A composition projection method for feasibility problems and applications to equilibrium problems

Jiawei Chen\textsuperscript{a,∗}, Yeong-Cheng Liou\textsuperscript{b}, Suhel Ahmad Khan\textsuperscript{c}, Zhongping Wan\textsuperscript{d}, Ching-Feng Wen\textsuperscript{e}

\textsuperscript{a}School of Mathematics and Statistics, Southwest University, Chongqing 400715, and College of Computer Science, Chongqing University, Chongqing 400044, China.
\textsuperscript{b}Department of Information Management, Cheng Shiu University, Kaohsiung 833, Taiwan.
\textsuperscript{c}Department of Mathematics, BITS-Pilani, Dubai Campus, Dubai-349055, UAE.
\textsuperscript{d}School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.
\textsuperscript{e}Center for General Education, Kaohsiung Medical University, Kaohsiung 807, Taiwan.

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Abstract

In this article, we propose a composition projection algorithm for solving feasibility problem in Hilbert space. The convergence of the proposed algorithm are established by using gap vector which does not involve the nonempty intersection assumption. Moreover, we provide the sufficient and necessary condition for the convergence of the proposed method. As an application, we investigate the split feasibility equilibrium problem. ©2016 All rights reserved.

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1. Introduction and Preliminaries

Throughout this paper, we assume that $X$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and that $A$ and $B$ are two nonempty closed and convex subsets of $X$.

The distance between the subsets $A, B$ of $X$ by

$$d(A, B) := \inf \{\|w - l\| : w \in A, l \in B\}.$$
If $B$ is empty, we set $\inf_{x \in A} d(x, B) = +\infty$.

Let $x \in X$. The metric projection of $x$ onto a nonempty closed and convex subset $B$ is defined by

$$P_B(x) = \arg \min_{y \in B} \|y - x\|.$$

It is well-known that $P_B$ is single-valued and nonexpansive. For $x \in X$, the metric projection $P_B(x)$ is characterized by

\textit{Kolmogorov’s criterion}:

$$P_B(x) \in B \text{ and } \langle y - P_B(x), x - P_B(x) \rangle \leq 0 \text{ for all } y \in B. \tag{1.1}$$

We are interested in the following feasibility problem (shortly, (FP)):

$$\text{Find } x \in A \bigcap B. \tag{1.2}$$

It is worth noting that many authors studied the common element for variational inequalities, equilibrium problem, maximal monotone operators and fixed points of nonlinear operators which can be considered as special cases of the problem (FP) (see, [2, 3, 11, 13, 14, 15, 17]). In many practical problems, the set $A \bigcap B$ is empty. A natural question arises: whether there exists a good substitute for $A \bigcap B$ when it is empty?

Bauschke and Borwein[1] introduced two good generalization of $A \bigcap B$:

$$E := \{a \in A : d(a, B) = d(A, B)\}, \quad F := \{b \in B : d(b, A) = d(B, A)\}. \tag{1.3}$$

Particularly, if $A \bigcap B \neq \emptyset$, then $E = F = A \bigcap B$.

For the reader’s convenience, we recall the following well-known definitions and results.

\textbf{Definition 1.1.} Let a mapping $T : X \rightrightarrows X$ with graph $\text{gr}T = \{(x, u) \in X \times X : u \in T(x)\}$. $T$ is said to be:

(i) monotone if $\langle x - y, \xi - \zeta \rangle \geq 0$ for all $(x, \xi), (y, \zeta) \in \text{gr}T$;

(ii) maximal monotone if $T$ is monotone and no proper enlargement of $\text{gr}T$ is monotone.

We also denote the set of fixed points of $T$ by $\text{Fix}(T) = \{x \in X : x \in T(x)\}$, and the resolvent of $T$ is defined as $J_T := (I + T)^{-1}$.

\textbf{Definition 1.2 ([1] 18).} Let $v \in X$. $v$ is said to be a gap vector from $A$ to $B$ if, $v = P_{B-A}(0)$.

It is easy to see that if $v$ is a gap vector from $A$ to $B$, then $-v$ is also a gap vector from $B$ to $A$, and $-v = P_{B-A}(0)$.

\textbf{Fact 1.3 ([1])}. Let $v$ be a gap vector from $A$ to $B$, and $E, F$ be defined by $[1,2]$ and $[1,3]$. Then

(i) $\|v\| = d(A, B)$, \quad $E + v = F$

(ii) $E = \text{Fix}(P_AP_B) = A \bigcap (B - v)$, \quad $F = \text{Fix}(P_BP_A) = B \bigcap (A + v)$

(iii) $P_B v = P_F e = e + v$ \quad ($e \in E$), \quad $P_A f = P_E f = f - v$ \quad ($f \in F$).

For more information on the gap vector see, for instance, [1] 18 and the references therein.

\textbf{Definition 1.4 ([16]).} Let $h : X \to (-\infty, +\infty]$ be a proper convex function. The subdifferential of $h$ at $x$ is defined by

$$\partial h(x) := \{\xi \in X : h(x + \tau) \geq h(x) + \langle \xi, \tau \rangle, \quad \forall \tau \in X\};$$

\textbf{Definition 1.5 ([1] 16).} Let $\Omega$ be a subset of $X$. The dual cone of $\Omega$ is

$$\Omega^* = \{\xi \in X : \langle \xi, x \rangle \geq 0, \quad \forall x \in \Omega\},$$

the polar cone of $\Omega$ is $\Omega^0 = -\Omega^*$, the tangent cone of $\Omega$ at $x$ is

$$T_\Omega(x) := \begin{cases} \text{cone}(\Omega - x), & \text{if } x \in \Omega, \\ \emptyset, & \text{otherwise}. \end{cases}$$
Fact 1.6 ([16, 18]). Let $\Omega \subseteq X$ and $h : X \to (-\infty, +\infty]$ be a proper convex function. Then

(i) the subdifferential operator $\partial h : X \rightrightarrows X$ is maximal monotone;

(ii) the proximal mapping of $h$, denoted by $\text{Prox}_h$, has a full domain and $\text{Prox}_h := J_{\partial h}$.

Particularly, if $h = \iota_\Omega$, then $\text{Prox}_\Omega = P_\Omega$ and $\partial \iota_\Omega = N_\Omega$, where $N_\Omega$ is the normal cone operator, and $\iota_\Omega$ is the indicator function of $\Omega$ defined by

$$N_\Omega(x) := \begin{cases} \{ \xi \in X : \langle \xi, y-x \rangle \leq 0, \forall y \in \Omega \}, & \text{if } x \in \Omega, \\ \emptyset, & \text{otherwise}, \end{cases}$$

and

$$\iota_\Omega(x) = \begin{cases} 0, & \text{if } x \in \Omega, \\ +\infty, & \text{otherwise}. \end{cases}$$

Fact 1.7 ([16, 18]). Let $\Omega$ be a nonempty convex subset of $X$ and let $x \in \Omega$. Then the following hold:

(i) $N_\Omega(x) = T^*_\Omega(x) = -T^*_\Omega(x)$ and $N^*_\Omega(x) = -N^*_\Omega(x) = T_\Omega(x)$;

(ii) $T_\Omega(x) = X \iff N_\Omega(x) = \{0\}$.

We now explore some properties of the gap vector.

Lemma 1.8. Let $\bar{a} \in A$ and $\bar{b} \in B$ and $v = \bar{a} - \bar{b}$. Then the following statements are equivalent:

(i) $v$ is a gap vector from $B$ to $A$;

(ii) $\bar{a} = P_A(\bar{b})$ and $\bar{b} = P_B(\bar{a})$;

(iii) $v \in N_B(\bar{b})$ and $-v \in N_A(\bar{a})$;

(iv) $v \in T^*_B(\bar{b}) \cap T^*_A(\bar{a})$;

(v) $(\bar{a}, \bar{b})$ is a solution of the following optimization problem:

$$\min_{(a,b)} \left[ \frac{1}{2} \|a-b\|^2 + \iota_{A \times B}(a,b) \right],$$

where $\iota_{A \times B}$ is the indicator function of $A \times B$.

Proof. (i)$\Rightarrow$(ii): Suppose that $v = \bar{a} - \bar{b}$ is a gap vector from $B$ to $A$. By Definition 1.2 we have

$$v = P_{A-B}(0).$$

Then

$$\langle y - v, -v \rangle \leq 0, \quad \forall y \in A - B.$$ 

That is,

$$\langle y - (\bar{a} - \bar{b}), \bar{b} - \bar{a} \rangle \leq 0, \quad \forall y \in A - B.$$ 

For any $x \in A$ and $z \in B$, $x - z \in A - B$. It follows from (1.5) that

$$\langle x - z - (\bar{a} - \bar{b}), \bar{b} - \bar{a} \rangle \leq 0, \quad \forall x \in A, z \in B.$$ 

Moreover, one has

$$\langle x - \bar{a}, \bar{b} - \bar{a} \rangle \leq \langle z - \bar{b}, \bar{b} - \bar{a} \rangle, \quad \forall x \in A, z \in B.$$ 

Take $z = \bar{b}$ and $x = \bar{a}$ in (1.6), respectively, we have

$$\langle x - \bar{a}, \bar{b} - \bar{a} \rangle \leq 0, \quad \forall x \in A$$

and

$$\langle z - \bar{b}, \bar{a} - \bar{b} \rangle \leq 0, \quad \forall z \in B.$$
Hence, we get Then
\[
\text{Fact 1.10 (1, 9)}.
\]

Lemma 1.9. Let \( a \in A \) and \( b \in B \). Then
\[
\bar{a} = P_A(\bar{b}), \bar{b} = P_B(\bar{a}) \iff \bar{a} = P_A P_B(\bar{a}), \bar{b} = P_B P_A(\bar{b}).
\]

Proof. The necessity is obvious. We only need to prove the sufficiency. Assume that \( \bar{a} = P_A P_B(\bar{a}) \) and \( \bar{b} = P_B P_A(\bar{b}) \). Then
\[
\bar{a} = P_A P_B(\bar{a}) = P_A P_B P_A(\bar{a}) = P_A(\bar{b})
\]
and
\[
\bar{b} = P_B P_A(\bar{b}) = P_B P_A P_B(\bar{b}) = P_B(\bar{a}).
\]
Consequently, \( \bar{a} = P_A(\bar{b}), \bar{b} = P_B(\bar{a}) \). This completes the proof.

Fact 1.10 (4, 8). Let \( \Omega \) be a nonempty closed and convex subset of \( X \), let \( T : \Omega \to X \) be nonexpansive, let \( (x_n)_{n \in N} \) be a sequence in \( \Omega \), and \( x \in X \). Suppose that \( x_n \rightharpoonup x \) and that \( x_n - T(x_n) \to 0 \). Then \( x \in \text{Fix}(T) \).
2. Main results

In this section, we propose a composition projection algorithm for solving feasibility problem in Hilbert space. The asymptotic behaviors of the proposed algorithm are established by using gap vector which does not involve the nonempty intersection assumption. Moreover, we provide the sufficient and necessary condition for the convergence of the proposed algorithm.

Theorem 2.1. Let $A, B$ be two nonempty closed convex subsets of a Hilbert space $X$, and let the sequence $(x_n)$ be generated by the following algorithm:

\[
\begin{align*}
    x_1 & \in X \text{ arbitrarily,} \\
    y_n & = P_A P_B(x_n), \\
    C_{n+1} & = \{ z \in C_n : \| z - y_n \| \leq \| z - x_n \| \}, \\
    x_{n+1} & = P_{C_{n+1}} x_1, \quad \forall n \geq 1,
\end{align*}
\]

(2.1)

where $C_1 = X$. Assume that there exist $a \in A$ and $b \in B$ such that $d(A, B) = \| a - b \|$. Then the following statements hold:

(i) the sequence $(x_n)$ generated by Algorithm 2.1 strongly converges to the point $p$, where $p = P_E(x_1)$ and $E = \text{Fix}(P_A P_B)$;

(ii) $p - P_B(p) = P_{A-B}(0)$;

(iii) $\lim_{n \to \infty} \| x_n - P_B(x_n) \| = \lim_{n \to \infty} \| y_n - P_B(x_n) \| = \| p - P_B(p) \| = d(A, B)$;

(iv) $p - P_B(p) \in N_B(P_B(p))$ and $P_B(p) - p \in N_A(p)$.

Proof. (i) We proceed in several steps.

Step 1. $E = \text{Fix}(P_A P_B)$ is nonempty closed and convex.

To this aim, we divide it into two cases:

(a1) Let $A \cap B \neq \emptyset$. Since $A$ and $B$ are two nonempty closed convex subsets of a Hilbert space $X$, from Fact 1.3 (ii), $E = \text{Fix}(P_A P_B) = A \cap B \neq \emptyset$.

(b1) Let $A \cap B = \emptyset$. Since there exist $a \in A$ and $b \in B$ such that $d(A, B) = \| a - b \|$, then $b - a$ is a gap vector from $A$ to $B$. Moreover, we conclude that $d(A, B) = \| a - b \| > 0$ and

\[ E = \text{Fix}(P_A P_B) = A \cap (B - v) \neq \emptyset, \]

where $v = P_{B-A}(0) = b - a$ and $\| v \| = d(A, B)$.

Combining (a1) and (b1) yield that $E = \text{Fix}(P_A P_B) \neq \emptyset$. Since $P_A$ and $P_B$ are nonexpansive, for any $x, y \in X$,

\[ \| P_A P_B(x) - P_A P_B(y) \| \leq \| P_B(x) - P_B(y) \| \leq \| x - y \|. \]

This means that $P_A P_B$ is also nonexpansive on $X$. Thus, $E = \text{Fix}(P_A P_B)$ is closed and convex.

Step 2. $E \subseteq C_n$ for all $n \geq 1$.

For any given $u \in E = \text{Fix}(P_A P_B)$, we have

\[ \| u - y_1 \| = \| P_A P_B(u) - P_A P_B(x_1) \| \leq \| u - x_1 \|. \]

So, $E = \text{Fix}(P_A P_B) \subseteq C_1$. Moreover, one has

\[ \| u - y_n \| = \| P_A P_B(u) - P_A P_B(x_n) \| \leq \| u - x_n \|, \quad \forall n \in N, \]

that is, $E \subseteq C_{n+1}$ for all $n \geq 1$. Therefore, $E \subseteq C_n$ for all $n \geq 1$.

Step 3. $(x_n)$ is well defined.

From Steps 1, 2 and Algorithm 2.1, it is easy to see that $C_n$ is nonempty closed and convex for all $n \geq 1$. Consequently, $(x_n)$ is well defined.
Step 4. The sequence \((x_n)\) is bounded and \(\lim_{n \to \infty} \|x_n - x_1\|\) exists.

Note that \(x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1} \subseteq C_n\). For \(x_n = P_{C_n}x_1 (n > 1)\), one has
\[
\|x_n - x_1\| \leq \|x_{n+1} - x_1\|. \tag{2.2}
\]
For any given \(u \in E\), from Step 2, \(u \in C_n\) for all \(n \geq 1\). It follows from \(x_n = P_{C_n}x_1\) that
\[
\langle u - x_n, x_1 - x_n \rangle \leq 0.
\]
Note that
\[
\|u - P_{C_n}x_1\|^2 + \|P_{C_n}x_1 - x_1\|^2 = \|u - x_n\|^2 + \|x_n - x_1\|^2 = \|u - x_1\|^2 + 2\langle u - x_n, x_1 - x_n \rangle \leq \|u - x_1\|^2,
\]
i.e.,
\[
\|u - x_n\|^2 + \|x_n - x_1\|^2 \leq \|u - x_1\|^2.
\]
Therefore, one has
\[
\|x_n\| - \|x_1\| \leq \|x_n - x_1\| \leq \|u - x_1\|
\]
and so, \(\|x_n\| \leq \|u - x_1\| + \|x_1\|\). These show that the sequences \((x_n)\) and \((x_n - x_1)\) are bounded. It follows from (2.2) that \(\lim_{n \to \infty} \|x_n - x_1\|\) exists.

Step 5. The sequence \((x_n)\) is a Cauchy sequence.

For any positive integer numbers \(m, n\) and \(m > n\), one has \(x_m \in C_m \subseteq C_n\). Again from \(x_n = P_{C_n}x_1\), we have
\[
\langle x_m - x_n, x_1 - x_n \rangle \leq 0. \tag{2.3}
\]
Taking into account \(\|x_m - x_n\|^2 + \|x_n - x_1\|^2 = \|x_m - x_1\|^2 + 2\langle x_m - x_n, x_1 - x_n \rangle\), from (2.3), we have
\[
\|x_m - x_n\|^2 + \|x_n - x_1\|^2 \leq \|x_m - x_1\|^2.
\]
Then
\[
\|x_m - x_n\|^2 \leq \|x_m - x_1\|^2 - \|x_n - x_1\|^2 = (\|x_m - x_1\| + \|x_n - x_1\|)(\|x_m - x_1\| - \|x_n - x_1\|).
\]
Therefore, \((x_n)\) is a Cauchy sequence and so, \(\|x_{n+1} - x_n\| \to 0\) as \(n \to \infty\). Without loss of generality, let \(x_n \to p \in X\).

Step 6. \(p \in E = \text{Fix}(P_AP_B)\).

Since \(x_{n+1} = P_{C_{n+1}}x_1, x_{n+1} \in C_{n+1}\). By the definition of \(C_{n+1}\), one has
\[
\|x_{n+1} - y_n\| \leq \|x_{n+1} - x_n\|. \tag{2.4}
\]
It follows from (2.4) and \(\|x_{n+1} - x_n\| \to 0\) as \(n \to \infty\) that \(\|x_{n+1} - y_n\| \to 0\) as \(n \to \infty\). Noticing that
\[
\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|.
\]
We have that \(\|x_n - y_n\| \to 0\) as \(n \to \infty\). Consequently, \(y_n \to p\) and \(x_n - P_AP_Bx_n \to 0\) as \(n \to \infty\). By the nonexpansiveness of \(P_AP_B\) and Corollary 4.18 ([4], p. 64), we conclude that \(p \in \text{Fix}(P_AP_B) = E\).

Last step. \(p = P_Ex_1\).

Without loss of generality, let \(q = P_Ex_1 = P_{\text{Fix}(P_AP_B)}x_1\). Then \(q \in E \subseteq C_n\) for all \(n \geq 1\). Since \(x_{n+1} = P_{C_{n+1}}x_1, \|x_{n+1} - x_1\| \leq \|z - x_1\|\) for all \(z \in C_{n+1}\). From this and \(E \subseteq C_{n+1}\), we have
\[
\|x_{n+1} - x_1\| \leq \|q - x_1\|. \tag{2.5}
\]
In view of \(x_{n+1} \to p \in E\). Take \(n \to \infty\) in (2.5), one has \(\|p - x_1\| \leq \|q - x_1\|\). This, together with \(q = P_Ex_1\), shows that \(p = q = P_Ex_1\).
(ii) Let us prove that \( p - P_B(p) = P_{A^\perp B}(0) \).
Indeed, since \( p \in E = \text{Fix}(P_A P_B) \), \( p = P_A P_B(p) \). Therefore, by Fact 1.3 and Lemma 1.8 we have
\[
v = p - P_B(p) = P_{A^\perp B}(0).
\]
(2.6)

(iii) Let us prove that
\[
\lim_{n \to \infty} \|x_n - P_B(x_n)\| = \lim_{n \to \infty} \|y_n - P_B(x_n)\| = \|p - P_B(p)\| = d(A, B).
\]
Since \( x_n, y_n \to p \) and from the continuity of \( P_B \), one has
\[
\lim_{n \to \infty} \|x_n - P_B(x_n)\| = \lim_{n \to \infty} \|y_n - P_B(x_n)\| = \|p - P_B(p)\|.
\]
It follows from (2.6) that \( p - P_B(p) \) is a gap vector from \( B \) to \( A \). This, together with Facts 1.3 shows that
\[
\|v\| = \|p - P_B(p)\| = d(A, B).
\]
As a consequence, we derive that
\[
\lim_{n \to \infty} \|x_n - P_B(x_n)\| = \lim_{n \to \infty} \|y_n - P_B(x_n)\| = \|p - P_B(p)\| = d(A, B).
\]
(iv) It follows from (2.6) and Lemma 1.8 that
\[
p - P_B(p) \in N_B(P_B(p)), \quad P_B(p) - p \in N_A(p).
\]
This completes the proof. \( \square \)

The next corollary shows the Algorithm (2.1) to solve a convex feasibility problem.

**Corollary 2.2.** Let \( A, B \) be two nonempty closed convex subsets of a Hilbert space \( X \) such that \( A \cap B \neq \emptyset \). Then the sequence \( (x_n) \) generated by Algorithm (2.1) strongly converges to some point \( p \) of \( A \cap B \), moreover, \( p = P_{A \cap B}(x_1) \).

Particularly, if \( A \) and \( B \) are two closed affine subspaces of \( X \), we have the following result.

**Corollary 2.3.** Let \( A, B \) be two closed affine subspaces of a Hilbert space \( X \). Then the sequence \( (x_n) \) generated by Algorithm (2.1) strongly converges to some point \( p \) of \( A \cap B \), moreover, \( p = P_{A \cap B}(x_1) \).

If \( A \cap B = \emptyset \), then we can find the distance between \( A \) and \( B \) from Algorithm (2.1).

**Corollary 2.4.** Let \( A, B \) be two nonempty closed convex subsets of a Hilbert space \( X \) such that \( A \cap B = \emptyset \), and let the sequences \( (x_n) \) and \( (y_n) \) be generated by Algorithm (2.1). Assume that there exist \( a \in A \) and \( b \in B \) such that \( d(A, B) = \|a - b\| \). Then
(i) \( d(A, B) = \lim_{n \to \infty} \|x_n - P_B(x_n)\| = \lim_{n \to \infty} \|y_n - P_B(x_n)\| = \|p - P_B(p)\| > 0; \)
(ii) \( p - P_B(p) = P_{A^\perp B}(0) \), where \( p = P_{\text{Fix}(P_A P_B)}(x_1) \).

**Remark 2.5.** (i) The assumption "there exist \( a \in A \) and \( b \in B \) such that \( d(A, B) = \|a - b\| \)" is reasonable.
On the one hand, from the computational viewpoint, we, in general, can only obtain approximate solutions (ε-optimal solutions) of nonlinear and linear problems by using the algorithms proposed in the literature, where \( \epsilon \) is the tolerance. So, we can view the assumption "there exist \( a \in A \) and \( b \in B \) such that \( d(A, B) = \|a - b\| \)" as the terminative condition or the tolerance \( \epsilon = d(A, B) = \|a - b\| \) of Algorithm (2.1) in the numerical experimentation. On the other hand, if \( A \cap B = \emptyset \), and \( A \) or \( B \) is bounded, we know that the assumption "there exist \( a \in A \) and \( b \in B \) such that \( d(A, B) = \|a - b\| \)" holds.
(ii) The assumption "there exist \( a \in A \) and \( b \in B \) such that \( d(A, B) = \|a - b\| \) " is essential in Theorem 2.1 and Corollary 2.4.

Example 2.6. Let \( A = \{(x, y) \in \mathbb{R}^2 : y \leq 0\} \) and \( \mathcal{Y}(x) = e^x \) for all \( x \in (-\infty, +\infty) \). Then the graph of \( \mathcal{Y} \), denoted by \( B, B = \{(x, y) \in \mathbb{R}^2 : e^x \leq y\} \), is a nonempty closed and convex subset of \( \mathbb{R}^2 \). It is easy to check that \( A \cap B = \emptyset \). But there does not exist \( a \in A \) and \( b \in B \) such that \( d(A, B) = \|a - b\| \) . Indeed, since the distance \( d(A, B) = 0 \) and \( E = \text{Fix}(P_A P_B) = \emptyset \).

Remark 2.7. Theorem 2.1 and Corollaries 2.2 and 2.3 develop and improve Corollaries 5.23, 5.25 and 5.28 of (Bauschke and Combettes [4], pages 84–85) in the following aspects:

(i) Theorem 2.1 and Corollaries 2.2 and 2.3 do not involve the assumptions \( \text{Fix}(P_A P_B) \neq \emptyset \) and \( A \cap B \neq \emptyset \);

(ii) The sequence \( \{x_n\} \) generated by Algorithm (2.1) can be guaranteed the strong convergence under the assumptions of Theorem 2.1 and Corollaries 2.2 and 2.3;

(iii) Compared with the Algorithms 6.1 and 6.2 of Kassay, Reich and Sabach [13] and Algorithm 6.1 of Sabach [17], the step \( Q_{k+1} = \{z \in A \cap B : (x_1 - x_{k+1}, z - x_{k+1}) \leq 0\} \) is removed.

In the proof of Theorem 2.1, we observe that

\[
\exists a \in A, b \in B \text{ such that } d(A, B) = \|a - b\| \Rightarrow \text{Fix}(P_A P_B) \neq \emptyset. \tag{2.7}
\]

Naturally, a question arises: whether the converse of (2.7) is true?

The next proposition presents some sufficient and necessary conditions for \( \text{Fix}(P_A P_B) \neq \emptyset \) as well as \( \text{Fix}(P_A P_B) = \emptyset \).

Proposition 2.8. \( \exists a \in A, b \in B \text{ such that } d(A, B) = \|a - b\| \Leftrightarrow \text{Fix}(P_A P_B) \neq \emptyset; \)

(ii) for any \( a \in A, b \in B \text{ such that } d(A, B) < \|a - b\| \Leftrightarrow \text{Fix}(P_A P_B) = \emptyset. \)

Proof. (i) By the proof of Theorem 2.1, we only need prove the sufficiency of (i).

Suppose that \( \text{Fix}(P_A P_B) \neq \emptyset \). We divide into two cases:

(a) If \( A \cap B \neq \emptyset \), then (i) holds;
(b) If \( A \cap B = \emptyset \). Since \( \text{Fix}(P_A P_B) \neq \emptyset \), for \( f^* \in \text{Fix}(P_A P_B) \), one has \( f^* = P_A P_B(f^*) \in A \) and \( P_B(f^*) = P_B P_A P_B(f^*) \in B \). By Lemmas 1.8 and 1.9, \( f^* - P_B(f^*) \) is a gap vector from \( B \) to \( A \), that is, \( f^* - P_B(f^*) = P_{A/B}(0) \). This shows that \( \|f^* - P_B(f^*)\| = d(A, B) \), as required.

(ii) It directly follows from (i). This completes the proof.

We now propose another question: what will happen of Algorithm (2.1) when \( \text{Fix}(P_A P_B) = \emptyset \)?

Theorem 2.9. Let \( A \) and \( B \) be two nonempty closed and convex subsets of a Hilbert space \( X \) such that \( E = \text{Fix}(P_A P_B) = \emptyset \). Assume that the sequence \( \{x_n\} \) is generated by Algorithm (2.1). Then exactly one of the following alternatives holds:

(i) \( \|x_n\| \rightarrow +\infty \) as \( n \rightarrow \infty \) whenever \( C_n \neq \emptyset \) for all \( n \geq 1; \)

(ii) Algorithm (2.1) stops at finite iteration \( n \geq 1 \) whenever \( C_n = \emptyset \) for some \( n \geq 1. \)

Proof. We only need to prove that (i) holds. Suppose that \( \|x_n\| \nrightarrow +\infty \) as \( n \rightarrow \infty \). That is, for some \( M > 0 \), there exists a subsequence \( \{x_{n_k}\}_{k \geq 1} \) of \( \{x_n\} \) such that \( \|x_{n_k}\| \leq M \). In other word, the subsequence \( \{x_{n_k}\}_{k \geq 1} \) is bounded. Then \( x_{n_k} \rightarrow x \in X \) (here we may take a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) if necessary).

Since \( C_n \) is closed and convex for all \( n \geq 1 \), \( v \in \bigcap_{k \geq 1} C_{n_k} \). Take into account \( C_{n+1} \subseteq C_n \) for all \( n \geq 1 \), one has \( \bigcap_{n \geq 1} C_n = \bigcap_{n \geq 1} C_{n_k} \). Moreover, \( v \in \bigcap_{n \geq 1} C_n \neq \emptyset \). Without loss of generality, let \( n_{k+1} \geq n_k + 1 \). In view of \( x_{n_{k+1}} = P_{C_{n_{k+1}}} x_{1} \in C_{n_{k+1}} \subseteq C_n \). It follows from \( x_{n_k} = P_{C_{n_k}} x_1 \) that

\[
\|x_{n_k} - x_1\| \leq \|v - x_1\| \tag{2.8}
\]
and
\[ \|x_{n_k} - x_1\| \leq \|x_{n_{k+1}} - x_1\|. \] (2.9)
Both (2.8) and (2.9) imply that \(\lim_{k \to \infty} \|x_{n_k} - x_1\|\) exists. Again, from \(x_{n_{k+1}} \in C_{n_k}\) and \(x_{n_k} = P_{C_{n_k}}x_1\), one has
\[ \langle x_{n_{k+1}} - x_{n_k}, x_1 - x_{n_k} \rangle \leq 0. \] (2.10)
Owing to \(\|x_{n_{k+1}} - x_{n_k}\|^2 + \|x_{n_k} - x_1\|^2 = \|x_{n_{k+1}} - x_1\|^2 + 2\langle x_{n_{k+1}} - x_{n_k}, x_1 - x_{n_k} \rangle\). This, together with (2.10), shows that
\[ \|x_{n_{k+1}} - x_{n_k}\|^2 + \|x_{n_k} - x_1\|^2 \leq \|x_{n_{k+1}} - x_1\|^2. \]
Furthermore, one has
\[ \|x_{n_{k+1}} - x_{n_k}\|^2 \leq \|x_{n_{k+1}} - x_1\|^2 - \|x_{n_k} - x_1\|^2. \]
This implies that
\[ \|x_{n_{k+1}} - x_{n_k}\| \to 0, \ k \to \infty. \]
Since \(x_{n_{k+1}} \in C_{n_{k+1}} \subseteq C_{n_k+1}\), by the definition of \(C_{n_k+1}\), we have
\[ \|x_{n_{k+1}} - y_{n_k}\| \leq \|x_{n_{k+1}} - x_{n_k}\|. \]
Moreover, one has
\[ \|x_{n_{k+1}} - y_{n_k}\| \to 0, \ k \to \infty. \]
In the light of \(\|x_{n_k} - y_{n_k}\| \leq \|x_{n_k} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - y_{n_k}\|\), we conclude that
\[ \|x_{n_k} - y_{n_k}\| \to 0, \ k \to \infty. \]
Consequently, one has
\[ \lim_{k \to \infty} \|x_{n_k} - y_{n_k}\| = \lim_{k \to \infty} \|x_{n_k} - P_AP_B(x_{n_k})\| = 0. \]
By Fact 1.10, we derived that \(v = P_AP_B(v)\) and so,
\[ v \in \text{Fix}(P_AP_B) \neq \emptyset, \]
which contradicts \(E = \text{Fix}(P_AP_B) = \emptyset\). This completes the proof. \(\square\)

3. An application to split feasibility equilibrium problem

Let \(C\) and \(D\) be nonempty closed and convex subsets of finite Euclidean spaces \(X\) and \(Y\), respectively, and let \(T : X \to Y\) be a bounded linear operator, \(g : C \times C \to \mathbb{R}\) and \(h : D \times D \to \mathbb{R}\) be two functions.

We consider the following split feasibility equilibrium problems (shortly, (SFEP)):

Find \(x^* \in C\) such that
\[ g(x^*, x) \geq 0 \quad (3.1) \]
for all \(x \in C\), and \(y^* = Tx^*\) is a solution of the following equilibrium problem:

Find \(y^* \in D\) such that
\[ h(y^*, y) \geq 0 \quad (3.2) \]
for all \(y \in D\).

Denote the solutions set of (SFEP) by \(S\). If we set \(S_1 = \{x \in C : g(x, z) \geq 0, \ \forall z \in C\}\) and \(S_2 = \{x \in C : y = Tx \in D, \ h(y, v) \geq 0, \ \forall v \in D\}\), then (SFEP) is equivalent to the following feasibility problem:

Find \(x \in S_1 \bigcap S_2\). \hspace{1cm} (3.3)

The characterizations of solution set of various equilibrium problem had been studied by many authors (see, [5 6 7 8 10 12]). Here we assume that \(S_1\) and \(S_2\) are nonempty, closed and convex with \(S_1 \bigcap S_2 \neq \emptyset\).

**Lemma 3.1.** Let \(A := S_1\) and \(B := S_2\) in Theorem 2.1. Assume that the sequence \((x_n)\) is generated by Algorithm (2.1). Then the sequence \((x_n)\) generated by Algorithm (2.1) converges to a solution of \(S\).

**Proof.** It directly follows from Theorem 2.1 (i). This completes the proof. \( \square \)
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