Solvability for integral boundary value problems of fractional differential equation on infinite intervals

Changlong Yu*, Jufang Wang, Yanping Guo

Abstract
In this paper, we establish the solvability for integral boundary value problems of fractional differential equation with the nonlinear term dependent in a fractional derivative of lower order on infinite intervals. The existence and uniqueness of solutions for the boundary value problem are proved by means of the Schauder’s fixed point theorem and Banach’s contraction mapping principle. Finally, we give two examples to demonstrate the use of the main results. ©2016 All rights reserved.

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1. Introduction
Boundary value problems on infinite intervals appear often in applied mathematics and physics. More examples and a collection of works on the existence of solutions of Boundary value problems on infinite intervals for differential, difference and integral equations may be found in the monographs [1,15]. For some works and various techniques dealing with such boundary value problems, see [2, 5, 7, 13, 21, 22, 23, 26] and the references therein.

The fractional differential equation has emerged as a new branch in the field of differential equations for their deep back grounds. For an extensive collection of such results, we refer the readers to the monographs [9, 11, 16, 17]. There has been a significant development in nonlocal problems for fractional differential equations or inclusions, see [3, 4, 8, 10, 12, 14, 19, 20, 24, 27] and the references therein.

*Corresponding author
Email addresses: changlongyu@126.com (Changlong Yu), wangjufang1981@126.com (Jufang Wang), guoyanping65@126.com (Yanping Guo)

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Boundary value problems for fractional differential equations on infinite intervals have been considered widely and there are some excellent results on the existence of solutions, see [6, 18, 25] and the references therein. However, to our knowledge, it is rare for works to be done on the solutions for integral boundary value problems (IBVPs) of fractional differential equations on infinite interval.

Recently, in [18], X. Su and S. Zhang considered the BVP:

\[
\begin{aligned}
D_{0+}^\alpha u(t) &= f(t, u(t), D_{0+}^{\alpha-1} u(t)), \quad t \in J := [0, +\infty), \\
u(0) &= 0, \quad D_{0+}^{\alpha-1} u(\infty) = u_\infty, \quad u_\infty \in R,
\end{aligned}
\]

where \( 1 < \alpha \leq 2 \), \( f \in C(J \times R \times R, R), D_{0+}^\alpha \) and \( D_{0+}^{\alpha-1} \) are the standard Riemann-Liouville fractional derivatives. The existence of unbounded positive solutions was obtained by the Schauder’s fixed point theorem on unbounded domain.

Motivated by the work above, in this paper, we will discuss the following IBVP:

\[
\begin{aligned}
D_{0+}^\alpha u(t) &= f(t, u(t), D_{0+}^{\alpha-1} u(t)), \quad t \in J, \\
u(0) &= 0, \quad D_{0+}^{\alpha-1} u(\infty) = \int_0^{+\infty} g(t) u(t) dt,
\end{aligned}
\]

where \( J = [0, +\infty), 1 < \alpha \leq 2, f \in C(J \times R \times R, R), \eta \geq 0, g(t) \in L^1[0, +\infty) \) and \( \int_0^{+\infty} g(t) t^{\alpha-1} dt < \Gamma(\alpha) \), \( D_{0+}^\alpha \) and \( D_{0+}^{\alpha-1} \) are the standard Riemann-Liouville fractional derivatives and \( D_{0+}^{\alpha-1} u(\infty) = \lim_{t \to +\infty} D_{0+}^{\alpha-1} u(t) \).

We deal with the existence and uniqueness of solutions for BVP (1.1) by using the Schauder’s fixed point theorem and Banach’s contraction mapping principle and obtain multiplicity results which extend and improve the known results.

2. Preliminary results

In this section, we introduce definitions and preliminary facts which are used throughout paper. Following, let us recall some basic concepts of fractional calculus, see [9, 16, 17] and the references therein.

**Definition 2.1.** The Riemann-Liouville fractional integral of order \( \delta > 0 \) of a function \( f(t) \) is defined by

\[
I_{a+}^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_a^t (t-s)^{\delta-1} f(s) ds, \quad t > a,
\]

provided that the right-hand side is pointwise defined.

It is well known that \( I_{a+}^\delta f(a) = 0 \), for \( f(t) \in C[a, b], \delta > 0 \) and \( I_{a+}^\delta : C[a, b] \to C[a, b] \) for \( \delta > 0 \).

**Definition 2.2.** The Riemann-Liouville fractional derivative of order \( \delta > 0 \) of a function \( f(t) \) is defined by

\[
D_{a+}^\delta f(t) = \left( \frac{d}{dt} \right)^n I_{a+}^{n-\delta} f(t) = \frac{1}{\Gamma(n-\delta)} \left( \frac{d}{dt} \right)^n \int_a^t (t-s)^{n-\delta-1} f(s) ds, \quad t > a,
\]

where \( n \) is the smallest integer greater than or equal to \( \delta \), provided that the right-hand side is pointwise defined. In particular, for \( \delta = n \), \( D_{a+}^n f(t) = f^{(n)}(t) \).

**Lemma 2.3** ([9]). In this work, we need the following composition relations:

(a) \( D_{a+}^\delta I_{a+}^\delta f(t) = f(t), \quad \delta > 0, \quad f(t) \in L^1[0, +\infty); \)
(b) \( D_{a+}^\delta I_{a+}^{\gamma-\delta} f(t) = I_{a+}^{\gamma-\delta} f(t), \quad \gamma > \delta > 0, \quad f(t) \in L^1[0, +\infty). \)

**Definition 2.4** ([13]). It holds that \( f : [0, \infty) \times R^2 \to R \) is called an S-Carathéodory function if and only if

(i) for each \( (u, v) \in R^2, t \in f(t, u, v) \) is measurable on \( [0, \infty); \)
(ii) for almost every \( t \in [0, \infty), \quad (u, v) \mapsto f(t, u, v) \) is continuous on \( R^2; \)
(iii) for each \( r > 0 \), there exist \( \varphi_r(t) \in L^1[0, \infty), \quad \varphi_r(t) > 0 \) on \( [0, \infty) \) such that \( \max\{|u|, |v|\} \leq r \) implies

\[
|f(t, u, v)| \leq \varphi_r(t), \quad \text{for a.e. } t \in [0, \infty).
\]
Lemma 2.5 ([18]). For $\delta > 0$, the equation $D_0^\delta x(t) = 0$ is valid if and only if, $x(t) = \sum_{j=1}^{n} c_j (t-a)^{\delta-j}$, where $c_j \in \mathbb{R}, \quad j = 1, 2, \cdots, n$ are arbitrary constants and $n$ is the smallest integer greater than or equal to $\delta$.

Lemma 2.6. Let $y(t) \in L^1[0, +\infty)$ and $\int_{\eta}^{+\infty} g(t) t^{\alpha-1} dt \neq \Gamma(\alpha)$, then IBVP

$$
\begin{align*}
\left\{ \begin{array}{ll}
D_0^{\alpha} u(t) = y(t), & t \in J, \\
u(0) = 0, & D_0^{\alpha-1} u(\infty) = \int_{\eta}^{+\infty} g(t) u(t) dt,
\end{array} \right.
\end{align*}
$$

(2.1)

has a unique solution

$$
u(t) = \int_{0}^{+\infty} G(t, s)y(s) ds,$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)(\Gamma(\alpha) - \int_{\eta}^{+\infty} g(t) t^{\alpha-1} dt)} \left\{ \begin{array}{ll}
[H(\eta, s) - \Gamma(\alpha)]_t^{\alpha-1} + (\Gamma(\alpha) - \int_{\eta}^{+\infty} g(t) t^{\alpha-1} dt)(t-s)^{\alpha-1}, & s \leq t, \\
[H(\eta, s) - \Gamma(\alpha)]_t^{\alpha-1}, & s \geq t,
\end{array} \right. $$

and

$$H(\eta, s) = \left\{ \begin{array}{ll}
\int_{\eta}^{+\infty} g(t)(t-s)^{\alpha-1} dt, & s \leq \eta, \\
\int_{\eta}^{+\infty} g(t)(t-s)^{\alpha-1} dt, & s \geq \eta.
\end{array} \right. $$

Proof. We may apply Lemma 2.5 to reduce the differential equation in (2.1) to the integral equation

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} y(s) ds.$$

In accordance with Lemma 2.3, Lemma 2.5 and the relation $D_0^{\alpha-1} t^{\alpha-1} = \Gamma(\alpha)$, we have

$$D_0^{\alpha-1} u(t) = c_1 \Gamma(\alpha) + \int_{0}^{t} y(s) ds.$$

The boundary condition in (2.1) imply that $c_2 = 0$ and

$$c_1 = \frac{\int_{0}^{+\infty} H(\eta, s) y(s) ds - \Gamma(\alpha) \int_{0}^{+\infty} y(s) ds}{\Gamma(\alpha)(\Gamma(\alpha) - \beta \int_{\eta}^{+\infty} g(t) t^{\alpha-1} dt)},$$

where

$$H(\eta, s) = \left\{ \begin{array}{ll}
\int_{\eta}^{+\infty} g(t)(t-s)^{\alpha-1} dt, & s \leq \eta, \\
\int_{\eta}^{+\infty} g(t)(t-s)^{\alpha-1} dt, & s \geq \eta.
\end{array} \right. $$

Hence,

$$u(t) = \int_{0}^{+\infty} H(\eta, s) y(s) ds - \Gamma(\alpha) \int_{0}^{+\infty} y(s) ds + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} y(s) ds$$

$$= \int_{0}^{+\infty} G(t, s)y(s) ds.$$

This proof is complete. \qed
Lemma 2.7. Let $\int_{\eta}^{+\infty} g(t) t^{\alpha-1} dt < \Gamma(\alpha)$, then

$$|G(t, s)| \leq \frac{2t^{\alpha-1}}{\Gamma(\alpha) - \int_{\eta}^{+\infty} g(t) t^{\alpha-1} dt}. \quad (2.2)$$

Proof. Obviously, it holds

$$0 \leq H(\eta, s) \leq \int_{\eta}^{+\infty} g(t) t^{\alpha-1} dt,$$

therefore, when $s \leq t$, then

$$|G(t, s)| \leq \frac{(\int_{\eta}^{+\infty} g(t) t^{\alpha-1} dt + \Gamma(\alpha)) t^{\alpha-1} + (\Gamma(\alpha) - \int_{\eta}^{+\infty} g(t) t^{\alpha-1} dt)t^{\alpha-1}}{\Gamma(\alpha)(\Gamma(\alpha) - \int_{\eta}^{+\infty} g(t) t^{\alpha-1} dt)} \leq \frac{2t^{\alpha-1}}{\Gamma(\alpha) - \int_{\eta}^{+\infty} g(t) t^{\alpha-1} dt}.$$

Obviously, when $s \geq t$, (2.2) holds. This proof is complete. \(\square\)

Remark 2.8. By Lemma 2.6, it is easy to get

$$D_{0+}^{\alpha-1} u(t) = \int_{0}^{+\infty} G_1(t, s) g(s) ds,$$

where

$$G_1(t, s) = \frac{1}{\Gamma(\alpha) - \int_{\eta}^{+\infty} g(t) t^{\alpha-1} dt} \left\{ \begin{array}{ll} H(\eta, s) - \int_{\eta}^{+\infty} g(t) t^{\alpha-1} dt, & s \leq t, \\ H(\eta, s) - \Gamma(\alpha), & s \geq t, \end{array} \right.$$ 

and if $\int_{\eta}^{+\infty} g(t) t^{\alpha-1} dt < \Gamma(\alpha)$, then

$$|G_1(t, s)| \leq \frac{2\Gamma(\alpha)}{\Gamma(\alpha) - \int_{\eta}^{+\infty} g(t) t^{\alpha-1} dt}. \quad (2.3)$$

Define the spaces by

$$X = \left\{ u(t) \in C(J, R) : \sup_{t \in J} \frac{|u(t)|}{1 + t^{\alpha-1}} < +\infty \right\},$$

$$C_{\infty} = \left\{ u(t) \in X : D_{0+}^{\alpha-1} u(t) \in C(J, R), \sup_{t \in J} |D_{0+}^{\alpha-1} u(t)| < +\infty \right\},$$

and the norm $\|u\|_X = \sup_{t \in J} \frac{|u(t)|}{1 + t^{\alpha-1}}$, $\|u\| = \max\{\sup_{t \in J} \frac{|u(t)|}{1 + t^{\alpha-1}}, \sup_{t \in J} |D_{0+}^{\alpha-1} u(t)|\}$.

Lemma 2.9 ([8]). $(X, \| \cdot \|_X)$ and $(C_{\infty}, \| \cdot \|)$ are Banach spaces.

Define operator $T : C_{\infty} \to C_{\infty}$,

$$Tu(t) := \int_{0}^{+\infty} G(t, s) f(s, u(s), D_{0+}^{\alpha-1} u(s)) ds, \quad (2.4)$$

and

$$D_{0+}^{\alpha-1} Tu(t) = \frac{\int_{0}^{+\infty} H(\eta, s) f(s, u(s), D_{0+}^{\alpha-1} u(s)) ds - \Gamma(\alpha) \int_{0}^{+\infty} f(s, u(s), D_{0+}^{\alpha-1} u(s)) ds}{\Gamma(\alpha) - \int_{\eta}^{+\infty} g(t) t^{\alpha-1} dt}$$

$$+ \int_{0}^{t} f(s, u(s), D_{0+}^{\alpha-1} u(s)) ds$$

$$= \int_{0}^{+\infty} G_1(t, s) f(s, u(s), D_{0+}^{\alpha-1} u(s)) ds. \quad (2.5)$$
In this paper, our basic space is $C_\infty$. Note that the Arzela-Ascoli theorem fails to work in $C_\infty$. Therefore, we need the following compactness criterion.

**Lemma 2.10** ([13]). Let $Z \subseteq Y$ be a bounded set, then $Z$ is relatively compact in $Y$ if the following conditions hold:

(i) For any $u(t) \in Z$, \( \frac{u(t)}{1 + t^{\alpha-1}} \) and $D_0^{\alpha-1}u(t)$ are equicontinuous on any compact interval of $J$; 
(ii) Given $\varepsilon > 0$, there exists a constant $T = T(\varepsilon) > 0$ such that
\[
|\frac{u(t_1)}{1 + t_1^{\alpha-1}} - \frac{u(t_2)}{1 + t_2^{\alpha-1}}| < \varepsilon \quad \text{and} \quad |D_0^{\alpha-1}u(t_1) - D_0^{\alpha-1}u(t_2)| < \varepsilon
\]
for any $t_1, t_2 \geq T$ and $u(t) \in Z$.

### 3. Main results

In this section, we apply fixed point theorem to IBVP [1.1]. First, we give the uniqueness result based on Banach’s contraction mapping principle.

**Theorem 3.1.** Let $f : J \times R^2 \rightarrow R$ be an $S$-Caratheodory function and there exist $L_1(t), L_2(t) \in L^1[0, +\infty)$ such that
\[
|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq L_1(t)|u_1 - u_2| + L_2(t)|v_1 - v_2|, \quad t \in I, \quad (u_1, v_1), (u_2, v_2) \in R^2.
\]
In addition, suppose that $\Lambda < 1$ holds, where
\[
\Lambda = \frac{2}{\Gamma(\alpha)} - \int_0^{+\infty} g(t)t^{\alpha-1}dt \int_0^{+\infty} [(1 + t^{\alpha-1})L_1(t) + L_2(t)]dt.
\]
Then IBVP [1.1] has a unique solution.

**Proof.** Let us choose
\[
r \geq \frac{2\int_0^{+\infty} \varphi_r(t)dt}{\Gamma(\alpha) - \int_0^{+\infty} g(t)t^{\alpha-1}dt - \int_0^{+\infty} [(1 + t^{\alpha-1})L_1(t) + L_2(t)]dt}.
\]
Now, we show that $TB_r \subset B_r$, where $B_r = \{u \in C_\infty : ||u|| \leq r\}$.

For each $u \in B_r$, we have
\[
\frac{|Tu(t)|}{1 + t^{\alpha-1}} = \int_0^{+\infty} \frac{|G(t, s)|}{1 + t^{\alpha-1}} |f(s, u(s), D_0^{\alpha-1}u(s))| ds \leq \frac{2}{\Gamma(\alpha)} - \int_0^{+\infty} g(t)t^{\alpha-1}dt \int_0^{+\infty} |f(s, u(s), D_0^{\alpha-1}u(s))|ds
\]
\[
\leq \frac{2}{\Gamma(\alpha)} - \int_0^{+\infty} g(t)t^{\alpha-1}dt \int_0^{+\infty} \left[ |f(s, u(s), D_0^{\alpha-1}u(s)) - f(s, 0, 0)| + |f(s, 0, 0)| \right] ds
\]
\[
\leq \frac{2}{\Gamma(\alpha)} - \int_0^{+\infty} g(t)t^{\alpha-1}dt \left( ||u|| \int_0^{+\infty} [(1 + t^{\alpha-1})L_1(t) + L_2(t)]dt + \int_0^{+\infty} \varphi_r(t)dt \right)
\]
\[
\leq r,
\]
and
\[
|D_0^{\alpha-1}Tu(t)| \leq \frac{2\Gamma(\alpha)}{\Gamma(\alpha) - \int_0^{+\infty} g(t)t^{\alpha-1}dt} \int_0^{+\infty} |f(s, u(s), D_0^{\alpha-1}u(s))|ds \leq \Gamma(\alpha)r \leq r.
\]
Hence, we obtain that $|Tu| \leq r$, so $TB_r \subset B_r$.

Next, for $u, v \in C_{\infty}$ and for each $t \in J$, we have

$$
\begin{align*}
\left| \frac{Tu(t)}{1 + t^\alpha} - \frac{Tv(t)}{1 + t^\alpha} \right| &= \frac{1}{1 + t^\alpha} \left| \int_0^{+\infty} G(t, s)(f(s, u(s), D_0^{\alpha-1} u(s)) - f(s, v(s), D_0^{\alpha-1} v(s)))ds \right| \\
&\leq \frac{2}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt} \int_0^{+\infty} |f(s, u(s), D_0^{\alpha-1} u(s)) - f(s, v(s), D_0^{\alpha-1} v(s))|ds \\
&\leq \frac{2}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt} \int_0^{+\infty} \left[ L_1(s)|u_1(s) - u_2(s)| + L_2(s)|v_1(s) - v_2(s)| \right]ds \\
&\leq \frac{2}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt} \int_0^{+\infty} [(1 + t^{\alpha-1})L_1(t) + L_2(t)]dt \|u - v\| \\
&\leq \Lambda \|u - v\| < \|u - v\|
\end{align*}
$$

and

$$
\begin{align*}
|D_0^{\alpha-1} Tu(t) - D_0^{\alpha-1} Tv(t)| &= \left| \int_0^{+\infty} G_1(t, s)(f(s, u(s), D_0^{\alpha-1} u(s)) - f(s, v(s), D_0^{\alpha-1} v(s)))ds \right| \\
&\leq \frac{2\Gamma(\alpha)}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt} \int_0^{+\infty} |f(s, u(s), D_0^{\alpha-1} u(s)) - f(s, v(s), D_0^{\alpha-1} v(s))|ds \\
&< \Gamma(\alpha)\|u - v\| \leq \|u - v\|.
\end{align*}
$$

Therefore, we obtain that $|Tu - Tv| < \|u - v\|$, $T$ is a contraction map. Thus, the conclusion of the theorem follows by Banach’s contraction mapping principle. \qed

The next existence result is based on the Schauder’s fixed-point theorem.

**Theorem 3.2.** Assume that there exists nonnegative functions $p(t), q(t), r(t) \in L^1(J, R^+)$ with $t^{\alpha-1}p(t) \in L^1(J, R^+)$ such that

$$
|f(t, u, v)| \leq p(t)|u| + q(t)|v| + r(t), \quad t \in J, \quad (u, v) \in R^2.
$$

(3.1)

Then BVP (1.1) has at least one solution provided

$$
\int_0^{+\infty} [(1 + t^{\alpha-1})p(t) + q(t)] \frac{dt}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt} < \frac{2}{2 + \int_0^{+\infty} r(s)ds}.
$$

Proof. First of all, by the continuity of $f$, we can conclude that $Tu(t)$ and $D_0^{\alpha-1} Tu(t)$ are continuous on $J$.

In what follows, we divide the proof into several steps.

Setp 1. Choose

$$
R \geq \frac{2 \int_0^{+\infty} r(s)ds}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt - 2 \int_0^{+\infty} [(1 + s^{\alpha-1})p(s) + q(s)]ds},
$$

and let

$$
U = \{u(t) \in C_{\infty} : \|u\| \leq R\}.
$$

Then, $T : U \to U$.

Indeed, for any $u(t) \in U$, by (2.2)-(2.5) and the condition of Theorem 3.2, we can get

$$
\frac{|Tu(t)|}{1 + t^{\alpha-1}} \leq \frac{2}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt} \int_0^{+\infty} |f(s, u(s), D_0^{\alpha-1} u(s))|ds
$$

and
Therefore, by Lebesgue’s dominated convergence theorem, we have
\[ ||\rightarrow 0 \text{ as } n \rightarrow \infty. \] Hence, \[ ||Tu|| \leq R \] and this show that \( T : U \rightarrow U \).

Step 2. \( T : U \rightarrow U \) is continuous operator.

Let \( u_n, u \in U, \ n = 1, 2, \cdots \), and \[ ||u_n - u|| \rightarrow 0 \text{ as } n \rightarrow \infty. \] Then by (2.2)–(2.5) and the condition of Theorem 3.2 we obtain that
\[
\left| \frac{T_{u_n(t)}}{1 + t^{\alpha-1}} - \frac{T_{u(t)}}{1 + t^{\alpha-1}} \right| = \frac{1}{1 + t^{\alpha-1}} \left| \int_0^{+\infty} G(t, s) (f(s, u_n(s), D_{0+}^{\alpha-1}u_n(s)) - f(s, u(s), D_{0+}^{\alpha-1}u(s))ds \right|
\leq \frac{2}{\Gamma(\alpha) - \int_{s}^{+\infty} g(t)t^{\alpha-1}dt} \int_0^{+\infty} |f(s, u_n(s), D_{0+}^{\alpha-1}u_n(s)) - f(s, u(s), D_{0+}^{\alpha-1}u(s))|ds
\leq \frac{2}{\Gamma(\alpha) - \int_{s}^{+\infty} g(t)t^{\alpha-1}dt} \left[ \int_0^{+\infty} |f(s, u_n(s), D_{0+}^{\alpha-1}u_n(s))|ds + \int_0^{+\infty} |f(s, u(s), D_{0+}^{\alpha-1}u(s))|ds \right]
\leq \frac{2}{\Gamma(\alpha) - \int_{s}^{+\infty} g(t)t^{\alpha-1}dt} \left[ ||u_n|| \int_0^{+\infty} [(1 + s^{\alpha-1})p(s) + q(s)]ds + \int_0^{+\infty} r(s)ds \right]
+ ||u|| \int_0^{+\infty} [(1 + s^{\alpha-1})p(s) + q(s)]ds + \int_0^{+\infty} r(s)ds
\leq \frac{4R \int_0^{+\infty} [(1 + s^{\alpha-1})p(s) + q(s)]ds + 4 \int_0^{+\infty} r(s)ds}{\Gamma(\alpha) - \int_{s}^{+\infty} g(t)t^{\alpha-1}dt},
\]
and
\[
|D_{0+}^{\alpha-1}Tu_n(t) - D_{0+}^{\alpha-1}Tu(t)|
\leq \frac{2\Gamma(\alpha)}{\Gamma(\alpha) - \int_{s}^{+\infty} g(t)t^{\alpha-1}dt} \int_0^{+\infty} |f(s, u_n(s), D_{0+}^{\alpha-1}u_n(s)) - f(s, u(s), D_{0+}^{\alpha-1}u(s))|ds
\leq \frac{4\Gamma(\alpha)}{\Gamma(\alpha) - \int_{s}^{+\infty} g(t)t^{\alpha-1}dt} \left[ R \int_0^{+\infty} [(1 + s^{\alpha-1})p(s) + q(s)]ds + \int_0^{+\infty} r(s)ds \right].
\]
Therefore, by Lebesgue’s dominated convergence theorem, we have \[ ||Tu_n - Tu|| \rightarrow 0 \text{ as } n \rightarrow \infty. \] Hence, \( T \) is continuous.

Step 3. Let \( V \) be a subset of \( U \). We apply Lemma 2.10 to verify that \( TV \) is relatively compact.
Let $I \in J$ be a compact interval, $t_1, t_2 \in I$ and $t_1 < t_2$. Then for any $u(t) \in V$, it is easy to know $f(t, u(t), D^{\alpha-1}_0 u(t))$ is bounded on $I$, so we can obtain that

$$\frac{|Tu(t_2)|}{1 + t_2^{\alpha-1}} - \frac{|Tu(t_1)|}{1 + t_1^{\alpha-1}} \leq \frac{\int_0^{+\infty} H(\eta, s) f(s, u(s), D^{\alpha-1}_0 u(s)) ds - \Gamma(\alpha) \int_0^{+\infty} f(s, u(s), D^{\alpha-1}_0 u(s)) ds}{\Gamma(\alpha) \left[ \frac{\Gamma(\alpha) - \int_0^{+\infty} g(t) t^{\alpha-1} dt}{1 + t_2^{\alpha-1}} \right] - \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{1 + t_1^{\alpha-1}}} + \frac{1}{\Gamma(\alpha)} \int_0^{t_1 \frac{(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{1 + t_2^{\alpha-1}} f(s, u(s), D^{\alpha-1}_0 u(s)) ds}$$

$$\leq \frac{\int_0^{+\infty} H(\eta, s) f(s, u(s), D^{\alpha-1}_0 u(s)) ds + \Gamma(\alpha) \int_0^{+\infty} f(s, u(s), D^{\alpha-1}_0 u(s)) ds}{\Gamma(\alpha) \left[ \frac{\Gamma(\alpha) - \int_0^{+\infty} g(t) t^{\alpha-1} dt}{1 + t_2^{\alpha-1}} \right] - \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{1 + t_1^{\alpha-1}}} + \frac{1}{\Gamma(\alpha)} \int_0^{t_2 \frac{(t_2 - s)^{\alpha-1}}{1 + t_2^{\alpha-1}} f(s, u(s), D^{\alpha-1}_0 u(s)) ds}$$

$$\leq \frac{2 R \int_0^{+\infty} [(1 + s^{-1}) p(s) + q(s)] ds + \int_0^{+\infty} r(s) ds}{\Gamma(\alpha) \left[ \frac{\Gamma(\alpha) - \int_0^{+\infty} g(t) t^{\alpha-1} dt}{1 + t_2^{\alpha-1}} \right] - \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{1 + t_1^{\alpha-1}}} + \frac{\max |f(s, u(s), D^{\alpha-1}_0 u(s))|}{\Gamma(\alpha)} \left[ \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{1 + t_2^{\alpha-1}} ds + \int_0^{t_1} \frac{(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{1 + t_1^{\alpha-1}} ds \right],$$

$$\to 0, \text{ uniformly as } t_1 \to t_2,$$

and

$$|D^{\alpha-1}_0 Tu(t_2) - D^{\alpha-1}_0 Tu(t_1)| \leq \int_{t_1}^{t_2} |f(t, u(t), D^{\alpha-1}_0 u(t))| dt \to 0, \text{ uniformly as } t_1 \to t_2.$$  

Then it is easy to see that $\frac{Tu(t)}{1 + t^{\alpha-1}}$ and $D^{\alpha-1}_0 Tu(t)$ are equicontinuous on $I$.

Now, we show that for any $u(t) \in V$, $\frac{Tu(t)}{1 + t^{\alpha-1}}$ and $D^{\alpha-1}_0 Tu(t)$ satisfy the condition (ii) of Lemma 2.10. Observing that by the condition of Theorem 3.2, we have

$$\int_0^{+\infty} |f(t, u(t), D^{\alpha-1}_0 u(t))| dt \leq ||u|| \int_0^{+\infty} [(1 + t^{-1}) p(t) + q(t)] dt + \int_0^{+\infty} r(t) dt$$

$$\leq \frac{2}{\Gamma(\alpha) \left[ \frac{\Gamma(\alpha) - \int_0^{+\infty} g(t) t^{\alpha-1} dt}{1 + t_2^{\alpha-1}} \right] - \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{1 + t_1^{\alpha-1}}} \left[ ||u|| \int_0^{+\infty} [(1 + t^{-1}) p(t) + q(t)] dt + \int_0^{+\infty} r(t) dt \right]$$

$$\leq R,$$

we know that for given $\varepsilon > 0$, there exists a constant $L > 0$, such that

$$\int_L^{+\infty} |f(t, u(t), D^{\alpha-1}_0 u(t))| dt < \varepsilon. \quad (3.2)$$

On the other hand, since $\lim_{t \to +\infty} \frac{t^{\alpha-1}}{1 + t^{\alpha-1}} = 1$, there exists a constant $T_1 > 0$, such that for any $t_1, t_2 \geq T_1$, we have

$$\left| \frac{t_1^{\alpha-1}}{1 + t_1^{\alpha-1}} - \frac{t_2^{\alpha-1}}{1 + t_2^{\alpha-1}} \right| < \varepsilon. \quad (3.3)$$
Similarly, \( \lim_{t \to +\infty} \frac{(t-L)^{\alpha-1}}{1+t^{\alpha-1}} = 1 \) and thus there exists a constant \( T_2 > L > 0 \), such that for any \( t_1, t_2 \geq T_2 \) and \( 0 \leq s \leq L \),

\[
\left| \frac{(t_1 - s)^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{(t_2 - s)^{\alpha-1}}{1+t_2^{\alpha-1}} \right| < \varepsilon. \tag{3.4}
\]

Now, choose \( T_0 \) such that \( T_0 > \max\{T_1, T_2\} \), then for any \( t_1, t_2 \geq T_0 \), by (3.2)-(3.4), we can obtain that

\[
\begin{align*}
\left| Tu(t_2) - T u(t_1) \right| & \leq 2 \left( R \int_0^{+\infty} \frac{\alpha-1}{1+t^{\alpha-1}} \left( f(t, u(t), D_0^{\alpha-1}u(t)) \right) dt \right) \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_0^L \left| \frac{(t_2 - s)^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{(t_1 - s)^{\alpha-1}}{1+t_1^{\alpha-1}} \right| |f(s, u(s), D_0^{\alpha-1}u(s))| ds \\
& \quad + \frac{1}{\Gamma(\alpha)} \left[ \int_0^{+\infty} |f(s, u(s), D_0^{\alpha-1}u(s))| ds + \int_L^{+\infty} |f(s, u(s), D_0^{\alpha-1}u(s))| ds \right] \\
& \leq \frac{2 R \int_0^{+\infty} \left( (1+s^{\alpha-1})p(s) + q(s) \right) ds + \int_0^{+\infty} r(s) ds}{\Gamma(\alpha)} \cdot \frac{\varepsilon}{\Gamma(\alpha)} + \frac{L}{\Gamma(\alpha)} \varepsilon,
\end{align*}
\]

and

\[
|D_0^{\alpha-1}Tu(t_2) - D_0^{\alpha-1}Tu(t_1)| \leq \int_{t_1}^{t_2} |f(t, u(t), D_0^{\alpha-1}u(t))| dt \leq \int_L^{+\infty} |f(t, u(t), D_0^{\alpha-1}u(t))| dt < \varepsilon.
\]

Consequently, lemma 2.10 yields that \( TV \) is relatively compact.

Therefore, by Schauder’s fixed point theorem, we conclude that the BVP (1.1) has at least one solutions in \( U \) and the proof is finished. □

4. Example

**Example 4.1.** Consider the following IBVP for fractional differential equation on infinite intervals:

\[
\begin{align*}
D_0^\frac{3}{4} u(t) &= e^{-t} + \frac{1}{10(1+t^2)(1+\sqrt{t})} \sin(u(t)) + \frac{1}{4(1+e^t)} \arctan(D_0^{\frac{1}{2}} u(t)), \\
u(0) &= 0, \quad D_0^{\frac{3}{4}} u(+\infty) = \int_1^{+\infty} \frac{1}{10} t^{-\frac{3}{2}} e^{-t} u(t) dt.
\end{align*}
\] \tag{4.1}

Here, \( \alpha = \frac{3}{4}, \eta = 1 \), \( f(t, u, v) = e^{-t} + \frac{1}{10(1+t^2)(1+\sqrt{t})} \sin(u) + \frac{1}{4(1+e^t)} \arctan(v) \), \( L_1(t) = \frac{1}{10(1+t^2)(1+\sqrt{t})} \), \( L_2(t) = \frac{1}{4(1+e^t)} \) and \( g(t) = \frac{1}{10} t^{-\frac{3}{2}} e^{-t} \). With the aid of simple computation, we can obtain that

\[
|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \frac{1}{10(1+t^2)(1+\sqrt{t})} |u_1 - u_2| + \frac{1}{4(1+e^t)} |v_1 - v_2|.
\]

\[
\int_1^{+\infty} g(t) t^{\frac{3}{2}} dt = \int_1^{+\infty} \frac{1}{10} t^{-\frac{3}{2}} e^{-t} dt < \Gamma(\frac{3}{2}) \text{ and } \Lambda \approx 0.77785 < 1. \] In the view of Theorem 3.1 then IBVP (4.1) has a unique solution.
**Example 4.2.** Consider the following IBVP for fractional differential equation on infinite intervals:

\[
\begin{aligned}
D_0^\alpha u(t) &= \frac{\ln(1+|u(t)|)}{10(1+t^2)(1+\sqrt{t})} + \frac{1}{10e^t} \sin(|D_0^\alpha u(t)|) + t e^{-t^2}, \quad t \in J, \\
u(0) &= 0, \quad D_0^\alpha u(\infty) = \int_1^{+\infty} \frac{1}{2} t^3 e^{-t} u(t) dt.
\end{aligned}
\]

Here, \(\alpha = \frac{5}{4}, \eta = 1, f(t, u, v) = \frac{\ln(1+|u|)}{10(1+t^2)(1+\sqrt{t})} + \frac{1}{10e^t} \sin(|v|) + t e^{-t^2}\) and \(g(t) = \frac{1}{2} e^{-t}\). Obviously,

\[
|f(t, u, v)| \leq \frac{|u|}{10(1+t^2)(1+\sqrt{t})} + \frac{1}{10e^t} |v| + t e^{-t^2},
\]

With the aid of simple computation, we find that

\[
\int_0^{+\infty} \left(1 + t^{\frac{1}{2}}\right) \frac{1}{10(1+t^2)} + \frac{1}{10e^t} dt = \frac{\pi}{20} + \frac{1}{10} \approx 0.25708,
\]

and

\[
\Gamma\left(\frac{5}{4}\right) - \frac{1}{2} \int_1^{+\infty} te^{-t} dt \approx \frac{0.906402 - \frac{1}{2} \times 0.73576}{2} = 0.26926,
\]

Hence, the conditions of Theorem 3.2 are satisfied, so IBVP (4.2) has at least one solution.

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**References**


