Brunn-Minkowski type inequalities for $L_p$ Blaschke-Minkowski homomorphisms

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Abstract

In this paper, the Brunn-Minkowski type inequalities for $L_p$ Blaschke-Minkowski homomorphisms and $L_p$ radial Minkowski homomorphisms are established. ©2016 All rights reserved.

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1. Introduction and preliminaries

The Brunn-Minkowski inequality is one of the most important geometric inequalities. There is a huge amount of work on its generalizations and on its connections with other areas (see [1, 5–7, 16, 18]). The excellent survey article of Gardner [5] gives a comprehensive account of various aspects and consequences of the Brunn-Minkowski inequality.

Projection bodies and intersection bodies played a critical role in the solution of the Shephard problem and the Busemann-Petty problem, respectively (see [14]). Through the work of Ludwig [12, 13], projection bodies and intersection bodies were characterized as continuous and $GL(n)$ contravariant valuations. Recently, Schuster [19, 20] introduced the Blaschke-Minkowski homomorphisms and radial Blaschke-Minkowski homomorphisms which are more general than the well-known projection body operators and intersection bodies, respectively. In order to state their definition, let $K^n$ denote the space of all convex bodies in $\mathbb{R}^n$ endowed with the Hausdorff topology.

A map $\Phi : K^n \rightarrow K^n$ is called a Blaschke-Minkowski homomorphism, if it satisfies the following conditions:

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(a) $\Phi$ is continuous with respect to the Hausdorff metric.

(b) For all $K_1, K_2 \in \mathcal{K}^n$,
\[ \Phi(K_1 \# K_2) = \Phi K_1 + \Phi K_2, \]
where $K_1 \# K_2$ denotes Blaschke addition (see [9]) of $K_1$ and $K_2$, and $\Phi K_1 + \Phi K_2$ is the Minkowski addition of $\Phi K_1$ and $\Phi K_2$.

(c) For all $K \in \mathcal{K}^n$ and every $\nu \in SO(n)$,
\[ \Phi(\nu K) = \nu \Phi K, \]
where $SO(n)$ is the group of rotations of $\mathbb{R}^n$.

Let $\mathcal{S}^n$ denote the space of all star bodies in $\mathbb{R}^n$ endowed with the radial metric. A map $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ is called a radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:

(a$^*$) $\Psi$ is continuous with respect to the radial metric.

(b$^*$) For all $L_1, L_2 \in \mathcal{S}^n$,
\[ \Psi(L_1 \tilde{\#} L_2) = \Psi L_1 \tilde{+} \Psi L_2, \]
where $L_1 \tilde{\#} L_2$ denotes the radial Blaschke addition (see [8]) of $L_1$ and $L_2$, and $\Psi L_1 \tilde{+} \Psi L_2$ is the radial Minkowski addition of $\Psi L_1$ and $\Psi L_2$.

(c$^*$) For all $L \in \mathcal{S}^n$ and every $\nu \in SO(n)$,
\[ \Psi(\nu L) = \nu \Psi L. \]

Volume inequalities for convex body and star body valued valuations are an active field of research (see [2–4, 17, 21, 23, 25]).

In the recent paper [22], Wang introduced the following concept of the $L_p$ Blaschke-Minkowski homomorphisms:

A map $\Phi_p : \mathcal{K}^n_s \to \mathcal{K}^n_s$ is called an $L_p$ Blaschke-Minkowski homomorphism, if it satisfies the following conditions:

(1) $\Phi_p$ is continuous with respect to the Hausdorff metric.

(2) For all $K_1, K_2 \in \mathcal{K}^n_s$,
\[ \Phi_p(K_1 \#_p K_2) = \Phi_p K_1 +_p \Phi_p K_2, \]
where $K_1 \#_p K_2$ denotes $L_p$ Blaschke addition of $K_1$ and $K_2$, and $\Phi_p K_1 +_p \Phi_p K_2$ is the $L_p$ Minkowski addition of $\Phi_p K_1$ and $\Phi_p K_2$.

(3) For all $K \in \mathcal{K}^n_s$ and every $\nu \in SO(n)$,
\[ \Phi_p(\nu K) = \nu \Phi_p K, \]
where $SO(n)$ is the group of rotations of $\mathbb{R}^n$.

In the paper [24], Wang et al. defined $L_p$ radial Minkowski homomorphisms as follows:

A map $\Psi_p : \mathcal{S}^n \to \mathcal{S}^n$ is called an $L_p$ radial Minkowski homomorphism, if it satisfies the following conditions:

(1$^*$) $\Psi_p$ is continuous with respect to the radial metric.

(2$^*$) For all $L_1, L_2 \in \mathcal{S}^n$,
\[ \Psi_p(L_1 \tilde{+}_{n-p} L_2) = \Psi_p L_1 \tilde{+}_{n-p} \Psi_p L_2, \]
where $L_1 \tilde{+}_{n-p} L_2$ denotes the radial addition of $L_1$ and $L_2$, and $\Psi_p L_1 \tilde{+}_{n-p} \Psi_p L_2$ is the radial Minkowski addition (see [8]) of $\Psi_p L_1$ and $\Psi_p L_2$. 
(3*) For all \( L \in S^n \) and every \( v \in SO(n) \), 
\[
\Psi_p(vL) = v\Psi_pL.
\]

In [19], Schuster has established the following Brunn-Minkowski type inequalities.

**Theorem 1.1** ([19]). Let \( \Phi : K^n \to K^n \) be a Blaschke-Minkowski homomorphism. If \( K_1, K_2 \in K^n \), then
\[
V(\Phi(K_1 + K_2))^{1/(n-1)} \geq V(\Phi(K_1))^{1/(n-1)} + V(\Phi(K_2))^{1/(n-1)},
\]
with equality, if and only if \( K_1 \) and \( K_2 \) are homothetic.

The operator \( \Phi \) is called even, if \( \Phi K = \Phi(-K) \) for all \( K \in K^n \).

**Theorem 1.2** ([19]). Let \( \Phi : K^n \to K^n \) be an even Blaschke-Minkowski homomorphism. If \( K_1, K_2 \in K^n \), then
\[
V(\Phi^+(K_1 + K_2))^{1/(n-1)} \geq V(\Phi^+(K_1))^{1/(n-1)} + V(\Phi^+(K_2))^{1/(n-1)},
\]
with equality, if and only if \( K_1 \) and \( K_2 \) are homothetic. Here \( \Phi^+K \) is the polar body of \( \Phi K \).

The aim of this paper is to establish Brunn-Minkowski type inequalities for \( L_p \) Blaschke-Minkowski homomorphisms and \( L_p \) radial Minkowski homomorphisms.

**Theorem 1.3.** Let \( \Phi_p : K^n_s \to K^n_s \) be an \( L_p \) Blaschke-Minkowski homomorphism. If \( K_1, K_2 \in K^n_s \) and \( n \neq p \geq 1 \), then
\[
V(\Phi_p(K_1 \#_p K_2))^{p/n} \geq V(\Phi_p(K_1))^{p/n} + V(\Phi_p(K_2))^{p/n}, \tag{1.1}
\]
with equality in (1.1), if and only if \( \Phi_pK_1 \) and \( \Phi_pK_2 \) are dilates.

**Theorem 1.4.** Let \( \Phi_p : K^n_s \to K^n_s \) be an \( L_p \) Blaschke-Minkowski homomorphism. If \( K_1, K_2 \in K^n_s \) and \( n \neq p \geq 1 \), then
\[
V(\Phi_p^+(K_1 \#_p K_2))^{-p/n} \geq V(\Phi_p^+(K_1))^{-p/n} + V(\Phi_p^+(K_2))^{-p/n}, \tag{1.2}
\]
with equality in (1.2), if and only if \( \Phi_pK_1 \) and \( \Phi_pK_2 \) are dilates.

**Theorem 1.5.** Let \( \Psi_p : S^n \to S^n \) be an \( L_p \) radial Minkowski homomorphism. If \( K_1, K_2 \in S^n_0 \) and \( 0 < p < n \), then
\[
V(\Psi_p(K_1 ^{+}_{p-n} K_2))^{p/n} \leq V(\Psi_p(K_1))^{p/n} + V(\Psi_p(K_2))^{p/n}, \tag{1.3}
\]
with equality in (1.3), if and only if \( \Psi_pK_1 \) and \( \Psi_pK_2 \) are dilates.

If \( p < 0 \) or \( p > n \), then we get
\[
V(\Psi_p(K_1 ^{+}_{p-n} K_2))^{p/n} \geq V(\Psi_p(K_1))^{p/n} + V(\Psi_p(K_2))^{p/n}, \tag{1.4}
\]
with equality (1.4), if and only if \( \Psi_pK_1 \) and \( \Psi_pK_2 \) are dilates.

2. Notation and background material

Let \( K^n \) denote the set of all convex bodies (compact, convex subsets with non-empty interiors) in \( \mathbb{R}^n \), and let \( K^n_0 \) denote the set of convex bodies that contain the origin in their interiors. The subset of \( K^n_0 \) consisting of the centered convex bodies will be denoted by \( K^n_s \). \( S^{n-1} \) is the unit sphere. A convex body is uniquely determined by its support function. The support function of \( K \in K^n, h(K, \cdot) \), is defined on \( S^{n-1} \) by
\[
h(K, u) = \max\{u \cdot x : x \in K\}.
\]

Let \( \delta \) denote the Hausdorff metric on \( K^n \), i.e., for \( K, L \in K^n \), \( \delta(K, L) = |h_K - h_L|_\infty \), where \(|\cdot|_\infty\) denotes the sup-norm on the space of continuous functions, \( C(S^{n-1}) \).
Associated with a compact subset \( L \in \mathbb{R}^n \), which is star-shaped with respect to the origin, is its radial function \( \rho(L, \cdot) : S^{n-1} \to \mathbb{R} \), defined by

\[
\rho(L, u) = \max \{ \lambda \geq 0 : \lambda u \in L \}.
\]

If \( \rho(L, \cdot) \) is positive and continuous, we call \( L \) a star body. Let \( S^n \) and \( S^n_0 \) denote the set of star bodies and the set of star bodies (about the origin) in \( \mathbb{R}^n \), respectively. Two star bodies \( K, L \) are said to be dilates (of one another), if \( \rho_K(u)/\rho_L(u) \) is independent of \( u \in S^{n-1} \).

If \( K \in \mathcal{K}^n_0 \), then the polar body of \( K, K^* \), is defined by

\[
K^* := \{ x \in \mathbb{R}^n : x \cdot y \leq 1, \forall y \in K \}.
\]

From (2.1), it follows that \( (K^*)^* = K \) and

\[
h_{K^*} = \frac{1}{\rho_K}, \quad \rho_{K^*} = \frac{1}{h_K}.
\]

Let \( K_1, K_2 \in \mathcal{K}^n_0, p \geq 1 \), and \( \lambda_1, \lambda_2 \geq 0 \) (not both 0). The \( L_p \) Minkowski sum \( \lambda_1 \cdot K_1 +_p \lambda_2 \cdot K_2 \) is the convex body whose support function is given by (see [15])

\[
h(\lambda_1 \cdot K_1 +_p \lambda_2 \cdot K_2, \cdot)^p = \lambda_1 h(K_1, \cdot)^p + \lambda_2 h(K_2, \cdot)^p.
\]

For \( p \geq 1 \), the \( L_p \)-mixed volume \( V_p(K, L) \) of \( K, L \in \mathcal{K}^n_0 \), can be defined by

\[
\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.
\]

In [15], Lutwak has shown that for \( p \geq 1 \), and each \( K \in \mathcal{K}^n_0 \), there exists a positive Borel measure \( S_p(K, \cdot) \) on \( S^{n-1} \), such that the \( L_p \)-mixed volume \( V_p(K, L) \) has the following integral representation:

\[
V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h^p(L, u)dS_p(K, u),
\]

for all \( L \in \mathcal{K}^n_0 \). The \( L_p \)-Minkowski inequality states that for \( K, L \in \mathcal{K}^n_0 \) and \( p \geq 1 \)

\[
V_p(K, L) \geq V(K)^{(n-p)/n} V(L)^{p/n},
\]

with equality, if and only if \( K \) and \( L \) are dilates.

For \( n \neq p \geq 1 \) and \( K, L \in \mathcal{K}^n_0 \), the \( L_p \)-Blaschke addition \( K +_p L \in \mathcal{K}^n_0 \) was defined in [15] by

\[
S_p(K +_p L, \cdot) = S_p(K, \cdot) + S_p(L, \cdot).
\]

Let \( K, L \in \mathcal{S}^n \), and \( p \in \mathbb{R} \) and \( p \neq 0 \). The \( L_p \) radial addition \( K +_p L \cdot \cdot \) is the star body defined by

\[
\rho(K +_p L \cdot \cdot , \cdot)^p = \rho(K, \cdot)^p + \rho(L, \cdot)^p.
\]

The \( L_p \) dual mixed volume \( V_p(K, L) \) of \( K, L \in \mathcal{K}^n_0 \), can be defined by

\[
\frac{n}{p} \tilde{V}_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.
\]

The definition above and the polar coordinate formula for volume give the following integral representation of the dual mixed volume \( \tilde{V}_p(K, L) \)

\[
\tilde{V}_p(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-p} \rho(L, u)^p dS(u).
\]
3. Proof of the main results

In this section, we give the proofs of our main results Theorems 1.3–1.5. First, we need the following lemma.

Lemma 3.1 ([10]). Let $K, L \in S^n$, if $0 < p < n$, then
\[
\tilde{V}_p(K, L) \leq V(K)^{(n-p)/n}V(L)^{p/n},
\]
with equality, if and only if $K$ and $L$ are dilates. If $p < 0$ or $p > n$, then
\[
\tilde{V}_p(K, L) \geq V(K)^{(n-p)/n}V(L)^{p/n},
\]
with equality, if and only if $K$ and $L$ are dilates.

Proof of Theorem 1.3 Let $K, L \in K^n_s$ and $n \neq p \geq 1$. From the definition of $L_p$ Blaschke-Minkowski homomorphisms and the $L_p$-Minkowski inequality, for any $M \in K^n_0$, it follows that
\[
V_p(M, \Phi_p(K_1\#_p K_2)) = V_p(M, \Phi_p K_1 + p \Phi_p K_2)

= V_p(M, \Phi_p K_1) + V_p(M, \Phi_p K_2)

\geq V(M)^{(n-p)/n}(V(\Phi_p K_1)^{p/n} + V(\Phi_p K_2)^{p/n}),
\]
with equality, if and only if $M, \Phi_p K_1$ and $\Phi_p K_2$ are dilates.

By taking $M = \Phi_p(K_1\#_p K_2)$, we get
\[
V(\Phi_p(K_1\#_p K_2))^{p/n} \geq V(\Phi_p K_1)^{p/n} + V(\Phi_p K_2)^{p/n},
\]
with equality, if and only if $\Phi_p K_1$ and $\Phi_p K_2$ are dilates.

Therefore we have proved inequality (1.1). \qed

Proof of Theorem 1.4 Let $K, L \in K^n_s$ and $n \neq p \geq 1$. From the polar coordinate formula for volume and the Minkowski integral inequality, it follows that
\[
V(\Phi_p(K_1\#_p K_2))^{p/n} = \left(\frac{1}{n} \int_{S^{n-1}} (h(\Phi_p(K_1\#_p K_2), u)^p)^{-n/p} dS(u)\right)^{-p/n}

= n^{p/n} \|h(\Phi_p(K_1, u))^p + h(\Phi_p(K_2, u))^p\|^{-n/p}
\]
\[
\geq n^{p/n} \|h(\Phi_p(K_1, u))^p\|^{-p/n} + n^{p/n} \|h(\Phi_p(K_2, u))^p\|^{-p/n},
\]
with equality, if and only if $\Phi_p K_1$ and $\Phi_p K_2$ are dilates.

Therefore we have proved inequality (1.2). \qed

Proof of Theorem 1.5 Let $K_1, K_2 \in S^n_0$ and $0 < p < n$. From Lemma 3.1 and the $L_p$-Minkowski inequality, for any $M \in S^n_0$, it follows that
\[
\tilde{V}_p(M, \Psi_p(K_1\#_{-p} K_2)) = \tilde{V}_p(M, \Psi_p K_1 + \Psi_p K_2)

= \tilde{V}_p(M, \Psi_p K_1) + \tilde{V}_p(M, \Psi_p K_2)

\leq V(M)^{(n-p)/n}(V(\Psi_p K_1)^{p/n} + V(\Psi_p K_2)^{p/n}),
\]
with equality, if and only if $M, \Psi_p K_1$ and $\Psi_p K_2$ are dilates.

By taking $M = \Psi_p(K_1\#_{-p} K_2)$, we get
\[
V(\Psi_p(K_1\#_{-p} K_2))^{p/n} \leq V(\Psi_p K_1)^{p/n} + V(\Psi_p K_2)^{p/n},
\]
with equality, if and only if $Ψ_pK_1$ and $Ψ_pK_2$ are dilates.

Therefore we have proved inequality (1.3).

If $p < 0$ or $p > n$, then we get

$$V(Ψ_p(K_1+_{n−p}K_2))^{p/n} ≥ V(Ψ_pK_1)^{p/n} + V(Ψ_pK_2)^{p/n},$$

with equality, if and only if $Ψ_pK_1$ and $Ψ_pK_2$ are dilates. The inequality (1.4) is proved. □

Since the $L_p$ projection body operator $Π_p$ is an $L_p$ Blaschke-Minkowski homomorphism, we get the following inequalities which were established by Lu and Leng in [11].

**Corollary 3.2 ([11]).** Let $Π_p : K^n_s → K^n_s$ be the $L_p$ projection body operator. If $K_1, K_2 ∈ K^n_s$ and $n ≠ p ≥ 1$, then

$$V(Π_p(K_1#_{p}K_2))^{p/n} ≥ V(Π_pK_1)^{p/n} + V(Π_pK_2)^{p/n},$$

(3.1)

$$V(Π_p(K_1#_{p}K_2))^{−p/n} ≥ V(Π_pK_1)^{−p/n} + V(Π_pK_2)^{−p/n},$$

(3.2)

with equality in (3.1) and (3.2), if and only if $Π_pK_1$ and $Π_pK_2$ are dilates.

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**References**


