Positive solutions for an impulsive boundary value problem with Caputo fractional derivative

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Abstract

In this work we use fixed point theorem method to discuss the existence of positive solutions for the impulsive boundary value problem with Caputo fractional derivative

\begin{equation}
\begin{cases}
\mathcal{D}_t^q u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, 1];
\Delta u(t_k) = I_k(u(t_k)), \quad \Delta u'(t_k) = J_k(u(t_k)), \quad k = 1, 2, \ldots, m;
au(0) - bu(1) = 0, \quad au'(0) - bu'(1) = 0,
\end{cases}
\end{equation}

where $q \in (1, 2)$ is a real number, $a, b$ are real constants with $a > b > 0$, and $\mathcal{D}_t^q$ is the Caputo’s fractional derivative of order $q$, $f : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$ and $I_k, J_k : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions, $k = 1, 2, \ldots, m$, $\mathbb{R}^+ := [0, +\infty)$. ©2016 All rights reserved.

Keywords: Caputo fractional derivative, impulsive boundary value problem, fixed point theorem, positive solution.


1. Introduction

In this work we study the impulsive boundary value problem with Caputo fractional derivative

\begin{equation}
\begin{cases}
\mathcal{D}_t^q u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, 1];
\Delta u(t_k) = I_k(u(t_k)), \quad \Delta u'(t_k) = J_k(u(t_k)), \quad k = 1, 2, \ldots, m;
au(0) - bu(1) = 0, \quad au'(0) - bu'(1) = 0,
\end{cases}
\end{equation}

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where \( q \in (1, 2) \) is a real number, \( a, b \) are real constants with \( a > b > 0 \), and \( \mathcal{D}_0^q \) is the Caputo’s fractional derivative of order \( q \); \( t_k(k = 1, 2, \ldots, m, m \geq 1 \) is a fixed integer) are constants with \( 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1 \), \( u(t_k^+) = \lim_{u \to -0} u(t_k + h) \) and \( u(t_k^-) = \lim_{u \to 0} u(t_k - h) \) represent the right-hand and left-hand limits of \( u(t) \) at \( t = t_k \), respectively. Moreover, \( f, I_k, J_k \) satisfy the condition:

\( \textbf{(H1)}. \ f: [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+ \) and \( I_k, J_k: \mathbb{R}^+ \to \mathbb{R}^+(k = 1, 2, \ldots, m) \) are continuous functions.

Denote \( J = [0, 1], J_0 = [0, t_1], J_k = (t_k, t_{k+1}][(k = 1, 2, \ldots, m)] \). Furthermore, we define

\[
P(C) = \{ u \in J : J \rightarrow \mathbb{R} \text{ is continuous and } u(t_k^+) \text{ and } u(t_k^-) \text{ exist, } u(t_k^-) = u(t_k), k = 1, 2, \ldots, m \}. \]

Clearly, \( P(C) \) is a Banach space with the norm \( \| u \| = \sup_{t \in J} |u(t)| \) for \( u \in P(C) \). Note that \( C(J) \), which represents the set of all continuous functions on \( J \), is also a Banach space with \( \| u \| \).

As is well known, it is an important method to express the solutions of differential equations by Green’s function. However, for impulsive differential equations of fractional order, there is no such approximate method. Therefore, it is a natural problem whether or not the same result holds for the fractional differential equations with impulses

\[
\begin{align*}
\mathcal{D}_0^\alpha u(t) &= f(t, u(t)), \quad 1 < \alpha \leq 2, \quad t \in J; \\
\Delta u(t_k) &= I_k(u(t_k)), \quad \Delta u'(t_k) = I_k^*(u(t_k)), \quad k = 1, 2, \ldots, p; \\
Tu'(0) &= -au(0) - bu(T), \quad Tu'(T) = cu(0) + du(T),
\end{align*}
\]

which can be written in the form

\[
u(t) = \int_{t_k}^{t} \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} f(s, u(s)) ds + \lambda_1(t) \int_{t_p}^{T} \frac{(T - s)^{\alpha - 1}}{\Gamma(\alpha)} f(s, u(s)) ds \\
- \lambda_2(t) \int_{t_p}^{T} \frac{(T - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} f(s, u(s)) ds + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha - 1}}{\Gamma(\alpha)} f(s, u(s)) ds + I_i(u(t_i))
\]

\[
+ \sum_{i=1}^{k-1} (t_k - t_i) \left[ \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} f(s, u(s)) ds + I_i^*(u(t_i)) \right]
\]

\[
+ \sum_{i=1}^{k} (t - t_k) \left[ \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} f(s, u(s)) ds + I_i^*(u(t_i)) \right]
\]

\[
+ \lambda_1(t) \sum_{i=1}^{p} \left[ \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha - 1}}{\Gamma(\alpha)} f(s, u(s)) ds + I_i(u(t_i)) \right]
\]

\[
+ \lambda_1(t) \sum_{i=1}^{p} (t_p - t_i) \left[ \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} f(s, u(s)) ds + I_i^*(u(t_i)) \right]
\]

\[
- \sum_{i=1}^{p} [\lambda_3(t) + \lambda_1(t)t_p] \left[ \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} f(s, u(s)) ds + I_i^*(u(t_i)) \right].
\]

We all know that impulsive differential equations with integer order can be expressed by Green’s function (see for example \([7, 10, 16, 20]\)). Therefore, it is a natural problem whether or not the same result holds for the fractional order case. To the best of our knowledge, only \([10, 21, 23]\) are devoted to this direction. In \([10]\), Liu and Jia considered the fractional impulsive differential equations:

\[
\begin{align*}
\mathcal{D}_0^{\alpha+} u(t) &= f(t, u(t), \mathcal{D}_0^{\alpha+} u(t)), \quad t \in J; \\
\Delta u(t_k) &= I_k(u(t_k), \mathcal{D}_0^{\alpha+} u(t_k)); \\
\Delta^c \mathcal{D}_0^{\alpha+} u(t_k) &= Q_k(u(t_k), \mathcal{D}_0^{\alpha+} u(t_k)), \quad k = 1, 2, \ldots, m; \\
u(0) &= 0, \quad u(1) = \int_{0}^{1} u(t) g(t) dt,
\end{align*}
\]
Definition 2.1. The Riemann-Liouville fractional integral operator of order \( \alpha > 0 \) is defined as

\[
I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} f(s)ds,
\]

where \( \Gamma(\cdot) \) is the Euler gamma function.

Definition 2.2. The fractional derivative of \( f \) in the Caputo sense is defined as

\[
^{c}D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \quad n-1 < \alpha < n,
\]

where \( n = [\alpha] + 1 \), \( [\alpha] \) denotes the integer part of the number \( \alpha \).

Lemma 2.3. Let \( \alpha > 0 \). Then the differential equation \( ^{c}D_{0+}^{\alpha} u(t) = 0 \) has a unique solution

\[
u(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}
\]

for some \( c_i \in \mathbb{R}(i = 0, 1, \ldots, n-1) \), where \( n = [\alpha] + 1 \).

Lemma 2.4. Assume that \( u \in C(0,1) \cap L(0,1) \) with a derivative of order \( \alpha > 0 \) that belongs to \( C(0,1) \cap L(0,1) \). Then

\[
I_{0+}^{\alpha} \, ^{c}D_{0+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}
\]

for some \( c_i \in \mathbb{R}(i = 0, 1, \ldots, n-1) \), where \( n = [\alpha] + 1 \).

Lemma 2.5 ([22, Lemma 2.5]). Let \( y \in C(J) \). Then the unique solution of the boundary value problem

\[
\begin{align*}
^{c}D_{0+}^{\alpha} u(t) &= y(t), \quad \text{a.e.} \ t \in [0,1]; \\
\Delta u(t_k) &= I_k(u(t_k)), \quad \Delta u'(t_k) = J_k(u(t_k)), \quad k = 1, 2, \ldots, m; \\
au(0) - bu(1) &= 0, \quad au'(0) - bu'(1) = 0,
\end{align*}
\]

is given by

\[
u(t) = \int_{0}^{1} G_{1}(t,s)y(s)ds + \sum_{i=1}^{m} G_{2}(t,t_i)J_{i}(u(t_i)) + \sum_{i=1}^{m} G_{3}(t,t_i)I_{i}(u(t_i)) \tag{2.2}
\]
where

\[
G_1(t, s) = \begin{cases} 
\frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{b(1-s)^{q-1}}{(a-b)\Gamma(q)} + \frac{b(q-1)t(1-s)^{q-2}}{(a-b)\Gamma(q)} + \frac{b^2(q-1)(1-s)^{q-2}}{(a-b)^2\Gamma(q)}, & 0 \leq s \leq t \leq 1; \\
\frac{b(1-s)^{q-1}}{(a-b)\Gamma(q)} + \frac{b(q-1)t(1-s)^{q-2}}{(a-b)\Gamma(q)} + \frac{b^2(q-1)(1-s)^{q-2}}{(a-b)^2\Gamma(q)}, & 0 \leq t \leq s \leq 1,
\end{cases}
\] (2.3)

\[
G_2(t, s) = \begin{cases} 
\frac{ab}{(a-b)^2} + \frac{a(t-t_i)}{a-b}, & 0 \leq t_i < t \leq 1, \quad i = 1, 2, \ldots, m; \\
\frac{ab}{(a-b)^2} + \frac{b(t-t_i)}{a-b}, & 0 \leq t \leq t_i \leq 1, \quad i = 1, 2, \ldots, m,
\end{cases}
\] (2.4)

\[
G_3(t, s) = \begin{cases} 
\frac{a}{a-b}, & 0 \leq t_i < t \leq 1, \quad i = 1, 2, \ldots, m; \\
\frac{b}{a-b}, & 0 \leq t \leq t_i \leq 1, \quad i = 1, 2, \ldots, m.
\end{cases}
\] (2.5)

**Lemma 2.6** ([22 Lemma 2.6]). Let \( a, b \) be real constants with \( a > b > 0 \). Then \( G_i(i = 1, 2, 3) \) have the following properties

(i) \( G_1(t, s) \in C(J \times J, \mathbb{R}^+) \) and \( G_1(t, s) > 0, G_2(t, t_i) > 0, G_3(t, t_i) > 0 \) for all \( t, t_i, s \in (0, 1) \),

(ii) there exists a negative function \( M(s), s \in [0, 1] \) such that

\[
\frac{b}{a} M(s) \leq G_1(t, s) \leq M(s),
\]

where

\[
M(s) = \frac{a[(1-s)a - (2-s-q)b](1-s)^{q-2}}{(a-b)^2\Gamma(q)}, \quad s \in [0, 1],
\]

(iii)

\[
\frac{b^2}{(a-b)^2} \leq G_2(t, t_i) \leq \frac{a^2}{(a-b)^2}, \quad \frac{b}{a-b} \leq G_3(t, t_i) \leq \frac{a}{a-b}, \quad \forall t, t_i \in [0, 1].
\]

For convenience, we need to calculate the following integral

\[
\kappa_1 := \int_0^1 M(s)ds = \frac{a^2(q-1) + abq(q-2) + abq(q-1)(a-b)^2\Gamma(q)}{a(q-1)(a-b)^2\Gamma(q)}.
\]

We define the operator \( \mathcal{A} : PC(J) \to PC(J) \) by

\[
(\mathcal{A}u)(t) := \int_0^t G_1(t, s)f(s, u(s))ds + \sum_{i=1}^m G_2(t, t_i)J_i(u(t_i)) + \sum_{i=1}^m G_3(t, t_i)I_i(u(t_i)),
\]

where \( G_i(i = 1, 2, 3) \) are defined in (2.3), (2.4) and (2.5). Then from Lemma 2.5 solving the solutions of (1.1) reduces to solve the fixed points of the operator equation \( u = \mathcal{A}u \). Furthermore, we can adopt the Ascoli-Arzela theorem to prove \( \mathcal{A} \) is a completely continuous operator.

Define \( P = \{ u \in PC(J) : u(t) \geq 0, t \in [0, 1] \} \), and \( P_0 = \{ u \in PC(J) : u(t) \geq \frac{b^2}{a-b} \| u \|, t \in [0, 1] \} \). Then \( P, P_0 \) are cone on \( PC(J) \). Moreover, we easily obtain the following lemma.

**Lemma 2.7.** \( \mathcal{A}(P) \subseteq P_0 \).

Let \( E \) be a Banach space, \( P \) be a cone on \( E \), and \( B_R := \{ u \in E : \| u \| < R \} \) for \( R > 0 \) in the sequel.

**Lemma 2.8** ([3]). Let \( \mathcal{A} : B_R \cap P \to P \) be a completely continuous operator. If there exists \( v_0 \in P \setminus \{ 0 \} \) such that \( v - \mathcal{A}v \neq \lambda v_0 \) for all \( v \in \partial B_R \cap P \) and \( \lambda \geq 0 \), then \( i(\mathcal{A}, B_R \cap P, P) = 0 \), where \( i \) is the fixed point index on \( P \).
Lemma 2.9. Let $A : \overline{B}_R \cap P \to P$ be a completely continuous operator. If $v \neq \lambda Av$ for all $v \in \partial B_R \cap P$ and $0 \leq \lambda \leq 1$, then $i(A, B_R \cap P, P) = 1$.

Lemma 2.10. Let $A : E \to E$ be a completely continuous operator. Assume that $T : E \to E$ is a bounded linear operator such that 1 is not an eigenvalue of $T$ and

$$\lim_{\|u\| \to \infty} \frac{\|Au - Tu\|}{\|u\|} = 0.$$

Then $A$ has a fixed point in $E$.

3. Main results

Theorem 3.1. Assume that

(H2). $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ and $I_k, J_k : \mathbb{R} \to \mathbb{R}(k = 1, 2, \ldots, m)$ are continuous functions, moreover,

$$\lim_{u \to \infty} \frac{f(t, u)}{u} = \lambda, \quad \text{uniformly in } t \in [0, 1],$$

and

$$\lim_{u \to \infty} \frac{I_k(u)}{u} = \lambda, \quad \lim_{u \to \infty} \frac{J_k(u)}{u} = \lambda, \quad k = 1, 2, \ldots, m.$$

If

$$|\lambda| < \left[ \kappa_1 + m \left( \frac{a^2}{(a-b)^2} + \frac{a}{a-b} \right) \right]^{-1},$$

then (1.1) has a nontrivial solution when $f(t, 0) \neq 0$ for $t \in [0, 1]$.

Proof. Define $T : PC(J) \to PC(J)$ by

$$(Tu)(t) := \lambda \left[ \int_0^1 G_1(t, s)u(s)ds + \sum_{i=1}^m G_2(t, t_i)u(t_i) + \sum_{i=1}^m G_3(t, t_i)u(t_i) \right]. \quad (3.1)$$

Then $T$ is a bounded linear operator. From Lemma 2.5 equation (3.1) is equivalent to

$$\begin{cases} \epsilon D_0^\alpha u(t) = \lambda u(t), & \text{a.e. } t \in [0, 1]; \\ \Delta u(t_k) = \lambda u(t_k), \quad \Delta u'(t_k) = \lambda u(t_k), & k = 1, 2, \ldots, m; \\ au(0) - bu(1) = 0, \quad au'(0) - bu'(1) = 0. \end{cases} \quad (3.2)$$

Next, we consider the following two cases.

Case 1. $\lambda = 0$. Equation (3.2) is a problem without impulse, and from Lemma 2.3 we have

$$u(t) = c_0 + c_1 t$$

for some $c_i \in \mathbb{R}, i = 0, 1$. In view of the boundary conditions (3.2), we have $c_0 = c_1 = 0$ and thus $u(t) \equiv 0$ for $t \in [0, 1]$. This shows (3.2) has only a trivial solution.

Case 2. $\lambda \neq 0$. From Case 1 we see (3.2) has nontrivial solutions. Let $u$ be a nontrivial solution for (3.2) and then $\|u\| > 0$. Suppose that 1 is an eigenvalue of $T$. Then we have

$$\|u\| = \|Tu\| \leq |\lambda| \|u\| \left[ \int_0^1 G_1(t, s)ds + \sum_{i=1}^m G_2(t, t_i) + \sum_{i=1}^m G_3(t, t_i) \right]$$

$$\leq |\lambda| \left[ \kappa_1 + m \left( \frac{a^2}{(a-b)^2} + \frac{a}{a-b} \right) \right] \|u\| < \|u\|.$$
This is impossible.

To sum up, 1 is not an eigenvalue of $T$.

From (H2), for all $\varepsilon > 0$, there exists $M > 0$ such that

$$|f(t, u) - \lambda u| \leq \varepsilon |u|, \ |I_k(u) - \lambda u| \leq \varepsilon |u|, \ |J_k(u) - \lambda u| \leq \varepsilon |u|, \text{ for } t \in [0, 1], |u| \geq M.$$  

Moreover, if $|u| \leq M$, then $|f(t, u) - \lambda u|, |I_k(u) - \lambda u|$ and $|J_k(u) - \lambda u|$ are bounded. Hence, there exists $M_1 > 0$ such that

$$|f(t, u) - \lambda u| \leq \varepsilon |u| + M_1, \ |I_k(u) - \lambda u| \leq \varepsilon |u| + M_1, \ |J_k(u) - \lambda u| \leq \varepsilon |u| + M_1, \text{ for } t \in [0, 1], u \in \mathbb{R}.$$  

Hence

$$\|Au - Tu\| = \sup_{t \in [0, 1]} \left| \int_0^1 G_1(t, s) [f(s, u(s)) - \lambda u(s)] \, ds \right|$$

$$+ \sum_{i=1}^m G_2(t, t_i) |J_i(u(t_i)) - \lambda u(t_i)| + \sum_{i=1}^m G_3(t, t_i) |I_i(u(t_i)) - \lambda u(t_i)|$$

$$\leq \sup_{t \in [0, 1]} \int_0^1 G_1(t, s) |f(s, u(s)) - \lambda u(s)| \, ds$$

$$+ \sup_{t \in [0, 1]} \sum_{i=1}^m G_2(t, t_i) |J_i(u(t_i)) - \lambda u(t_i)| + \sup_{t \in [0, 1]} \sum_{i=1}^m G_3(t, t_i) |I_i(u(t_i)) - \lambda u(t_i)|$$

$$\leq (\varepsilon \|u\| + M_1) \left[ \kappa_1 + m \left( \frac{a^2}{(a-b)^2} + \frac{a}{a-b} \right) \right],$$

which implies that

$$\lim_{\|u\| \to \infty} \frac{\|Au - Tu\|}{\|u\|} \leq \lim_{\|u\| \to \infty} \frac{(\varepsilon \|u\| + M_1) \left[ \kappa_1 + m \left( \frac{a^2}{(a-b)^2} + \frac{a}{a-b} \right) \right]}{\|u\|} = \varepsilon \left[ \kappa_1 + m \left( \frac{a^2}{(a-b)^2} + \frac{a}{a-b} \right) \right].$$

Note that the arbitrariness of $\varepsilon$, so

$$\lim_{\|u\| \to \infty} \frac{\|Au - Tu\|}{\|u\|} = 0.$$  

Therefore, from Lemma 2.10, $A$ has a fixed point in $PC(J)$, that is, 1 has at least one solution $u$. Further, we can assert that $u$ is nontrivial when $f(t, 0) \neq 0$ for $t \in [0, 1]$. This completes the proof.  

In order to establish the following two theorems, we need some conditions as follows:

(H3). There exist $c > 0$ and $a_1 \geq 0, a_2 \geq 0, a_3 \geq 0$ satisfying

$$a_3b^4m + a_2b^3(a-b)m > (a-b)^2(a^2 - aba_1 \kappa_1)$$

such that

$$f(t, u) \geq a_1 u - c, \ I_k(u) \geq a_2 u - c, \ J_k(u) \geq a_3 u - c, \text{ for all } t \in [0, 1], u \in \mathbb{R}^+.$$  

(H4). There exist $r > 0$ and $b_1 \geq 0, b_2 \geq 0, b_3 \geq 0$ satisfying

$$(1 - \kappa_1 b_1)b^2(a-b)^2 > m \left[ a^4b_3 + a^3(a-b)b_2 \right]$$

such that

$$f(t, u) \leq b_1 u, \ I_k(u) \leq b_2 u, \ J_k(u) \leq b_3 u, \text{ for all } t \in [0, 1], u \in [0, r].$$
(H5). There exist $r > 0$ and $a_4 \geq 0, a_5 \geq 0, a_6 \geq 0$ satisfying

$$a_6 b^4 m + a_5 b^3 (a - b) m > (a - b)^2 (a^2 - a b_4 \kappa_1)$$

such that

$$f(t, u) \geq a_4 u, \quad I_k(u) \geq a_5 u, \quad J_k(u) \geq a_6 u, \quad \text{for all } t \in [0, 1], u \in [0, r].$$

(H6). There exist $c > 0$ and $b_4 \geq 0, b_5 \geq 0, b_6 \geq 0$ satisfying

$$(1 - \kappa_1 b_4) b^2 (a - b)^2 > m \left[ a_4 b_6 + a_3 (a - b) b_5 \right]$$

such that

$$f(t, u) \leq b_4 u + c, \quad I_k(u) \leq b_5 u + c, \quad J_k(u) \leq b_6 u + c, \quad \text{for all } t \in [0, 1], u \in \mathbb{R}^+. \leqno{(3.6)}$$

**Theorem 3.2.** Suppose that (H1), (H3) and (H4) hold. Then $[1, 1]$ has at least one positive solution.

**Proof.** Let $M_1 = \{ u \in P : u = A u + \lambda \psi, \lambda \geq 0 \}$, where $\psi \in P_0$ is a given element. From Lemma 2.7, $u \in M_1$ implies that $u \in P_0$. We shall prove that $M_1$ is bounded. If $u \in M_1$, then $u \geq A u$. This shows

$$u(t) \geq \int_0^1 G_1(t, s) f(s, u(s)) ds + \sum_{i=1}^m G_2(t, t_i) J_i(u(t_i)) + \sum_{i=1}^m G_3(t, t_i) I_i(u(t_i)). \leqno{(3.3)}$$

Multiplying by $M(t)$ on both sides of the above and integrating over $[0, 1]$, we obtain

$$\int_0^1 u(t) M(t) dt \geq \int_0^1 M(t) \left[ \int_0^1 G_1(t, s) f(s, u(s)) ds + \sum_{i=1}^m G_2(t, t_i) J_i(u(t_i)) + \sum_{i=1}^m G_3(t, t_i) I_i(u(t_i)) \right] dt \geq b a_\kappa \int_0^1 f(t, u(t)) M(t) dt + \frac{b^2}{(a - b)^2} \kappa_1 \sum_{i=1}^m J_i(u(t_i)) + \frac{b}{a - b} \kappa_1 \sum_{i=1}^m I_i(u(t_i)). \leqno{(3.4)}$$

Combining this and (H3), we find

$$\int_0^1 u(t) M(t) dt \geq b a_\kappa \int_0^1 M(t) (a_1 u(t) - c) dt + \frac{b^2}{(a - b)^2} \kappa_1 \sum_{i=1}^m (a_3 u(t_i) - c) + \frac{b}{a - b} \kappa_1 \sum_{i=1}^m (a_2 u(t_i) - c) \leqno{(3.5)}$$

$$= \frac{b}{a} a_1 \kappa_1 \int_0^1 u(t) M(t) dt + \frac{b^2}{(a - b)^2} a_3 \kappa_1 \sum_{i=1}^m u(t_i) + \frac{b}{a - b} a_2 \kappa_1 \sum_{i=1}^m u(t_i) - c_1,$$

where $c_1 = \frac{b}{a} a_1 \kappa_1 + \frac{b^2 c m}{(a - b)^2} \kappa_1 + \frac{b c m}{a - b} \kappa_1$. Next we consider the following two cases.

**Case 1.** $\frac{b}{a} a_1 \kappa_1 \geq 1$. From (3.5) and $u \in P_0$, we obtain

$c_1 \geq \frac{b}{a} a_1 \kappa_1 - 1 \int_0^1 u(t) M(t) dt + \frac{b^2}{(a - b)^2} a_3 \kappa_1 \sum_{i=1}^m u(t_i) + \frac{b}{a - b} a_2 \kappa_1 \sum_{i=1}^m u(t_i) \leqno{(3.6)}$

This shows that there exists $M_2 > 0$ such that

$$\|u\| \leq \frac{a^2 (a - b)^2}{b^2} \cdot \frac{c_1}{a_3 b^2 \kappa_1 m + a_2 b (a - b) \kappa_1 m + \kappa_1 (a - b)^2 (\frac{b}{a} a_1 \kappa_1 - 1)} = M_2, \text{ for all } u \in M_1.
Case 2. \( \frac{b}{a_1} \kappa_1 < 1 \). From (3.5), we have
\[
c_1 + (1 - \frac{b}{a_1} \kappa_1) \int_0^1 u(t)M(t)dt \geq \frac{b^2}{(a - b)^2} a_3 \kappa_1 \sum_{i=1}^{m} u(t_i) + \frac{b}{a - b} a_2 \kappa_1 \sum_{i=1}^{m} u(t_i).
\]
Note that \( u \in P_0 \), we have
\[
c_1 + (1 - \frac{b}{a_1} \kappa_1) \|u\| \geq \frac{b^2}{(a - b)^2} a_3 \kappa_1 \sum_{i=1}^{m} \frac{b^2}{a^2} \|u\| + \frac{b}{a - b} a_2 \kappa_1 \sum_{i=1}^{m} \frac{b^2}{a^2} \|u\|.
\]
Therefore,
\[
\|u\| \leq \frac{c_1 a^2 (a - b)^2}{a_3 b^2 \kappa_1 a + a_2 b (a - b) \kappa_1 m - (a - b)^2 (a^2 - a_1 b_1) \kappa_1} =: M_3, \quad \text{for all} \ u \in \mathcal{M}_1.
\]
To sum up, \( \mathcal{M}_1 \) is a bounded set, as required. Taking \( R > \max\{M_2, M_3\} \), we obtain
\[
u \neq Au + \lambda \psi, \quad \text{for all} \ u \in \partial B_R \cap P, \quad \lambda \geq 0.
\]
Lemma 2.8 yields
\[
i(A, B_R \cap P, P) = 0.
\]
Let \( \mathcal{M}_2 := \{u \in \overline{B}_r \cap P : u = \lambda Au, \lambda \in [0, 1]\} \). We shall prove \( \mathcal{M}_2 = \{0\} \). Indeed, if \( u \in \mathcal{M}_2 \), we have \( u \in P_0 \) and
\[
u(t) \leq \int_0^1 G_1(t, s)f(s, u(s))ds + \sum_{i=1}^{m} G_2(t, t_i)J_i(u(t_i)) + \sum_{i=1}^{m} G_3(t, t_i)I_i(u(t_i)), \quad \text{for all} \ u \in \overline{B}_r \cap P.
\]
Similar to (3.4), multiplying by \( M(t) \) on both sides of the above and integrating over \([0, 1]\), we obtain
\[
\int_0^1 u(t)M(t)dt \leq \kappa_1 \int_0^1 M(t)f(t, u(t))dt + \frac{a^2}{(a - b)^2} \kappa_1 \sum_{i=1}^{m} J_i(u(t_i)) + \frac{a}{a - b} \kappa_1 \sum_{i=1}^{m} I_i(u(t_i)), \quad \text{for all} \ u \in \overline{B}_r \cap P.
\]
This, together with (H4), implies that
\[
\int_0^1 u(t)M(t)dt \leq \kappa_1 b_1 \int_0^1 u(t)M(t)dt + \frac{a^2}{(a - b)^2} b_3 \kappa_1 \sum_{i=1}^{m} u(t_i) + \frac{a}{a - b} b_2 \kappa_1 \sum_{i=1}^{m} u(t_i).
\]
From \( u \in P_0 \) we have
\[
\frac{(1 - \kappa_1 b_1)}{a^2} \kappa_1 \|u\| \leq (1 - \kappa_1 b_1) \int_0^1 u(t)M(t)dt \leq \frac{a^2}{(a - b)^2} b_3 \kappa_1 \sum_{i=1}^{m} \|u\| + \frac{a}{a - b} b_2 \kappa_1 \sum_{i=1}^{m} \|u\|,
\]
which contradicts the condition \( \frac{(1 - \kappa_1 b_1)}{a^2} \kappa_1 > m \left[ \frac{a^2}{(a - b)^2} b_3 \kappa_1 + \frac{a}{a - b} b_2 \kappa_1 \right] \). This implies \( \mathcal{M}_2 = \{0\} \) and thus \( u \neq \lambda Au \) for all \( u \in \partial B_r \cap P \) and \( \lambda \in [0, 1] \). Lemma 2.9 yields
\[
i(A, B_r \cap P, P) = 1.
\]
Equations (3.9) and (3.12) imply that
\[
i(A, (B_R \setminus \overline{B}_r) \cap P, P) = 0 - 1 = -1.
\]
Hence the operator \( A \) has at least one fixed point on \( (B_R \setminus \overline{B}_r) \cap P \) and therefore (1.1) has at least one positive solution. This completes the proof.
Theorem 3.3. Suppose that (H1), (H5) and (H6) hold. Then [1.1] has at least one positive solution.

Proof. Let $\mathcal{M}_3 := \{u \in B_r \cap P : u = Au + \lambda \psi, \lambda \geq 0\}$, where $\psi \in P_0$ is a given element. We claim $\mathcal{M}_3 \subset \{0\}$. Indeed, if $u \in \mathcal{M}_3$, then $u \in P_0$ and $u \geq Au$. By (H5) and (3.5), we have

$$\int_0^1 u(t)M(t)dt \geq \frac{b}{a}a_4 \kappa_1 \int_0^1 u(t)M(t)dt + \frac{b^2}{(a-b)^2}a_6 \kappa_1 \sum_{i=1}^m u(t_i) + \frac{b}{a-b} a_5 \kappa_1 \sum_{i=1}^m u(t_i).$$  \hspace{1cm} (3.13)

If $\frac{b}{a}a_4 \kappa_1 \geq 1$, note that $u \in P_0$, then

$$0 \geq \left(1 - \frac{b}{a}a_4 \kappa_1\right) \int_0^1 u(t)M(t)dt \geq \left(1 - \frac{b}{a}a_4 \kappa_1\right) \int_0^1 u(t)M(t)dt + \frac{b^2}{(a-b)^2}a_6 \kappa_1 \sum_{i=1}^m u(t_i) + \frac{b}{a-b} a_5 \kappa_1 \sum_{i=1}^m u(t_i)$$

$$\geq \frac{b^2}{(a-b)^2}a_6 \kappa_1 \sum_{i=1}^m \frac{b^2}{a^2} ||u|| + \frac{b}{a-b} a_5 \kappa_1 \sum_{i=1}^m \frac{b^2}{a^2} ||u||,$$

which contradicts the property $(1 - \frac{b}{a}a_4 \kappa_1) \kappa_1 < \kappa_1 m \left[ \frac{b^2}{(a-b)^2}a_6 \kappa_1 + \frac{b}{a-b} a_5 \kappa_1 \right]$. This also verify $||u|| \equiv 0, \forall u \in \mathcal{M}_3$.

Hence $\mathcal{M}_3 \subset \{0\}$, as claimed. As a result, we have $u - Au \neq \lambda \psi$ for all $u \in \partial B_r \cap P$ and $\lambda \geq 0$. Lemma 2.8 gives

$$i(A, B_r \cap P, P) = 0.$$  \hspace{1cm} (3.14)

Let $\mathcal{M}_4 := \{u \in P : u = \lambda Au, \lambda \in [0, 1]\}$. We assert $\mathcal{M}_4$ is bounded. Indeed, if $u \in \mathcal{M}_4$, then we have $u \in P_0$ and $u \leq Au$, which can be written in the form

$$u(t) \leq \int_0^1 G_1(t, s)f(s, u(s))ds + \sum_{i=1}^m G_2(t, t_i)J_i(u(t_i)) + \sum_{i=1}^m G_3(t, t_i)I_i(u(t_i)).$$

By (H6) and (3.10), we obtain

$$\int_0^1 u(t)M(t)dt \leq b_4 \kappa_1 \int_0^1 u(t)M(t)dt + \frac{a^2}{(a-b)^2}b_6 \kappa_1 \sum_{i=1}^m u(t_i) + \frac{a}{a-b} b_5 \kappa_1 \sum_{i=1}^m u(t_i) + c_2,$$

where $c_2 = \kappa_2 c + \frac{a^2 c m}{(a-b)^2} \kappa_1 + \frac{a c m}{a-b} \kappa_1$.

From $u \in P_0$, we get

$$\frac{(1 - b_4 \kappa_1)b^2}{a^2} \kappa_1 ||u|| \leq \left(1 - b_4 \kappa_1\right) \int_0^1 u(t)M(t)dt \leq \frac{a^2}{(a-b)^2}b_6 \kappa_1 \sum_{i=1}^m u(t_i) + \frac{a}{a-b} b_5 \kappa_1 \sum_{i=1}^m u(t_i) + c_2$$

$$\leq \frac{a^2}{(a-b)^2}b_6 \kappa_1 \sum_{i=1}^m ||u|| + \frac{a}{a-b} b_5 \kappa_1 \sum_{i=1}^m ||u|| + c_2.$$
Consequently, we see
\[
\|u\| \leq c_2 a^2 (a - b)^2 \left( 1 - \kappa_1 b^2 \right),
\]
Now \( M_4 \) is a bounded set, as asserted. Taking \( R > M_4 \), we have \( u \neq \lambda A u \) for all \( u \in \partial B_R \cap P \) and \( \lambda \in [0, 1] \). Lemma 2.9 yields
\[
i(A, B_R \cap P, P) = 1.
\]
Equations (3.14) and (3.15) imply that
\[
i(A, (B_R \setminus B_r) \cap P, P) = 1 - 0 = 1.
\]
Hence the operator \( A \) has at least one fixed point on \((B_R \setminus B_r) \cap P\) and therefore, (1.1) has at least one positive solution. This completes the proof.

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References


