat equation governing the process of heat transfer in tissues during the thermal therapy. The corresponding linear system is given. The stability and convergence are analyzed. Some numerical examples are given to demonstrate the efficiency of this method. ©2016 All rights reserved.

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1. Introduction

Hyperthermia treatment is a promising approach to cancer therapy in recent years. It has been proven the combination of hyperthermia and chemotherapy/radiotherapy can greatly improve clinical effect in various randomized trials [5]. The inevitable technical problem of hyperthermia is a difficult issue in heating only the local tumor region to the intended temperature without damaging the surrounding healthy tissue. Consequently, improvement and accurate simulation of mathematics models of hear transfer are becoming more and more important.

Recently, fractional differential equations have gained much attention and appreciation due to their particular efficiencies in describing some phenomena in fluid mechanics, biology, physics, viscoelasticity, and engineering [1, 4, 11, 12, 15, 21, 22, 29]. The main advantage of fractional order systems is that they allow greater degrees of freedom in the model. So fractals and fractional calculus have been used to improve the

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modeling accuracy of many phenomena in natural sciences [10, 14, 18, 24, 26, 27, 31–33]. Pennes' bioheat equation is a widely used model for studying the bioheat transfer model nowadays. In this paper, we consider a fractional differential form of heat transfer model by replacing time derivative with the Caputo one.

Up to now, many analytic techniques and numerical methods have been presented for the solution of Pennes' equation. Yue et al. [30] obtained an analytic solution of one-dimensional steady-state Pennes' bioheat transfer equation in cylindrical coordinates. Gupta et al. [9] gave some approximate analytic solutions of Pennes' bioheat equation in thermal therapy. Ooi et al. [19] presented a boundary element model of the human eye undergoing laser thermokeratoplasty. Xiao-Zhou et al. [28] gave estimation of temperature elevation generated by ultrasonic irradiation in biological tissues. Recently, Singh et al. [23] gave the solution by finite difference and homotopy perturbation method for the fractional bioheat equation, Damor et al. [7] obtained numerical simulation of fractional bioheat equation in hyperthermia treatment. In all of these presented methods, we noticed there is a lack of theoretical analysis about the convergence and stability.

In this paper, we present a new numerical scheme by quadratic spline collocation (QSC) methods for the fractional bioheat equation. Spline collocation methods have been used to solve the boundary problems for ordinary and partial differential equations over the past several decades [2]. But the methods based on basic smoothest spline collocation are not of optimal accuracy. The optimal QSC method was derived by Christara [6], which was globally optimal in L^∞ and superconvergent. The approximate solution is fourth-order accurate at the nodes. In 2008, Bialecki et al. [3] developed the QSC methods and presented a new QSC method for the Helmholtz equation with homogeneous Dirichlet boundary conditions. Later, Fairweather et al. [8] extended this method to the solutions of Helmholtz equation with non-homogeneous Dirichlet, Neumann and mixed boundary conditions. Luo et al. [17] studied the QSC method and efficient preconditioner for the Helmholtz equation with Robbins boundary conditions. Lakestani et al. [13] developed a collocation and finite difference-collocation methods for the solution of nonlinear Klein-Gordon equation. In these literatures, the studied models covering the QSC methods are mostly related to the integer order equations. Recently, Luo et al. [16] exploited the QSC method to solve the time fractional subdiffusion equation with Dirichlet boundary value conditions, where the authors established a novel collocation method via taking the quadratic spline polynomials as basic functions, and found the proposed technique can enjoy the global error bound of $O(\tau^3 + h^3)$ and fourth-order accuracy at collocation points. To avoid the singular matrix, the initial time collocation point τ_1 is replaced by an extra parameter $\theta \in (0, \frac{1}{2})$ in [16].

The contribution of the present paper is to extend the QSC method [16] to solve the fractional bioheat equation with mixed boundary value conditions for thermal therapy. This new introduced method does not need any extra parameter, and the obtained matrix is still nonsingular. The existence and uniqueness of solution are analyzed. The convergence and the stability of the new QSC method are discussed. The theoretical analyses and numerical computation demonstrate the proposed technique can enjoy the local error bound with $O(h^4 + \tau^4)$ at collocation points, and the global error bound can achieve the accuracy of $O(h^3 + \tau^3)$ under the L_∞ norm. Over the entire affected region, the temperature profile is given for different values of α.

2. Heat transfer model

For the one-dimensional time fractional Pennes' bioheat transfer equation

$$\begin{cases} \rho c \frac{\partial^\alpha T}{\partial t^\alpha} = K \frac{\partial^2 T}{\partial r^2} + W_b C_b (T_a - T) + q_m + q_s, \quad 0 < \alpha \leq 1, \\ T(r, 0) = T_0, \\ \frac{\partial T(r,t)}{\partial r}\big|_{r=0} = 0, \\ T(r, t)|_{r=R} = T_w, \end{cases} \tag{2.1}$$

where $T = T(r, t)$ is the local tissue temperature, r is the space coordinate, t is time, ρ, c, K represent density, specific heat and thermal conductivity, respectively, and the subscript b is used for blood. T_a stands for the arterial blood temperature which is taken constant. W_b stands for the blood perfusion rate. T_w and L represent the tissue wall temperature and the tissue length, respectively. The metabolic heat generation is given by

$$q_m = q_{m0}(1 + d\theta),$$

where $d = 0.1T_0$, and $q_{m0} = q_{00}[1 + 0.1(T_0 - 37)]$. The q_s represents the heat generated per unit volume of tissue due to the heat source and it is given by

$$q_s = \rho SPe^{a(\overline{r}-0.01)},$$

where S and a are antenna constants, P is the transmitted power, and $\overline{r} = R - r$ is the distance of tissue from outer surface.

The $\frac{\partial^\alpha T}{\partial t^\alpha}$ is the αth-order Caputo derivative [20] operator of the form

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial T(x,s)}{\partial t}(t-s)^{-\alpha} ds, & \text{for } 0 < \alpha < 1, \\ \frac{\partial T(x,t)}{\partial t}, & \text{for } \alpha = 1. \end{cases} \tag{2.2}$$

For convenience, we introduce the following variables.

$$\begin{cases} x = \frac{r}{R}, t* = (\frac{K}{\rho CR^2})^{\frac{1}{\alpha}} t, \theta = \frac{T-T_0}{T_0}, \theta_a = \frac{T_a-T_0}{T_0}, \theta_w = \frac{T_w-T_0}{T_0}, \\ P_m = \frac{q_{m0}}{T_0K}R^2, P_f = \sqrt{\frac{W_bC_b}{K}}R, P_r = \frac{\rho SP}{T_0K}R^2, a_0 = 0.04a, b_0 = aR. \end{cases}$$

Then Eq. (2.1) reads

$$\begin{cases} \frac{\partial^\alpha \theta(x,t*)}{\partial t*^\alpha} = c_1 \frac{\partial^2\theta(x,t*)}{\partial x^2} + c_2\theta(x,t*) + f(x,t*), \quad 0 < \alpha \le 1, \\ \theta(x,0) = \phi(x), \\ \frac{\partial\theta(x,t*)}{\partial x}|_{x=0} = \varphi(t*), \\ \theta(x,t*)|_{x=1} = \psi(t*), \end{cases} \tag{2.3}$$

where $c_1 = 1$, $c_2 = (P_m d - P_f^2)$, $\phi(x) = 0$, $\varphi(t*) = 0$, $\psi(t*) = \theta_w$, and $f(x,t*) = P_m + P_f^2\theta_a + P_r \exp(a_0 - b_0x)$.

To facilitate the narrative, we substitute t for $t*$ in (2.3), then

$$\begin{cases} \frac{\partial^\alpha \theta(x,t)}{\partial t^\alpha} = c_1 \frac{\partial^2\theta(x,t)}{\partial x^2} + c_2\theta(x,t) + f(x,t), \quad 0 < \alpha \le 1, \\ \theta(x,0) = \phi(x), \\ \frac{\partial\theta(x,t)}{\partial x}|_{x=0} = \varphi(t), \\ \theta(x,t)|_{x=1} = \psi(t). \end{cases} \tag{2.4}$$

In order to give general enough scheme, we do not limit to $\phi(x) = 0, \varphi(t) = 0$ and $\psi(t) = \theta_w$ in the following sections.

3. Preliminaries

In this section, we introduce some notations and results of biquadratic spline interpolation. Define $x \in [0,1], t \in [0, t_{end}], h = \frac{1}{N_h}, \tau = \frac{t_{end}}{N_t}$,

$$x_i = ih, \ t_j = j\tau, \ i = 0,1,\cdots,N_h, \ j = 0,1,\cdots,N_t,$$

and $\varrho_h = \{x_i\}_{i=0}^{N_h}, \varrho_t = \{t_j\}_{j=0}^{N_t}$.

Let the collocation points $\{(\eta_i, \tau_j)\}$, $i = 1,2,\cdots,N_h$, $j = 1,2,\cdots,N_t$ in $(0,1) \times (0, t_{end})$ be the center of each gridding cell, i.e.

$$\eta_i = \frac{x_{i-1} + x_i}{2}, \ i = 1,2,\cdots,N_h, \ \tau_j = \frac{t_{j-1} + t_j}{2}, \ j = 1,2,\cdots,N_t,$$

and

$$\eta_0 = 0,\ \eta_{N_h+1} = 1,\ \tau_0 = 0,\ \tau_{N_t+1} = t_{end}.$$

Thus, the collocation points in $[0,1]\times[0,t_{end}]$ are taken to be $Q = \{(\eta_i,\tau_j)\}$, $i = 0,1,\cdots,N_h+1$, $j = 0,1,\cdots,N_t+1$.

Take the basis of S_2 to be $\{D_n\}_{n=0}^{N}$, $N = N_h+1$ or $N = N_t+1$, where

$$D_n(x) = \tfrac{1}{2}\xi(\tfrac{x}{\tau} - n + 2), \quad n = 0,1,2,\cdots,N_t+1, \tag{3.1}$$

and ξ is the quadratic spline defined by

$$\xi(x) = \begin{cases} x^2, & x\in[0,1],\\ -3+6x-2x^2, & x\in[1,2],\\ 9-6x+x^2, & x\in[2,3],\\ 0, & \text{otherwise}. \end{cases} \tag{3.2}$$

From Eqs. (3.1) and (3.2), it follows that, for all $i = 1,2,\cdots,N$,

$$D_n(\eta_i) = \begin{cases} \frac{1}{8}, & i = n\pm 1,\\ \frac{3}{4}, & i = n,\\ 0, & \text{otherwise}, \end{cases} \quad D'_n(\eta_i) = \begin{cases} \mp\frac{1}{2h}, & i = n\pm 1,\\ 0, & \text{otherwise}, \end{cases} \quad D''_n(\eta_i) = \begin{cases} \frac{1}{h^2}, & i = n\pm 1,\\ \frac{-2}{h^2}, & i = n,\\ 0, & \text{otherwise}. \end{cases}$$

Let $\theta_s(x,t)\in S_2$ be the biquadratic spline interpolant of the function $\theta(x,t)$, which was introduced in [6, 8], so that

$$\theta_s(\eta_i,\tau_j) = \begin{cases} \theta(\eta_i,\tau_j), & i = 1,2,\cdots,N_h, j = 1,2,\cdots,N_t;\\ \theta(\eta_i,\tau_j) - \frac{h^4}{128}\frac{\partial^4\theta}{\partial x^4}(\eta_i,\tau_j), & i = 0, N_h+1; j = 1,2,\cdots,N_t;\\ \theta(\eta_i,\tau_j) - \frac{\tau^4}{128}\frac{\partial^4\theta}{\partial t^4}(\eta_i,\tau_j), & i = 1,2,\cdots,N_h, j = 0, N_t+1;\\ \theta(\eta_i,\tau_j) - \frac{h^4}{128}\frac{\partial^4\theta}{\partial x^4}(\eta_i,\tau_j), & i = 0, N_h+1; j = 0, N_t+1. \end{cases} \tag{3.3}$$

Suppose the function $\theta(x,t)$ is smooth enough, this interpolant $\theta_s(x,t)$ satisfies

$$\frac{\partial^2\theta_s}{\partial x^2}(\eta_i,\tau_j) = \frac{\partial^2\theta}{\partial x^2}(\eta_i,\tau_j) - \frac{h^2}{24}\frac{\partial^4\theta}{\partial x^4}(\eta_i,\tau_j) + O(h^4) \tag{3.4}$$

for all $i = 1,2,\cdots,N_h, j = 1,2,\cdots,N_t$.

4. The QSC method

In this section, we introduce the QSC method for the model (2.4), with the basis

$$\{B_m(x)\}_{m=0}^{N_h+1} = \{D_0(x), D_0(x)+D_1(x), D_2(x),\cdots, D_{N_h}(x) - D_{N_h+1}(x), D_{N_h+1}(x)\}$$

and $\{D_n(t)\}_{n=0}^{N_t+1} = \{D_0(t), D_0(t)-D_1(t),\cdots,D_{N_h}(t), D_{N_h+1}(t)\}$ for S_2. Substituting Eq. (2.2) into Eq. (2.4) we get

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial\theta(x,s)}{\partial s}ds - c_1\frac{\partial^2\theta(x,t)}{\partial x^2} - c_2\theta(x,t) = f(x,t), \tag{4.1}$$

from which we can immediately obtain

$$c_1\frac{\partial^4\theta(x,t)}{\partial x^4} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial^3\theta(x,s)}{\partial x^2\partial s}ds - c_2\frac{\partial^2\theta(x,t)}{\partial x^2} - \frac{\partial^2 f(x,t)}{\partial x^2}. \tag{4.2}$$

Using Eqs. (3.4) and (4.2) leads to

$$c_1\frac{\partial^4\theta(x,t)}{\partial x^4} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial^3\theta_s(x,s)}{\partial x^2\partial s}ds - c_2\frac{\partial^2\theta_s(x,t)}{\partial x^2} - \frac{\partial^2 f(x,t)}{\partial x^2} + O(h^2). \tag{4.3}$$

Using Eqs. (3.3), (3.4), and (4.1) becomes

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial \theta_s}{\partial s}(\eta_i,s)ds - c_1\frac{\partial^2\theta_s}{\partial x^2}(\eta_i,t) - \frac{c_1h^2}{24}\frac{\partial^4\theta}{\partial x^4}(\eta_i,t) - c_2\theta_s(\eta_i,t)ds = f(\eta_i,t)+O(h^4). \quad (4.4)$$

As a result, from Eq. (4.3), Eq. (4.4) can be reduced as

$$\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial \theta_s}{\partial s}(\eta_i,s)ds - \frac{h^2}{24\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial^3\theta_s}{\partial x^2\partial s}(\eta_i,s)ds \\
&\quad -(c_1-\frac{c_2h^2}{24})\frac{\partial^2\theta_s}{\partial x^2}(\eta_i,t) - c_2\theta_s(\eta_i,t) \\
&= -\frac{h^2}{24}\frac{\partial^2 f}{\partial x^2}(\eta_i,t)+f(\eta_i,t)+O(h^4).
\end{aligned} \quad (4.5)$$

Denoting

$$\begin{aligned}
\Phi_1 &= \frac{1}{\Gamma(1-\alpha)}\int_0^{\tau_j}(\tau_j-s)^{-\alpha}\frac{\partial\theta_h}{\partial s}(\eta_i,s)ds,\\
\Phi_2 &= -\frac{h^2}{24\Gamma(1-\alpha)}\int_0^{\tau_j}(\tau_j-s)^{-\alpha}\frac{\partial^3\theta_h}{\partial x^2\partial s}(\eta_i,s)ds,\\
\Phi_3 &= -(c_1-\frac{c_2h^2}{24})\frac{\partial^2\theta_h}{\partial x^2}(\eta_i,\tau_j),\\
\Phi_4 &= -c_2\theta_h(\eta_i,\tau_j),
\end{aligned} \quad (4.6)$$

where $\theta_h(x,t)$ is the approximation of $\theta(x,t)$, and

$$\theta_h(x,t)=\sum_{m=0}^{N_h+1}\sum_{n=0}^{N_t+1} v_{m,n}B_m(x)D_n(t), \quad (4.7)$$

the collocation formulation for (2.4) can be written as

$$\begin{cases}
\sum_{i=1}^{4}\Phi_i = -\frac{h^2}{24}\frac{\partial^2 f}{\partial x^2}(\eta_i,\tau_j)+f(\eta_i,\tau_j),\ i=1,2,\cdots,N_h,\ j=1,2,\cdots,N_t+1,\\
\frac{\partial\theta_h(x,\tau_j)}{\partial x}|_{x=0}=\varphi(\tau_j),\theta_h(1,\tau_j)=\psi(\tau_j),\ j=0,1,\cdots,N_t+1,\\
\theta(\eta_i,0)=\phi(\eta_i)\ i=0,1,\cdots,N_h+1.
\end{cases} \quad (4.8)$$

Using the boundary value conditions $\frac{\partial\theta(x,t)}{\partial x}|_{x=0}=\varphi(t),\theta(1,t)=\psi(t)$, we get

$$-\frac{1}{h}\boldsymbol{A}\boldsymbol{v}_0=\boldsymbol{\varphi},\ \ \frac{1}{2}\boldsymbol{A}\boldsymbol{v}_{N_h+1}=\boldsymbol{\psi},$$

where

$$\boldsymbol{A}=\begin{pmatrix} D_0(\tau_0) & D_1(\tau_0) & \cdots & D_{N_t+1}(\tau_0)\\ D_0(\tau_1) & D_1(\tau_1) & \cdots & D_{N_t+1}(\tau_1)\\ \vdots & \vdots & \vdots & \vdots\\ D_0(\tau_{N_t+1}) & D_1(\tau_{N_t+1}) & \cdots & D_{N_t+1}(\tau_{N_t+1})\end{pmatrix} = \frac{1}{8}\begin{pmatrix} 4 & & & & & \\ 1 & 5 & 1 & & & \\ & 1 & 6 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 6 & 1\\ & & & & 4 & 4\end{pmatrix},$$

and

$$\boldsymbol{v}_0=(v_{0,0},v_{0,1},\cdots,v_{0,N_t+1})^T, \qquad \boldsymbol{v}_{N_h+1}=(v_{N_h+1,0},v_{N_h+1,1},\cdots,v_{N_h+1,N_t+1})^T,$$

$$\boldsymbol{\varphi}=(\varphi(\tau_0),\varphi(\tau_1),\cdots,\varphi(\tau_{N_t+1}))^T, \qquad \boldsymbol{\psi}=(\psi(\tau_0),\psi(\tau_1),\cdots,\psi(\tau_{N_t+1}))^T.$$

The initial condition $\theta(x,0)=\phi(x)$ gives

$$\tfrac{1}{2}\boldsymbol{B}\boldsymbol{v}^0=\boldsymbol{\phi}-v_{0,0}\boldsymbol{B}_0-v_{N_h+1,0}\boldsymbol{B}_{N_h+1}, \tag{4.9}$$

where

$$\boldsymbol{B}=\frac{1}{8}\begin{pmatrix}7&1&&&\\1&6&1&&\\&\ddots&\ddots&\ddots&\\&&1&6&1\\&&&1&5\end{pmatrix},\boldsymbol{B}_0=\frac{1}{8}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix},\boldsymbol{B}_{N_h+1}=\frac{1}{8}\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix},$$

and $\boldsymbol{v}^0=(v_{1,0},v_{2,0},\cdots,v_{N_h,0})^T$, $\boldsymbol{\phi}=(\phi(\eta_1),\phi(\eta_2),\cdots,\phi(\eta_{N_h}))^T$. Substituting Eq. (4.7) into Eq. (4.6) we have

$$\begin{aligned}
\Phi_1&=\frac{1}{\Gamma(1-\alpha)}\sum_{m=0}^{N_h+1}B_m(\eta_i)\sum_{n=0}^{N_t+1}\int_0^{\tau_j}(\tau_j-s)^{-\alpha}\frac{\partial D_n(s)}{\partial s}dsv_{m,n}\\
&=\frac{1}{\Gamma(1-\alpha)}[(\boldsymbol{B}\bigotimes\widetilde{\boldsymbol{PP}})\boldsymbol{v}+(\boldsymbol{B}\bigotimes\widetilde{\boldsymbol{P}}_0)\boldsymbol{v}^0+(\boldsymbol{B}_0\bigotimes\widetilde{\boldsymbol{P}})\boldsymbol{v}_0+(\boldsymbol{B}_{N_h+1}\bigotimes\widetilde{\boldsymbol{P}})\boldsymbol{v}_{N_h+1}],\\
\Phi_2&=-\frac{h^2}{24\Gamma(1-\alpha)}\sum_{m=0}^{N_h+1}\frac{\partial^2B_m}{\partial x^2}(\eta_i)\sum_{n=0}^{N_t+1}\int_0^{\tau_j}(\tau_j-s)^{-\alpha}\frac{\partial D_n(s)}{\partial s}dsv_{m,n}\\
&=-\frac{h^2}{24\Gamma(1-\alpha)}[(\widetilde{\boldsymbol{B}}\bigotimes\widetilde{\boldsymbol{PP}})\boldsymbol{v}+(\widetilde{\boldsymbol{B}}\bigotimes\widetilde{\boldsymbol{P}}_0)\boldsymbol{v}^0+(\widetilde{\boldsymbol{B}}_0\bigotimes\widetilde{\boldsymbol{P}})\boldsymbol{v}_0+(\widetilde{\boldsymbol{B}}_{N_h+1}\bigotimes\widetilde{\boldsymbol{P}})\boldsymbol{v}_{N_h+1}],\\
\Phi_3&=(\frac{c_2h^2}{24}-c_1)\sum_{m=0}^{N_h+1}\frac{\partial^2B_m}{\partial x^2}(\eta_i)\sum_{n=0}^{N_t+1}D_n(\tau_j)v_{m,n}\\
&=(\frac{c_2h^2}{24}-c_1)[(\widetilde{\boldsymbol{B}}\bigotimes\boldsymbol{DD})\boldsymbol{v}+(\widetilde{\boldsymbol{B}}\bigotimes\boldsymbol{D}_0)\boldsymbol{v}^0+(\widetilde{\boldsymbol{B}}_0\bigotimes\boldsymbol{D})\boldsymbol{v}_0+(\widetilde{\boldsymbol{B}}_{N_h+1}\bigotimes\boldsymbol{D})\boldsymbol{v}_{N_h+1}],\\
\Phi_4&=-c_2\sum_{m=0}^{N_h+1}B_m(\eta_i)\sum_{n=0}^{N_t+1}D_n(\tau_j)v_{m,n}\\
&=-c_2[(\boldsymbol{B}\bigotimes\boldsymbol{DD})\boldsymbol{v}+(\boldsymbol{B}\bigotimes\boldsymbol{D}_0)\boldsymbol{v}^0+(\boldsymbol{B}_0\bigotimes\boldsymbol{D})\boldsymbol{v}_0+(\boldsymbol{B}_{N_h+1}\bigotimes\boldsymbol{D})\boldsymbol{v}_{N_h+1}].
\end{aligned}$$

where

$$\widetilde{\boldsymbol{B}}=\frac{1}{h^2}\begin{pmatrix}-1&1&&&\\1&-2&1&&\\&\ddots&\ddots&\ddots&\\&&1&-2&1\\&&&1&-3\end{pmatrix},\widetilde{\boldsymbol{B}}_0=\frac{1}{h^2}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix},\widetilde{\boldsymbol{B}}_{N_h+1}=\frac{1}{h^2}\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix},$$

$$\boldsymbol{D}=\frac{1}{8}\begin{pmatrix}1&5&1&&&\\&1&6&1&&\\&&\ddots&\ddots&\ddots&\\&&&1&6&1\\&&&&4&4\end{pmatrix},\boldsymbol{DD}=\frac{1}{8}\begin{pmatrix}5&1&&&\\1&6&1&&\\&\ddots&\ddots&\ddots&\\&&1&6&1\\&&&4&4\end{pmatrix},\boldsymbol{D}_0=\frac{1}{8}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix},$$

and

$$\begin{aligned}
\boldsymbol{v}&=(\boldsymbol{v}_1,\boldsymbol{v}_2,\cdots,\boldsymbol{v}_{N_h})^T,\ \boldsymbol{v}_i=(v_{i,1},v_{i,2},\cdots,v_{i,N_t+2})^T,\ i=1,2,\cdots,N_h,\\
\widetilde{P}_{i,j}&=\int_0^{\tau_i}(\tau_i-s)^{-\beta}\frac{\partial D_j(s)}{\partial s}ds,\ i=1,2,\cdots,N_t+1,\ j=0,1,\cdots,N_t+1,\\
\widetilde{\boldsymbol{P}}&=(\widetilde{\boldsymbol{P}}_0,\widetilde{\boldsymbol{PP}}),\ \widetilde{\boldsymbol{PP}}=(\widetilde{\boldsymbol{P}}_1,\cdots,\widetilde{\boldsymbol{P}}_{N_t+1}),\widetilde{\boldsymbol{P}}_j=(\widetilde{P}_{1,j},\widetilde{P}_{2,j},\cdots,\widetilde{P}_{N_t+1,j})^T,\ j=0,1,\cdots,N_t+1.
\end{aligned}$$

Noting

$$\boldsymbol{H} = \frac{1}{\Gamma(1-\alpha)}(\boldsymbol{B}\bigotimes \widetilde{\boldsymbol{PP}}) - \frac{h^2}{24\Gamma(1-\alpha)}(\widetilde{\boldsymbol{B}}\bigotimes \widetilde{\boldsymbol{PP}}) + (\frac{c_2h^2}{24} - c_1)(\widetilde{\boldsymbol{B}}\bigotimes \boldsymbol{DD}) - c_2(B\bigotimes \boldsymbol{DD}),$$

and

$$g_1(j+(i-1)(N_t+1)) = -\frac{h^2}{24}\frac{\partial^2 f}{\partial x^2}(\eta_i,\tau_j) + f(\eta_i,\tau_j),\ i=1,2,\cdots,N_h,\ \ j=1,2,\cdots,N_t+1,$$

$$\begin{aligned}\boldsymbol{g}_2 = &-\frac{1}{\Gamma(1-\alpha)}[(\boldsymbol{B}\bigotimes \widetilde{\boldsymbol{P}}_0)\boldsymbol{v}^0 + (\boldsymbol{B}_0\bigotimes \widetilde{\boldsymbol{P}})\boldsymbol{v}_0 + (\boldsymbol{B}_{N_h+1}\bigotimes \widetilde{\boldsymbol{P}})\boldsymbol{v}_{N_h+1}] \\ &+ \frac{h^2}{24\Gamma(1-\alpha)}[(\widetilde{\boldsymbol{B}}\bigotimes \widetilde{\boldsymbol{P}}_0)\boldsymbol{v}^0 + (\widetilde{\boldsymbol{B}}_0\bigotimes \widetilde{\boldsymbol{P}})\boldsymbol{v}_0 + (\widetilde{\boldsymbol{B}}_{N_h+1}\bigotimes \widetilde{\boldsymbol{P}})\boldsymbol{v}_{N_h+1}] \\ &- (\frac{c_2h^2}{24} - c_1)[(\widetilde{\boldsymbol{B}}\bigotimes \boldsymbol{D}_0)\boldsymbol{v}^0 + (\widetilde{\boldsymbol{B}}_0\bigotimes \boldsymbol{D})\boldsymbol{v}_0 + (\widetilde{\boldsymbol{B}}_{N_h+1}\bigotimes \boldsymbol{D})\boldsymbol{v}_{N_h+1}] \\ &+ c_2[(\boldsymbol{B}\bigotimes \boldsymbol{D}_0)\boldsymbol{v}^0 + (\boldsymbol{B}_0\bigotimes \boldsymbol{D})\boldsymbol{v}_0 + (\boldsymbol{B}_{N_h+1}\bigotimes \boldsymbol{D})\boldsymbol{v}_{N_h+1}],\end{aligned}$$

Eq. (4.8) reads

$$\boldsymbol{Hv} = \boldsymbol{g}_1 + \boldsymbol{g}_2. \tag{4.10}$$

Eq. (4.10) is a linear system. According to Eqs. (4.6), (4.8) and (4.10), we can compute $v_{m,n}$ for all $m = 0,1,\cdots,N_h+1, n = 0,1,\cdots,N_t+1$, which yields $\theta_h(\eta_i,\tau_j)$ for all $i = 0,1,\cdots,N_h+1, j = 0,1,\cdots,N_t+1$.

5. Convergence and stability

In this section, we discuss the existence and uniqueness of the solution of the new QSC scheme, and give the error estimation and stability of the numerical solution.

Lemma 5.1 ([16]). *Given $\theta(x,y) \in C^{4,4}_{x,y}[a,b]$, let $\wedge xy\theta(x,y)$ be the biquadratic spline interpolant of function $\theta(x,y)$ in the sense of*

$$\wedge xy\theta(\eta_i^x,\eta_j^y) = \theta(\eta_i^x,\eta_j^y), i,j = 0,1,\cdots,N+1,$$

with the collocation points, then

$$\| \wedge xy\theta(\eta_i^x,\eta_j^y) - \theta(\eta_i^x,\eta_j^y)\| = O(h^4), \|\theta - \wedge xy\theta\|_\infty = O(h^3),$$

where $\|\wedge xy\theta(\eta_i^x,\eta_j^y) - \theta(\eta_i^x,\eta_j^y)\| = max\{|\wedge xy\theta(\eta_i^x,\eta_j^y) - \theta(\eta_i^x,\eta_j^y)|, i,j = 0,1,\cdots,N+1\}$, and $\|\wedge xy\theta - \theta\|_\infty = max\{|\wedge xy\theta(x,y) - \theta(x,y)|,(x,y) \in [a,b]\times[a,b]\}$.

Theorem 5.2. *For sufficiently small h and τ, the collocation (4.8) has a unique solution $\theta_h(x,t)$, and for all $i = 0,1,\cdots,N_h+1, j = 0,1,\cdots,N_t+1$, the following error bound holds:*

$$\begin{aligned}\|\theta(\eta_i,\tau_j) - \theta_h(\eta_i,\tau_j)\| &= O(h^4+\tau^4), \\ \|\theta - \theta_h\|_\infty &= O(h^3+\tau^3).\end{aligned} \tag{5.1}$$

Proof. First, it is easy to prove

$$\boldsymbol{DD} = \begin{pmatrix} 5 & 1 & & & \\ 1 & 6 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 6 & 1 \\ & & & 4 & 4 \end{pmatrix}$$

and

$$\widetilde{\boldsymbol{BB}} = \begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -3 \end{pmatrix}$$

are invertible, which means $\widetilde{\boldsymbol{BB}} \bigotimes \boldsymbol{DD}$ is invertible, and $(\widetilde{\boldsymbol{BB}} \bigotimes \boldsymbol{DD})^{-1}$ is bounded. Let

$$\boldsymbol{Z} = (-c_1 \widetilde{\boldsymbol{B}} \bigotimes \boldsymbol{DD})^{-1} = h^2(-c_1 \widetilde{\boldsymbol{BB}} \bigotimes \boldsymbol{DD})^{-1},$$

then $\boldsymbol{Z} \to \boldsymbol{0}$, when $h \to 0$. Multiplying $\boldsymbol{H}$ in (4.10), we have

$$\boldsymbol{ZH} = \boldsymbol{I} + \frac{1}{\Gamma(1-\alpha)} \boldsymbol{Z}(\boldsymbol{B} \bigotimes \widetilde{\boldsymbol{PP}}) - \frac{h^2}{24\Gamma(1-\alpha)} \boldsymbol{Z}(\widetilde{\boldsymbol{B}} \bigotimes \widetilde{\boldsymbol{PP}}) + \frac{c_2 h^2}{24} \boldsymbol{Z}(\widetilde{\boldsymbol{B}} \bigotimes \boldsymbol{DD}) - c_2 \boldsymbol{Z}(\boldsymbol{B} \bigotimes \boldsymbol{DD}).$$

Obviously, $\boldsymbol{ZH}$ is a strictly diagonally dominant matrix, if h and τ are small enough. Therefore, $\boldsymbol{H}$ is a nonsingular matrix, $\|\boldsymbol{H}^{-1}\|$ is bounded, and Eq. (4.10) has a unique solution $\boldsymbol{v}$. Then Eq. (4.8) has a unique solution $\theta_h(x,t)$. Denoting

$$\begin{aligned} \ell(\theta) = {} & \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial \theta}{\partial s}(\eta_i, s) ds - \frac{h^2}{24\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial^3 \theta}{\partial x^2 \partial s}(\eta_i, s) ds \\ & - (c_1 - \frac{c_2 h^2}{24}) \frac{\partial^2 \theta}{\partial x^2}(\eta_i, t) - c_2 \theta(\eta_i, t), \end{aligned}$$

and subtracting Eq. (4.8) from Eq. (4.5), we can obtain

$$\ell(\theta_s - \theta_h) = O(h^4).$$

Using the boundedness of $\boldsymbol{H}^{-1}$ we get

$$\|\theta_s - \theta_h\|_\infty = O(h^4). \tag{5.2}$$

Let $\theta_s(x,y) = \wedge_{xy}\theta(x,y)$. Using the triangular inequality, Lemma 5.1 and Eq. (5.2) one can deduce Eq. (5.1). □

Theorem 5.3. *If the initial value function $\phi(x) \in C^4([a,b])$ is given, then the collocation scheme* (4.10) *is stable, and*

$$\|\boldsymbol{v} - \tilde{\boldsymbol{v}}\| \leq C\|\widetilde{\boldsymbol{g}}\|, \tag{5.3}$$

where $\boldsymbol{v}, \tilde{\boldsymbol{v}}$ are the numerical solutions of (4.10) *subject to the initial values $\phi(x), \widetilde{\phi}(x)$, respectively, C is a positive constant independent of τ, and $\tilde{\boldsymbol{g}}$ is defined as* (5.4).

Proof. Obviously,

$$\boldsymbol{B} = \frac{1}{8} \begin{pmatrix} 7 & 1 & & & \\ 1 & 6 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 6 & 1 \\ & & & 1 & 5 \end{pmatrix}$$

is invertible. Let $\phi(x) - \widetilde{\phi}(x) = e(x)$, then by Eq. (4.9), we get

$$\boldsymbol{v}^0 - \tilde{\boldsymbol{v}}^0 = 2\boldsymbol{B}^{-1}\boldsymbol{e},$$

where $\boldsymbol{e} = (e(\eta_1), e(\eta_2), \cdots, e(\eta_{Nh}))^T$.

Denoting

$$\widetilde{\boldsymbol{g}} = 2[-\frac{1}{\Gamma(1-\alpha)}(\boldsymbol{B}\bigotimes \widetilde{\boldsymbol{P}}_0) + \frac{h^2}{24\Gamma(1-\alpha)}(\widetilde{\boldsymbol{B}}\bigotimes \widetilde{\boldsymbol{P}}_0) - (\frac{c_2h^2}{24} - c_1)(\widetilde{\boldsymbol{B}}\bigotimes \boldsymbol{D}_0) + c_2(\boldsymbol{B}\bigotimes \boldsymbol{D}_0)]\boldsymbol{B}^{-1}\boldsymbol{e}, \quad (5.4)$$

and by Eq. (4.10), we have

$$\boldsymbol{H}(\boldsymbol{v} - \widetilde{\boldsymbol{v}}) = \widetilde{\boldsymbol{g}}.$$

Using the boundedness of $\|\boldsymbol{H}^{-1}\|$, we can obtain (5.3), then the scheme (4.10) is stable. □

6. Numerical computations

In this section, we give some numerical computations. First, we test the accuracy of the new QSC scheme (4.10) by two examples.

Example 6.1.

$$\theta(x,t) = (t^\mu - \Gamma(2-\alpha)/\pi^2)sin(\pi x), x \in [0,1], t \in [0,T], \mu = 2+\alpha,$$
$$f(x,t) = \frac{\Gamma(\mu+1)t^{\mu-\alpha}sin(\pi x)}{\Gamma(\mu+1-\alpha)} + (\pi^2-1)(t^\mu - \Gamma(2-\alpha)/\pi^2)sin(\pi x).$$

Example 6.2.

$$\theta(x,t) = sin(x)(t^2+2), x \in [0,1], t \in [0,T], c_1 = c_2 = 1, f(x,t) = \frac{2sin(x)t^{2-\alpha}}{\Gamma(1-\alpha)(1-\alpha)(2-\alpha)}.$$

In examples, we take the step length $h = \tau = \frac{1}{N}$, compute all the errors via using L_∞ norm, and denote by E_c and E_n the errors at all collocation points $\{(\eta_i, \tau_j)\}, i,j = 0,1,\cdots,N+1$ and all nodes $\{(x_i,t_j)\}, i,j = 0,1,\cdots,N$, respectively. The convergence rate is defined as

$$R_c = \log(E_c(N/2)/E_c(N))/\log(2), \; R_n = \log(E_n(N/2)/E_n(N))/\log(2).$$

Table 1: Results of Example 6.1.

α	N	E_c	R_c	E_n	R_n
0.9	8	4.2218e-5	–	4.0206e-4	–
0.9	16	2.6145e-6	4.01325	4.6427e-5	3.11438
0.9	32	1.6230e-7	4.00980	5.8239e-6	2.99491
0.9	64	1.7504e-8	3.212906	5.8239e-7	2.88422
0.5	8	5.2613e-5	–	5.7651e-4	–
0.5	16	4.7774e-6	3.46112	8.5295e-5	2.75681
0.5	32	9.9548e-7	2.26276	1.2178e-5	2.80818
0.5	64	2.0088e-7	2.30906	1.7212e-6	2.82280
0.2	8	5.7330e-5	–	6.9474e-4	–
0.2	16	3.6393e-6	3.97756	2.4915e-4	1.47946
0.2	32	2.5586e-7	3.83023	9.9905e-5	1.31839
0.2	64	2.64110e-8	3.27614	3.5502e-5	1.49266

Table 2: Results of Example 6.2.

α	N	E_c	R_c	E_n	R_n
0.9	8	1.4804e-6	–	8.72241e-7	–
0.9	16	9.6350e-8	3.94156	1.0347e-7	3.07580
0.9	32	6.1446e-9	3.97089	1.2525e-8	3.04633
0.9	64	3.8792e-10	3.98549	1.5418e-9	3.02212
0.5	8	1.4975e-6	–	5.4828e-7	–
0.5	16	9.6908e-8	3.94980	6.4541e-8	3.08663
0.5	32	6.1624e-9	3.97505	7.7767e-9	3.05299
0.5	64	3.8885e-10	3.98758	9.5488e-10	3.02577
0.2	8	1.5095e-6	–	3.5565e-7	–
0.2	16	9.7300e-8	3.95549	4.1561e-8	3.09716
0.2	32	6.1675e-9	3.97795	4.9856e-9	3.05939
0.2	64	3.8889e-10	3.98902	6.1041e-10	3.02992

The results in Tables 1 and 2 show the convergence and stability of our methods with different fractional orders. The numerical experiment illustrates and confirms our theoretical analysis.

Now, we check the efficiency of the QSC method using the following parameters [25], $S = 12.5\,\text{m}^{-1}$, $P = 10\,\text{W}$, $a = -127$, $\rho = 1050\,\text{kgm}^{-3}$, $c = 4180\,\text{Jkg}^{-1}\text{K}^{-1}$, $C_b = 3344\,\text{Jkg}^{-1}\text{K}^{-1}$, $K_t = 0.5\,\text{Wm}^{-1}\text{K}^{-1}$, $T_0 = T_a = T_w = T_f = 37\,^\circ\text{C}, Q_{m0} = 1091\,\text{Wm}^{-3}, b_0 = -6.35, a_0 = -5.08, W_b = 8\,\text{kgm}^{-1}\text{s}^{-1}$, and $R = 0.05\,\text{m}$. We consider the following cases:

Case I. Standard two equations $\alpha = 1$;

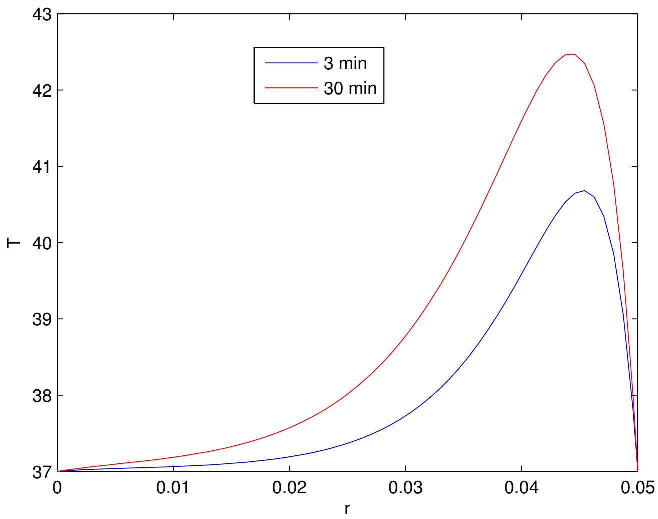

Figure 1: Temperature distribution of Case I, $\alpha = 1$.

Figure 1 shows the temperature distribution for the standard case. In this case, time taken to reach the temperature 43°C is 30 min.

Case II. Time fractional means $\alpha = 0.9$;

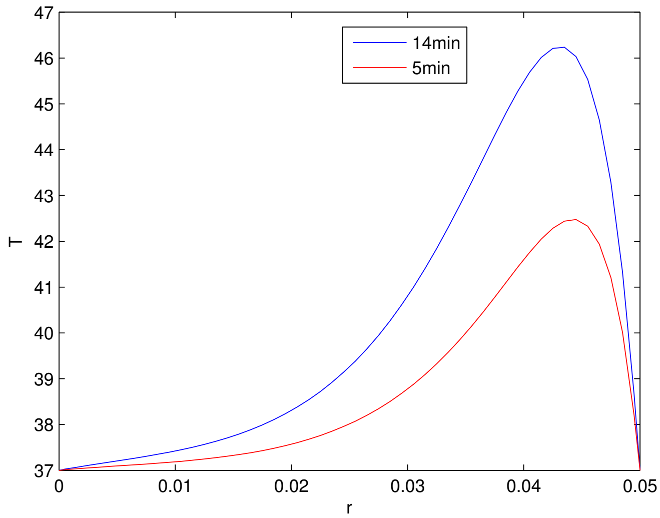

Figure 2: Temperature distribution of Case II, $\alpha = 0.9$.

In this case, we find time taken to reach the temperature 46.1°C is 14 min.

7. Conclusions

In this paper, we present a new numerical scheme by QSC methods to solve the fractional bioheat equation with mixed boundary value conditions for thermal therapy. This new introduced method does not need any extra parameter, and the obtained matrix is still nonsingular. We discuss the existence and uniqueness of the solution of the new QSC scheme, prove that the proposed technique can enjoy the local error bound with $O(h^4 + \tau^4)$ at collocation points, and find the global error bound can achieve the accuracy of $O(h^3 + \tau^3)$ under the L_∞ norm.

To verify our theoretical results, we present some numerical experiments with different α by two examples. The numerical experiments show that the QSC scheme is convergent and stable for the fractional diffusion equation. For the fractional Pennes' equation, we give the temperature curve over the entire affected region.

Our future work will focus on fractionalize the space derivative. For the space-time fractional Pennes' bioheat transfer equations, we will study the QSC method with high precision and better stability. As a result, we will provide more theoretical reference for cancer therapy.

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